Functional Data Analysis of Generalized Quantile Regression

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Generalized Quantile Regression (GQR)

- Quantiles and Expectiles are generalized quantiles, Jones (1994).
- Capture the tail behaviour of conditional distributions.
- Applications in finance, weather, demography, $\cdots$
- Some applications involve MANY GQR curves.

Applications

- Finance: VaR and expected shortfall (ES)
- Labor: Wage with education levels
- Weather: Temperature and rainfall
    - Energy Company
    - Tourist Company
    - Farmers

Figure 1: Land before and after the flood

Weather Derivatives

A CAT temperature future under the non-arbitrage pricing setting:

$$\begin{aligned} F_{CAT(t,\tau_1,\tau_2)} &= \mathsf{E}^{\,Q_\lambda}\left[\int_{\tau_1}^{\tau_2} T_u du \middle| \mathcal{F}_t\right] \\ &= \int_{\tau_1}^{\tau_2} \Lambda_u du + \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{X}_t + \int_{t}^{\tau_1} \lambda_u \sigma_u \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{e}_L du \\ &+ \int_{\tau_1}^{\tau_2} \lambda_u \sigma_u \mathbf{e}_1^\top \mathbf{A}^{-1} \left[\exp\left\{\mathbf{A}(\tau_2 - u)\right\} - I_L\right] \mathbf{e}_L du \end{aligned} \quad (1)$$

where λ_u is market price of risk, and σ_u is the volatility of temperature.

To estimate σ_u more accurately is to price the futures more precisely.

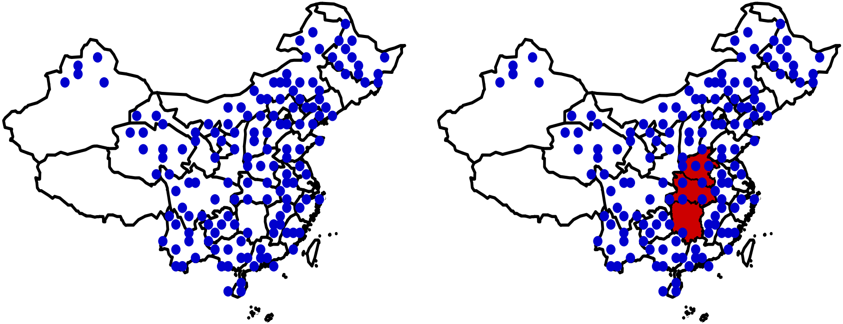


Figure 2: Weather Stations in China

Statistical Challenges

- Traditional: estimate GQR individually
- Directly: estimate GQR jointly

- common structure neglected
- too many parameters, curse of dimensionality

Functional Data Analysis (FDA)

- a tool to capture random curves
- consider dependencies between individuals
- FPCA a tool to reduce dimensionality
- interpretation of factors
- apply “FPCA” and least asymmetric weighted squares (LAWS)

Outline

1. Motivation ✓
2. Generalized Quantile Estimation
3. FDA for GQR
4. Simulation
5. Application
6. Conclusion

Quantile and Expectile

Quantile

$$F(l) = \int_{-\infty}^{l} dF(y) = \tau$$

$$l = F^{-1}(\tau)$$

Expectile

$$G(l) = \frac{\int_{-\infty}^{l} |y - l| \, dF(y)}{\int_{-\infty}^{\infty} |y - l| \, dF(y)} = \tau$$

$$l = G^{-1}(\tau)$$

Loss Function

Loss function:

$$L(y, \theta) = |y - \theta|^{\alpha} \qquad (2)$$

Asymmetric loss function for generalized quantiles:

$$\rho_{\tau}(u) = |\,\mathbf{I}(u \leq 0) - \tau||u|^{\alpha}, \qquad \tau \in (0, 1) \qquad (3)$$

with $\alpha \in \{1, 2\}$ and $u = y - \theta$.

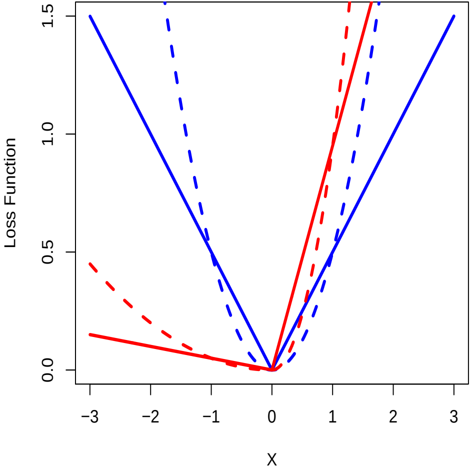


Figure 3: Loss functions for $\tau = 0.9$ (red); $\tau = 0.5$ (blue); $\alpha = 1$ (solid line); $\alpha = 2$ (dashed line).

Weight

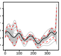
$$w_\alpha(u) = |\mathbf{I}(u \le 0) - \tau||u|^{(\alpha-2)} \tag{4}$$

Minimum contrast approach:

$$\begin{aligned} l_\tau &= \arg\min_\theta \; \mathsf{E}\{\rho_\tau(Y-\theta)\} \\ &= \arg\min_\theta \; \mathsf{E}\, w_\alpha(Y-\theta)|Y-\theta|^2 \end{aligned}$$

Generalized quantile regression curve:

$$\begin{aligned} l_\tau(t) &= \arg\min_\theta \; \mathsf{E}\{\rho_\tau(Y-\theta)|X=t\} \\ &= \arg\min_\theta \; \mathsf{E}\{w_\alpha(Y-\theta)|Y-\theta|^2|X=t\} \end{aligned}$$



Estimation Method

- Kernel Smoothing
  - Quantile: Fan et.al (1994)
  - Expectile: Zhang (1994)
- Penalized Spline Smoothing
  - Quantile: Koenker et.al (1994)
  - Expectile: Schnabel and Eilers (2009)

GQR can be estimated by Iterated Reweighted Asymmetric Least Square.

Single Curve Estimation

Rewrite as regression pb:

$$Y_t = l(t) + \varepsilon_t \tag{5}$$

where $F^{-1}_{\varepsilon|t}(\tau) = 0$ and $G^{-1}_{\varepsilon|t}(\tau) = 0$.
Approximate $l(\cdot)$ by a B-spline basis:

$$l(t) = b(t)^\top \theta_\mu \tag{6}$$

where $b(t) = \{b_1(t), \cdots, b_q(t)\}^\top$ is a vector of cubic B-spline basis and θ_μ is a vector with dimension q.

Estimation

Employ a roughness penalty:

$$S(\theta_\mu) = \sum_{t=1}^{T} w_t\{Y_t - b(t)^\top\theta_\mu\}^2 + \lambda\{\theta_\mu^\top \int \ddot{b}(t)\ddot{b}(t)^\top dt\ \theta_\mu\} \qquad (7)$$

where $Y = (Y_1, Y_2, \cdots, Y_T)^\top$, $\ddot{b}(t) = \frac{\partial^2 b(t)}{\partial t^2}$ and $w_t = w_\alpha\{Y_t - l(t)\}$ ($l(t)$ known).

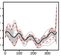
Estimation

The generalized quantile curve:

$$
\begin{aligned}
\widehat{\theta}_\mu &= \arg\min_{\theta_\mu} S(\theta_\mu) \\
&= \{B^\top W B + \lambda \int \ddot{b}(t)\ddot{b}(t)^\top dt\}^{-1}(B^\top W Y)
\end{aligned}
$$

$B = \{b(t)\}_{t=1}^{T}$ is the spline basis matrix with dimension $T \times q$, and $W = \mathrm{diag}\{w_t\}$ defined in (4):

$$\widehat{l}(t) = b(t)\widehat{\theta}_\mu \tag{8}$$



Regression Model

$$Y_{ij} = l_i(t_{ij}) + \varepsilon_{ij} \tag{9}$$

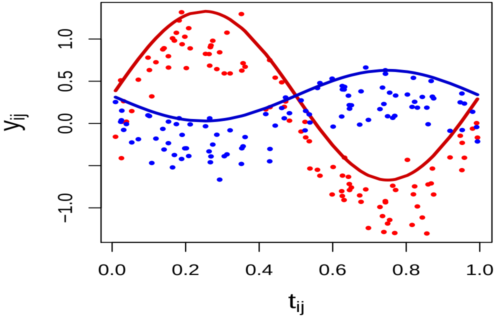
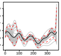


Figure 4: Data design with $\tau = 0.95$. design



Mixed effect Model

Observe $i = 1, \cdots, N$ individual curves:

$$l_i(t) = \mu(t) + v_i(t) \tag{10}$$

- $\mu(t)$ common shape
- $v_i(t)$ departure from $\mu(t)$.

Approximate via

$$l_{ij} = l_i(t_{ij}) = b(t_{ij})^\top \theta_\mu + b(t_{ij})^\top \gamma_{ij} \tag{11}$$

where $i = 1, \cdots, N$ and $j = 1, \cdots, T_i$.

- Too many parameters to estimate.
- Very volatile for sparse data, James et.al (2000).

Reduced Model

▸ Mercer's Lemma
▸ Karhunen-Loève Theorem

$$l_i(t) = \mu(t) + \sum_{k=1}^{K} f_k(t)\alpha_{ik} \tag{12}$$

- K the number of factors and f_k k-th factor:

$$f(t) = \{f_1(t), \cdots, f_K(t)\}^\top$$

- $\alpha_i = (\alpha_{i1}, \cdots, \alpha_{iK})^\top$ random scores.

Representation of μ and f:

$$\begin{aligned} \mu(t) &= b(t)^\top \theta_\mu \\ f(t)^\top &= b(t)^\top \Theta_f \end{aligned}$$

where $\theta_\mu \in R^q$ and Θ_f with dimension $q \times K$.

Reduced Model

Rewrite (12)

$$l_{ij} = l_i(t_{ij}) = b(t_{ij})^\top \theta_\mu + b(t_{ij})^\top \Theta_f \alpha_i \tag{13}$$

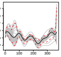
With $L_i = \{l_i(t_1), \cdots, l_i(T_i)\}^\top$, $B_i = \{b(t_1), \cdots, b(T_i)\}^\top$, the GQR curves:

$$L_i = B_i \theta_\mu + B_i \Theta_f \alpha_i \tag{14}$$

Then the model reads:

$$Y_i = L_i + \varepsilon_i = B_i \theta_\mu + B_i \Theta_f \alpha_i + \varepsilon_i \tag{15}$$

with Y_i is $T_i \times 1$ and α_i is $K \times 1$.



Constraints

Orthogonality requirements of the factors:

$$\int f(t)f(t)^\top dt = \Theta_f^\top \int b(t)^\top b(t)dt \; \Theta_f = \mathrm{I}_K$$

That is to say

$$\begin{aligned} \Theta_f^\top \Theta_f &= \mathrm{I}_K \\ \int b(t)^\top b(t)dt &= \mathrm{I}_q \end{aligned}$$

"Empirical" Loss Function

For the GQR regression:

$$S = \sum_{i=1}^{N} \sum_{j=1}^{T_i} w_{ij} \{ Y_{ij} - b(t_j)^\top \theta_\mu - b(t_j)^\top \Theta_f \alpha_i \}^2 \qquad (16)$$

Roughness penalty:

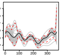
$$\begin{aligned} M_\mu &= \theta_\mu^\top \int \ddot{b}(t) \ddot{b}(t)^\top dt \ \theta_\mu \\ M_f &= \sum_{k=1}^{K} \theta_{kf}^\top \int \ddot{b}(t) \ddot{b}(t)^\top dt \ \theta_{kf} \end{aligned}$$

And $w_{ij} = w_\alpha(Y_{ij} - l_{ij})$, where l_{ij} defined in (13).

LAWS

$$
\begin{aligned}
S^* &= S + \lambda_\mu M_\mu + \lambda_f M_f \\
&= \sum_{i=1}^{N} (Y_i - B_i\theta_\mu - B_i\Theta_f\alpha_i)^\top W_i (Y_i - B_i\theta_\mu - B_i\Theta_f\alpha_i) \\
&\quad + \lambda_\mu \{\theta_\mu^\top \int \ddot{b}(t)\ddot{b}(t)^\top dt\ \theta_\mu\} \\
&\quad + \lambda_f \{\sum_{k=1}^{K} \theta_{f,k}^\top \int \ddot{b}(t)\ddot{b}(t)^\top dt\ \theta_{f,k}\} \qquad (17)
\end{aligned}
$$

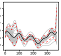
where $\theta_{f,k}$ is the k-th column in Θ_f.



Solutions

Minimizing S^*:

$$\begin{aligned}
\widehat{\theta}_\mu &= \left\{ \sum_{i=1}^{N} B_i^\top W_i B_i + \lambda_\mu \int \ddot{b}(t)\ddot{b}(t)^\top dt \right\}^{-1} \\
&\quad \left\{ \sum_{i=1}^{N} B_i^\top W_i (Y_i - B_i \widehat{\Theta}_f \widehat{\alpha}_i) \right\} \\
\widehat{\theta}_{f,j} &= \left\{ \sum_{i=1}^{N} \widehat{\alpha}_{ij}^2 B_i^\top W_i B_i + \lambda_f \int \ddot{b}(t)\ddot{b}(t)^\top dt \right\}^{-1} \\
&\quad \left\{ \sum_{i=1}^{N} \widehat{\alpha}_{ij} B_i^\top W_i (Y_i - B_i \widehat{\theta}_\mu - B_i Q_{ij}) \right\} \qquad (18)
\end{aligned}$$



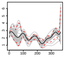
$$\widehat{\alpha}_i = \left\{\widehat{\Theta}_f^\top B_i^\top W_i B_i \widehat{\Theta}_f\right\}^{-1} \left\{\widehat{\Theta}_f^\top B_i^\top W_i (Y_i - B_i \widehat{\theta}_\mu)\right\} \quad (19)$$

Where

$$Q_{ij} = \sum_{k \neq j} \hat{\theta}_{f,k} \hat{\alpha}_{ik}$$

and $i = 1, \cdots, N$, $j = 1, \cdots, K$.

- initial values ▸ Details
- updated procedure ▸ Details



Auxiliary Parameters

- Number of knots is not crucial, James et.al (2000)
- Use 5-fold cross validation (CV) to choose the number of factors and the penalty parameters
- Define an index function $\kappa : \{1, \cdots, N\} \rightarrow \{1, \cdots, 5\}$ and $\widehat{l}_i(\cdot)^{-\kappa(i)}$ denote the estimated ith generalized quantile function using data excluding those in the fold of $\kappa(i)$.

$$CV(K, \lambda_\mu, \lambda_f) = \frac{1}{\sum_{i=1}^{N} T_i} \sum_{i=1}^{N} \sum_{j=1}^{T_i} \rho_\tau\{Y_{ij} - \widehat{l}_i^{-\kappa(i)}(t_{ij})\}$$

Simulation

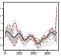
$$Y_{ij} = \mu(t_j) + f_1(t_j)\alpha_{1i} + f_2(t_j)\alpha_{2i} + e_{ij} \tag{20}$$

with $i = 1, \cdots, N$, $j = 1, \cdots, T_i$ and t_j is equal distanced on $[0, 1]$.

The common shape curve and factor functions:

$$\begin{aligned} \mu(t) &= 1 + t + \exp\{-(t-0.6)^2/0.05\} \\ f_1(t) &= \sin(2\pi t)/\sqrt{0.5} \\ f_2(t) &= \cos(2\pi t)/\sqrt{0.5} \end{aligned}$$

where $\alpha_{1i} \sim \mathrm{N}(0, 36)$, $\alpha_{2i} \sim \mathrm{N}(0, 9)$.



Scenarios

- $e_{ij} \sim N(0, 0.5)$
- $e_{ij} \sim N(0, \mu(t) \times 0.5)$
- $e_{ij} \sim t(5)$

- small sample: $N = 20$, $T = T_i = 100$
- large sample: $N = 40$, $T = T_i = 150$

Theoretical τ quantile and expectile for individual i:

$$l_{ij} = \mu(t_j) + f_1(t_j)\alpha_{1i} + f_2(t_j)\alpha_{2i} + \varepsilon_{ij}$$

where ε_{ij} represents the corresponding theoretical τ-th quantile and expectile of the distribution of e_{ij} ($\varepsilon_{ij} = e_{ij} + \sqrt{0.5} \cdot \Phi^{-1}(\tau)$).

Estimators

- The individual curve:

$$\begin{aligned} l_i &= \mu + \sum_{k=1}^{K} f_k \alpha_{ik} \\ \widehat{l}_{i,fp} &= B_i \widehat{\theta}_\mu + B_i \widehat{\Theta}_f \widehat{\alpha}_i \\ \widehat{l}_{i,in} &: \text{Single curve, see (8)} \end{aligned}$$

- The mean curve:

$$\begin{aligned} m &= \mu(t) + e_\tau \\ m_{fp} &= \frac{1}{N} \sum_{i=1}^{N} B_i \widehat{\theta}_\mu \\ m_{in} &= \frac{1}{N} \sum_{i=1}^{N} \widehat{l}_{i,in} \end{aligned} \tag{21}$$

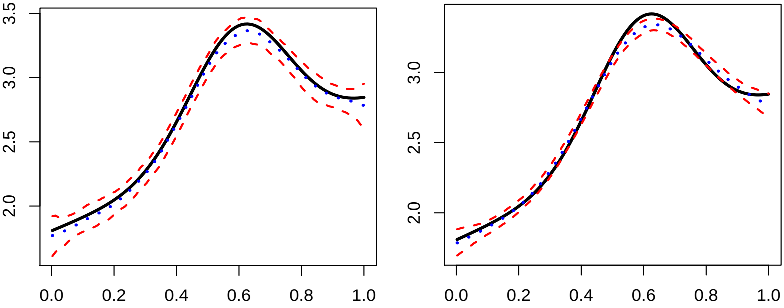


Figure 5: The estimated μ (blue dotted), the real μ (black solid) and the $5\% - 95\%$ pointwise confidence intervals (red dashed) for 95% expectile curves when the error term is normally distributed with mean 0 and variance 0.5. The sample size are respectively $N = 20, M = 100$ (Left) and $N = 40, M = 150$ (Right).

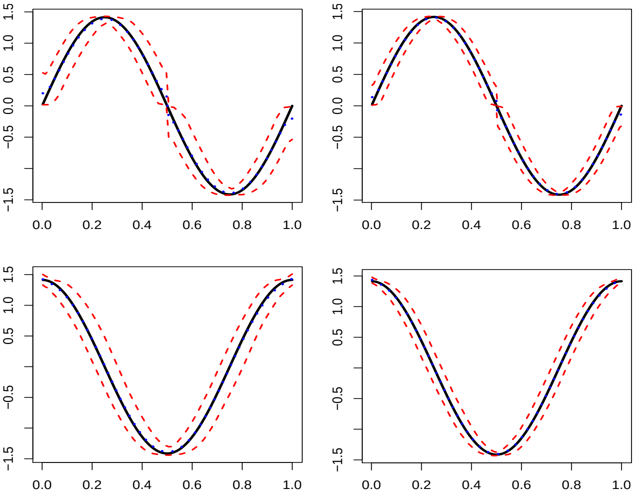


Figure 6: The estimated first and seond principal components f_1 and f_2.

Scenario	Sample Size	Expectile curves FDA	Expectile curves Separate	Quantile curves FDA	Quantile curves Separate
1	$N = 20, T = 100$	0.0815	0.1407	0.1733	0.2539
		(0.0296)	(0.0149)	(0.0283)	(0.0227)
	$N = 40, T = 150$	0.0189	0.0709	0.0723	0.1875
		(0.0025)	(0.0052)	(0.1205)	(0.0127)
2	$N = 20, T = 100$	0.1436	0.3188	0.2769	0.8039
		(0.0248)	(0.0339)	(0.1061)	(0.0860)
	$N = 40, T = 150$	0.0931	0.2751	0.1785	0.6029
		(0.0106)	(0.0188)	(0.0813)	(0.0503)
3	$N = 20, T = 100$	0.2859	0.5194	0.4490	1.2227
		(0.0525)	(0.1284)	(0.2867)	(0.2290)
	$N = 40, T = 150$	0.1531	0.4087	0.2340	0.8683
		(0.0212)	(0.0707)	(0.1259)	(0.1085)

Table 1: The summary statistics (mean and SD) of the MSEs for estimating 95% generalized quantile curves by the FDA approach and the separate estimation approach. Scenario 1 with $\varepsilon_{ij} \sim \mathrm{N}(0, 0.5)$, Scenario 2 with $\varepsilon_{ij} \sim \mathrm{N}(0, \mu(t) \times 0.5)$ and Scenario 3 with $\varepsilon_{ij} \sim t(5)$.

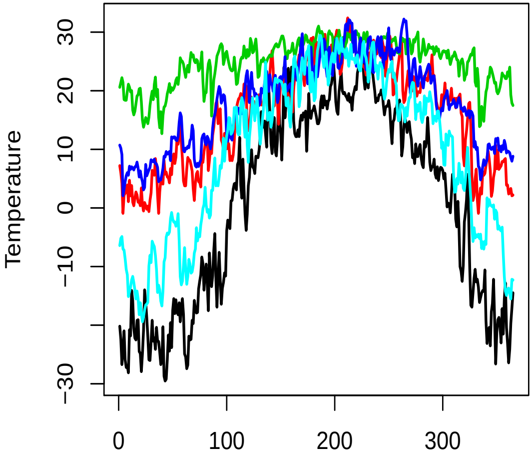


Figure 7: Time series plot of 5 selected weather stations (south, north, east, west and middle) from 150 weather stations in China

Data

Let T_{it} denote the average temperature on day t for city (station) i. The standard model described is:

$$
\begin{aligned}
T_{it} &= X_{it} + \Lambda_{it}, \\
\Lambda_{it} &= a_i + b_i t + \sum_{m=1}^{M} c_{im} \cos\left\{ \frac{2\pi(t - d_{im})}{m \cdot 365} \right\}, \\
X_{it} &= \sum_{j=1}^{p_i} \beta_{ij} X_{i,t-j} + \varepsilon_{it}.
\end{aligned}
\tag{22}
$$

	Expectile levels			
PC index	5%	25%	75%	95%
1	0.3833	0.0596	0.0659	0.4421
2	0.0665	0.0131	0.0194	0.1102
3	0.0471	0.0077	0.0158	0.0746
4	0.0415	0.0074	0.0123	0.0657
5	0.0306	0.0072	0.0056	0.0455
6	0.0262	0.0051	0.0050	0.0226

Table 2: The empirical variances of PC scores for the Chinese temperature data.

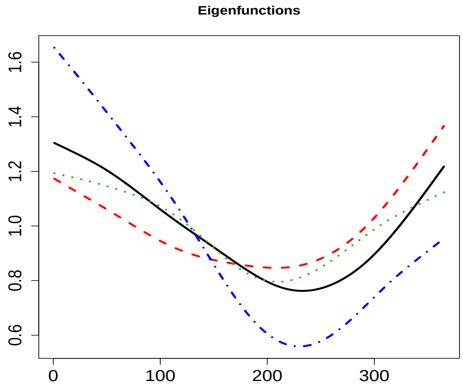


Figure 8: The estimated first principal component for the 5% (black solid), 25% (red dashed), 75% (green dotted), 95% (blue dash-dotted) expectiles curves of the volatility of the temperature of China in 2010 with the data from 150 weather stations.

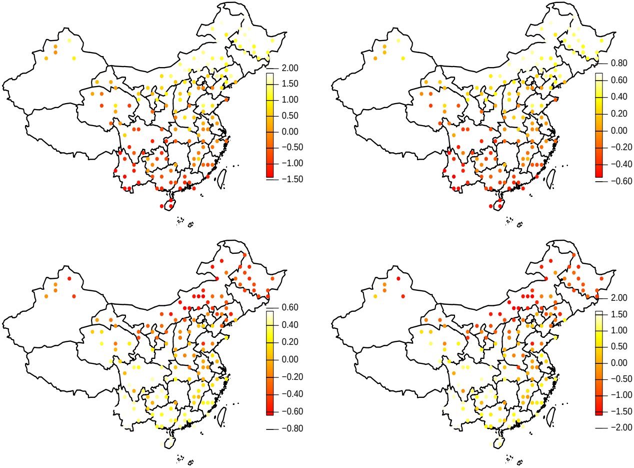
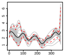


Figure 9: The estimated first principal component scores α_1 for the 5%, 25%, 75% and 95% expectile curves of the temperature distribution.

Conclusion

- Dimension Reduction technique applied to a nonlinear object.
- Provides a novel way to estimate several generalized quantile curves simultaneously.
- Outperforms the single curve estimation, especially when the data is very volatile.
- Pricing weather derivatives more precisely can be possible.



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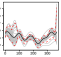

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Volatility of Temperature

Return

- The temperature T_{it} on day t for city i:

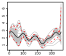
$$T_{it} = X_{it} + \Lambda_{it}$$

- The seasonal effect Λ_{it}:

$$\Lambda_{it} = a_i + b_i t + \sum_{m=1}^{M} c_{im} \cos\left\{\frac{2\pi(t - d_{im})}{365}\right\}$$

- X_{it} follows an $AR(p_i)$ process:

$$\begin{aligned} X_{it} &= \sum_{j=1}^{p_i} \beta_{ij} X_{i,t-j} + \varepsilon_{it} \qquad (23) \\ \widehat{\varepsilon}_{it} &= X_{it} - \sum_{j=1}^{p_i} \hat{\beta}_{ij} X_{i,t-j} \end{aligned}$$



Initial Values

▸ Return

1. Estimate N single curves $\widehat{l}_i$ individually.
2. Linear regression for $\widehat{\theta}_{\mu 0}$: $\quad \widehat{l}_i = B_i \theta_\mu + \varepsilon_i$
3. Calculate $\widetilde{l}_{i0} = \widehat{l}_i - B_i \widehat{\theta}_{\mu 0}$, and $\widehat{\Gamma}_0 = (\widehat{\Gamma}_{10}, \cdots, \widehat{\Gamma}_{N0})$.
$$\widetilde{l}_{i0} = B_i \Gamma_i + \varepsilon_i$$
4. Apply SVD to decompose $\widehat{\Gamma}_{i0}$:
$$\widehat{\Gamma}_{i0} = UDV^\top = \Theta_{f0} \alpha_{i0}$$
5. Choose the first K factors from U as $\widehat{\Theta}_{f0}$, and regress $\widehat{\Gamma}_{i0}$ on $\widehat{\Theta}_{f0}$ to get $\widehat{\alpha}_{i0}$:
$$\widehat{\Gamma}_{i0} = \widehat{\Theta}_{f0}(\alpha_{i1}, \cdots, \alpha_{iK}) + \varepsilon_i \tag{24}$$

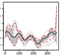
Update Procedure

▸ Return

1. Plug $\widehat{\Theta}_{f0}$ and $\widehat{\alpha}_{i0}$ into (18) to update θ_μ, and get $\widehat{\theta}_{\mu 1}$.
2. Plugging $\hat{\theta}_{\mu 1}$ and $\widehat{\alpha}_{i0}$ into the second equation of (18) gives $\widehat{\Theta}_{f1}$.
3. Given $\widehat{\theta}_{\mu 1}$ and $\widehat{\Theta}_{f1}$, estimate $\widehat{\alpha}_i$.
4. Recalculate the weight matrix:

$$w_{ij}' = w_\alpha(Y_{ij} - \widehat{l_{ij}})$$

   where $\widehat{l_{ij}}$ is the j-th element in $\widehat{l_i} = B_i\widehat{\theta}_{\mu 1} + B_i\widehat{\Theta}_{f1}\widehat{\alpha}_i$
5. Repeat step (1) to (4) until the solutions converge.



Mercer's Lemma

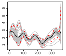
The covariance operator K

$$K(s,t) = \mathrm{Cov}\{l(s), l(t)\}, \mathrm{E}\{l(t)\} = \mu(t), s, t \in \mathcal{T} \tag{25}$$

There exists an orthonormal sequence (ψ_j) and non-increasing and non-negative sequence (κ_j),

$$\begin{aligned}(K\psi_j)(s) &= \kappa_j \psi_j(s) \\ K(s,t) &= \sum_{j=1}^{\infty} \kappa_j \psi_j(s) \psi_j(t) \\ \sum_{j=1}^{\infty} \kappa_j &= \int_{\mathrm{I}} K(t,t) dt < \infty \end{aligned} \tag{26}$$

Return



Karhunen-Loève Theorem

Under assumptions of Mercer's lemma

$$l(t) = \mu(t) + \sum_{j=1}^{\infty} \sqrt{\kappa_j} \xi_j \psi_j(t) \qquad (27)$$

where $\xi_j \stackrel{\text{def}}{=} \frac{1}{\sqrt{\kappa_j}} \int \; l(t)\psi_j(s)ds$, and $\mathsf{E}(\xi_j) = 0$

$$\mathsf{E}(\xi_j \xi_k) = \delta_{j,k} \qquad j, k \in \mathbb{N}$$

and $\delta_{j,k}$ is the Kronecker delta.

Return