Empirical Pricing Kernels and Investors' Preferences

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Financial Market

Riskless bond with constant interest rate r, stock price process $(S_t)_{t\in[0,T]}$ with continuously distributed marginals S_t

- examples:
  - Black-Scholes model (Nobel prize 1997)
  - GARCH model (Nobel prize 2003, Engle)
  - non-parametric diffusion model (Ait-Sahalia (2000))
- **risk neutral valuation principle for pay offs $\psi(S_T)$:**

$$\int_0^\infty e^{-Tr}\psi(s_T)\,\frac{q(s_T)}{p(s_T)}p(s_T)\,ds_T$$

where q is some probability density function and p is the probability density function of S_T.

Pricing Kernels & Preferences

- representative investor with strictly increasing, concave, indirect von Neumann-Morgenstern utility u dependent on realizations of S_T
- **relationship between representative investor's preferences and pricing kernel:**

$$\frac{du}{dx} \propto \frac{q}{p}$$

Empirical Pricing Kernel (EPK)

- EPK: any estimation of pricing kernel $\frac{q}{p}$
- different estimation methods and models for stock prices, Ait-Sahalia & Lo (2000), Engle & Rosenberg (2002), Brown & Jackwerth (2004)

$\downarrow$

some paradoxa

EPK paradoxon

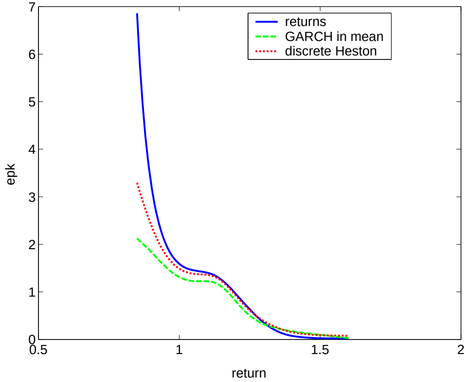

Figure 1: Estimated PK on 24 March 2000 for $\tau = 0.5$ year, $r_{0.5} = 4.06\%$.

EPK paradoxon: across maturities

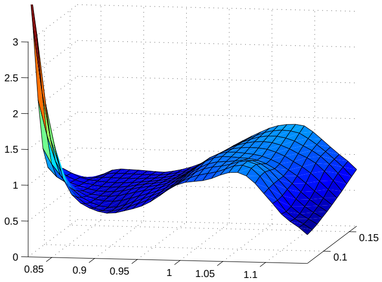

Figure 2: Estimated PK across moneyness $\varkappa$ and maturity τ, DAX on 20010710

EPK paradoxon: across time

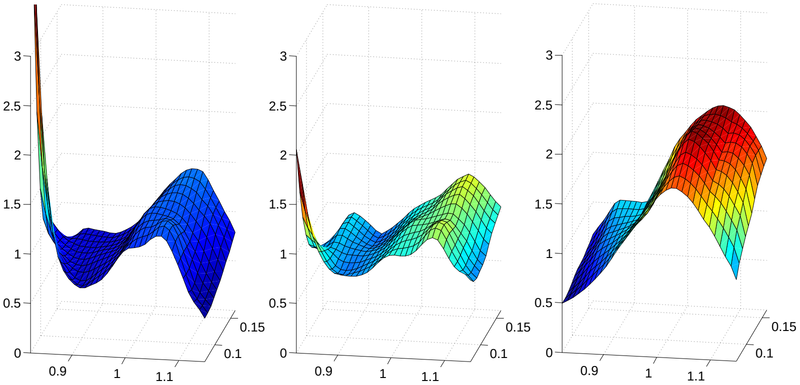

Figure 3: Empirical PK across $\varkappa$ and τ, estimated form DAX on 20010710, 20010904 and 20011130

Financial market with regime switch

Chabi-Yo, Garcia and Renault (2007). Discrete time period $\{0, \dots, T\}$

- **two basic financial markets:** Two types of price processes for risky asset $(S_0^0, \dots, S_T^0), (S_0^1, \dots, S_T^1)$ of continuous random vectors constituting separately, together with the riskyless bond, arbitrage free financial market
- **latent regime switching state variable** $(U_0, \dots, U_T)$ Markov-chain of Bernoulli-distributed random variables (unobservable)
- $S_t = S_t^i$ if $U_t = i$ $(i = 0, 1)$

EPK paradoxon: aims

Empirical pricing kernels are not monotone decreasing across strikes, vary across maturities and time:

Regime switch for prices vs. switch of agents' preferences

- **What could be a microeconomic explanation for the empirical pricing kernel paradoxon?**
- **How to explain empirical pricing kernel dynamics ?**

Outline

The Financial Market

1. time interval $[0, T]$ of investment with finite horizon T
2. one riskless bond with deterministic Riemannian integrable process $(r_t)_{0 \leq t \leq T}$ of interest rates
3. one risky assets with nonnegative price process $(S_t)_{0 \leq t \leq T}$, semimartingale, S_0 constant

Stochastic Discount Factor

1. arbitrage free market, there exists at least one **state price density** (SPD) i.e. a positive random variable π s.t.

$$E[\pi] = 1$$

$$E\left[S_{t_2} \frac{\pi}{E[\pi | S_t, t \leq t_1]} \middle| S_t, t \leq t_1 \right] = e^{\int_{t_1}^{t_2} r_x \, dx} S_{t_1} \qquad 0 \leq t_1 < t_2 \leq T$$

2. **stochastic discount factor at time** t_1

$$\pi_{t_1} = \frac{\pi}{E[\pi | S_t, t \leq t_1]}$$

Risk Neutral Pricing Rules

Nonnegative pay off $\psi(S_T)$, state price density π, family $(\pi_t)_{0\leq t\leq T}$ of stochastic discount factors:

1. **risk neutral price of $\psi(S_T)$ at time t_1** (w.r.t. π):

$$E\left[\; e^{-\int_{t_1}^{T} r_x dx}\; \psi(S_T)\; \pi_{t_1} \;\middle|\; S_t, t \leq t_1 \;\right]$$

2. **risk neutral price of $\psi(S_T)$ at time t = 0** (w.r.t. π):

$$E\left[e^{-\int_0^T r_x dx}\; \psi(S_T)\; \pi\;\right] \;\; = \;\; E\left[e^{-\int_0^T r_x dx}\; \psi(S_T)\; E[\pi|S_T]\;\right]$$

The Pricing Kernel(s)

1. **pricing kernel** (w.r.t. π), positive random variable $\mathcal{K}_\pi$. s.t.

$$E[\pi|S_T] = \mathcal{K}_\pi(S_T)$$

2. **risk neutral distribution** Q_{S_T} of S_T (w.r.t. π):

$$Q_{S_T}([S_T \leq x]) \stackrel{\text{def}}{=} \int_{-\infty}^{x} \mathcal{K}_\pi \, dP_{S_T} \quad (P_{S_T} \text{ the distribution of } S_T).$$

3. **risk neutral price** of $\psi(S_T) \mathrel{\widehat{=}}$ expected value of $e^{-\int_0^T r_x dx}\,\psi$ w.r.t. Q_{S_T}

Intertemporal Pricing Kernels 1

Assumption:

$$E\left[\psi(S_T)\pi_{t_1}|S_t, t \leq t_1\right] = E\left[\psi(S_T)\,\frac{\pi}{E\left[\pi|S_{t_1}\right]}\;\Big|\; S_{t_1}\right] \quad (1)$$

Risk neutral price of $\psi(S_T)$ at time t_1 (w.r.t. π):

$$E\left[e^{-\int_{t_1}^{T} r_x dx}\;\psi(S_T)\,\frac{E\left[\pi|S_{t_1}, S_T\right]}{E\left[\pi|S_{t_1}\right]}\;\Big|\; S_{t_1}\right]$$

1. **intertemporal pricing kernel at time t_1** (w.r.t. π): positive random variable $\mathcal{K}_\pi^t$ s.t.

$$\frac{E\left[\pi|S_{t_1}, S_T\right]}{E\left[\pi|S_{t_1}\right]} = \mathcal{K}_\pi^t(S_{t_1}, S_T)$$

2. **conditional risk neutral distributions $Q_{S_T|S_t}$** (w.r.t. π):

$$Q_{S_T|S_t=s_t}([S_T \leq x]) \stackrel{\text{def}}{=} \int_{-\infty}^{x} \mathcal{K}_\pi^t(s_t, \cdot)\, dP_{S_T|S_t=s_t} \qquad (2)$$

where $P_{S_T|S_t=s_t}$ is the conditional distribution of S_T under S_t.

3. **risk neutral price of $\psi(S_T)$ under** $(S_t = s_t) \mathrel{\widehat{=}}$ expected value of $e^{\int_0^T r_x dx}\, \psi$ w.r.t. $Q_{S_T|S_t=s_t}$

Intertemporal Pricing Kernels 2

Assume a two factor financial market where the prices $(S_t)_t$ follow the diffusion

$$dS_t = S_t \mu(Y_t) dt + S_t \sigma(Y_t) dW_t^1$$

where W^1 is standard Brownian motion, Y denotes an external economic factor process following

$$dY_t = g(Y_t) + \rho dW_t^1 + \overline{\rho} dW_t^2$$

$\rho \in [-1, 1]$, $\overline{\rho} \stackrel{\text{def}}{=} \sqrt{1 - \rho^2}$ and W^2 is standard Brownian motion independent of W^1

Intertemporal Pricing Kernels 3

Assumption (1) is fulfilled, Hernández-Hernández and Schied (2007). From (2) the intertemporal pricing kernel at time t (w.r.t. π) can be written as

$$\mathcal{K}_\pi^t(s_t, S_T) = \frac{q_t(S_T)}{p_t(S_T)}$$

where

1. $q_t(S_T) \stackrel{\text{def.}}{=} q_{S_T|S_t=s_t}(S_T)$ and $p_t(S_T) \stackrel{\text{def.}}{=} p_{S_T|S_t=s_t}(S_T)$ are density functions of $Q_{S_T|S_t=s_t}$ and $P_{S_T|S_t=s_t}$
2. q_t is called the risk neutral density function (RND), p_t the objective density function.

Pricing Kernel Estimation

Ait-Sahalia and Lo (2000) estimate the estimate PK as the ratio between the estimated RND and the estimated objective density:

$$\widehat{\mathcal{K}_{\pi}^{t}}(s_t, S_T) \quad = \quad \frac{\widehat{q}_t(S_T)}{\widehat{p}_t(S_T)}$$

q_t is estimated from option and p_t from underlying prices

RND Estimation

Breeden and Litzenberger (1978), RND from option prices

$$q_t(S_T) = e^{r\tau} \left. \frac{\partial^2 C_t(\varkappa, \tau)}{\partial K^2} \right|_{K=S_T} \tag{3}$$

Ait-Sahalia and Lo (1998) used the estimate

$$\widehat{q}_t(S_T) = e^{r\tau} \left. \frac{\partial^2 C_{t,BS}\{S_t, K, \tau, r_t, \widehat{\sigma}_t(\varkappa, \tau)\}}{\partial K^2} \right|_{K=S_T} \tag{4}$$

1. $C_{t,BS}$ is the Black-Scholes price at time t
2. $\widehat{\sigma}_t(\varkappa, \tau)$ is a nonparametric estimator for the implied volatility surface (IVS)

Implied Volatility Surface

Implied volatility surface is the function $\sigma_t : \mathbb{R}_+^2 \to \mathbb{R}_+$ satisfying for all $(K, \tau) \in \mathbb{R}_+^2$

$$C_t(K,\tau) \quad = \quad C_{BS}\{S_t, r_t, K, \tau, \sigma_t(K,\tau)\} \qquad (5)$$

$C_{BS}(v) = C_{BS}(S_t, r_t, K, \tau, v)$ is continuous increasing on v and $\sigma_t(K,\tau) = C_{BS}^{-1}\{C_t(K,\tau)\}$. At day $t = 1, \ldots, T$ there are $j = 1, \ldots, J_t$ options traded. Each trade j at day t corresponds to

1. an implied volatility σ_{jt}
2. and a pair of strike and maturity $X_{jt} = (\varkappa_{jt}, \tau_{jt})^\top$

IV - Degenerated Design

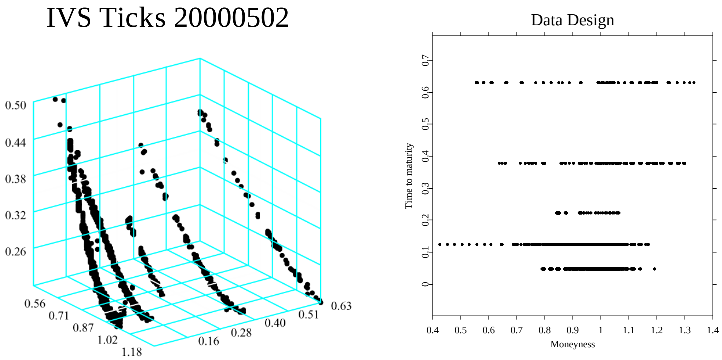

Figure 4: Left panel: call and put implied volatilities observed on 20000502. Right panel: data design on 20000502; ODAX, difference-dividend correction according to Hafner and Wallmeier (2001) applied.

Dynamic Semiparametric Factor Models (DSFM)

regress log implied volatilities $Y_{jt} = \log \sigma_{jt}$ on X_{jt}

$$Y_{jt} = \sum_{l=0}^{L} z_{lt} m_l(X_{jt}) + \varepsilon_{jt}$$

1. $m_l(\cdot)$ are smooth basis functions, $l = 0, \ldots, L$
2. z_{lt} are time dependent factors
3. ε_{jt} is noise

Time Invariant Smooth Basis Functions

Basis functions expanded using a series estimator, Borak et al. (2008). The lth basis function is written as

$$m_l(X_{jt}) = \sum_{k=1}^{K} \gamma_{lk} \psi_k(X_{jt})$$

for functions $\psi_k : \mathbb{R}^2 \rightarrow \mathbb{R}$ and coefficients $\gamma_{lk} \in \mathbb{R}$, $k = 1, \ldots, K$.

DSFM Estimation

Defining $Z = (z_{t,l})$, $\Gamma = (\gamma_{lk})$ the least square estimators are

$$(\widehat{\Gamma}, \widehat{Z}) = \arg \min_{\Gamma \in \mathcal{G}, Z \in \mathcal{Z}} \sum_{t=1}^{T} \sum_{j=1}^{J} \left\{ Y_{jt} - z_t^\top \Gamma \psi(X_{jt}) \right\}^2$$

where

1. $z_t = (z_{0t}, \ldots, z_{Lt})^\top$, $\psi = (\psi_1, \ldots, \psi_K)^\top$
2. $\mathcal{G} = \mathcal{M}(L+1, K)$, $\mathcal{Z} = \{Z \in \mathcal{M}(I, L+1) : z_{0t} \equiv 1\}$, $\mathcal{M}(a, b)$ is the set of $(a \times b)$ matrices

IVS and DSFM

The implied volatility surface at day t is estimated as

$$\widehat{\sigma}_t(\varkappa, \tau) = exp\left\{\widehat{z}_t^\top \widehat{m}(\varkappa, \tau)\right\} \tag{6}$$

where

1. $\widehat{m} = (\widehat{m}_0, \ldots, \widehat{m}_L)^\top$
2. $\widehat{m}_l = \widehat{\gamma}_l^\top \psi$
3. $\widehat{\gamma}_l = (\widehat{\gamma}_{l,1}, \ldots, \widehat{\gamma}_{l,K})^\top$

Implied RND and DSFM

Using (4) the implied RND may be approximated by

$$\widehat{q}_t(\varkappa, \tau, \widehat{z}_t, \widehat{m}) = \varphi(d_2) \left\{ \frac{1}{K \widehat{\sigma}_t \sqrt{\tau}} + \frac{2 d_1}{\widehat{\sigma}_t} \frac{\partial \widehat{\sigma}_t}{\partial K} + \frac{K \sqrt{\tau} d_1 d_2}{\widehat{\sigma}_t} \left(\frac{\partial \widehat{\sigma}_t}{\partial K} \right)^2 + K \sqrt{\tau} \frac{\partial^2 \widehat{\sigma}_t}{\partial K^2} \right\} \Bigg|_{K = S_T}$$

where $\varphi(x)$ is the standard normal pdf, $d_1 = \frac{\log\left(\frac{S_t}{K}\right) + (r + \frac{1}{2}\widehat{\sigma}_t^2)\tau}{\widehat{\sigma}_t \sqrt{\tau}}$ and $d_2 = d_1 - \widehat{\sigma}_t \sqrt{\tau}$

EPK and DSFM

The EPK $\widehat{\mathcal{K}}^t(\varkappa, \tau)$ is constructed as the ratio between the estimated RND and the estimated p:

$$\widehat{\mathcal{K}}^t(\varkappa, \tau, \widehat{z}_t, \widehat{m}) = \frac{\widehat{q}_t(\varkappa, \tau, \widehat{z}_t, \widehat{m})}{\widehat{p}_t(\varkappa, \tau)}$$

Here $\widehat{p}_t$ is estimated by a GARCH(1,1) model.

Empirical Results

Intraday DAX index and option data

1. from 20010101 to 20020101
2. 253 trading days
3. $L = 3$, see Borak et al. (2008)
4. $\widehat{q}_t$ estimated with DSFM
5. $\widehat{p}_t$ estimated from last 240 days with GARCH(1,1)

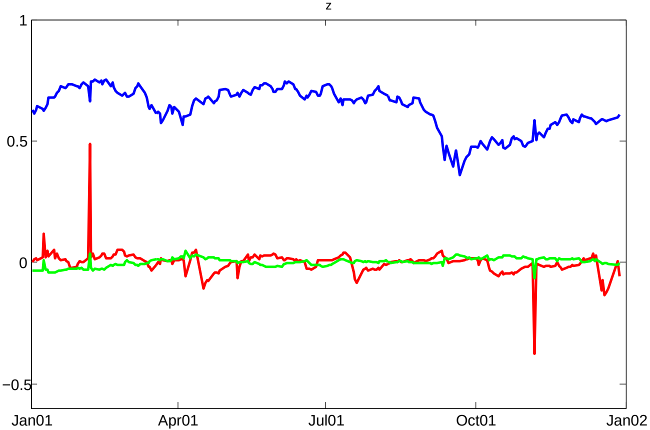

Figure 5: Loading factors $\widehat{z}_{lt}$, $l = 1, 2, 3$ from the top

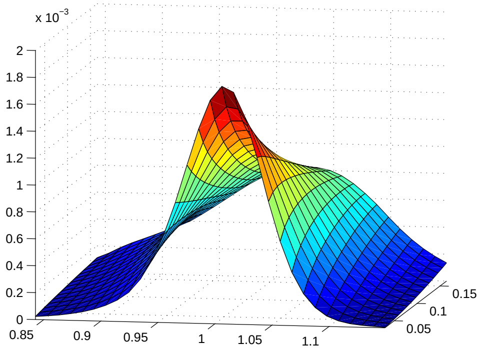

Figure 6: Estimated RND across $\varkappa$ and τ at $t = 20010710$

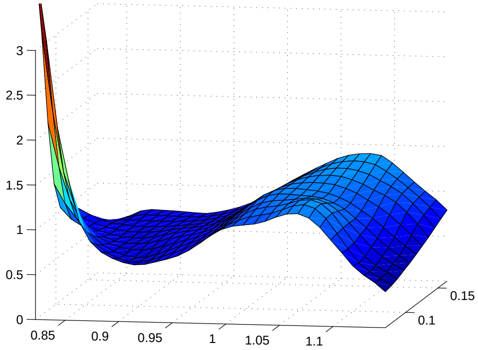

Figure 7: Estimated PK across $\varkappa$ and τ at $t = 20010710$

IV, RND and PK dynamics

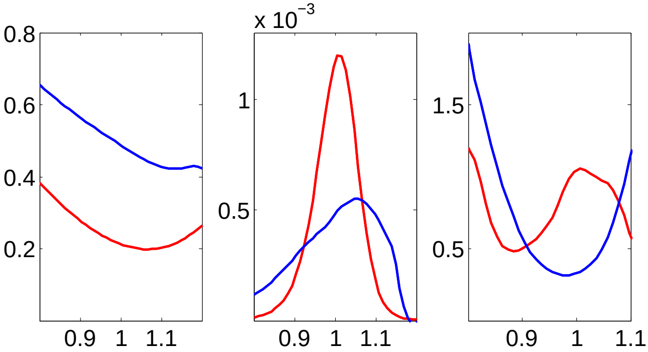

Figure 8: IV (left), RND (middle) and PK (right), $\tau = 20$ days. Red: $t = 20010824$, $\widehat{z}_{t1} = 0.68$, blue: $t = 20010921$, $\widehat{z}_{t1} = 0.36$

Comparative Statics

IV, RND and EPK estimated with loadings W^l:

1. typical effect of variation in loading l, remaining factors constant at median
2. observed changes in skewness and excess kurtosis

Scenario loadings W^l

1. linear increase in $N = 50$ steps on loading l
2. from levels $d_l = \min \widehat{z}_{lt} - 0.5|\min \widehat{z}_{lt}|$ to $u_l = \max \widehat{z}_{lt} + 0.5|\max \widehat{z}_{lt}|$
3. remaining loadings constant at median
4. scenario loadings to factor l in matrices $W^l = (w^l_{n,j})$, $l, j = 0, \ldots, 3$, and $n = 1, \ldots, N$ with

$$w^l_{n,j} = \left\{ d_j + \frac{n-1}{N-1}(u_j - d_j) \right\} \mathbf{1}(j = l) + med(\widehat{z}_{jt})\mathbf{1}(j \neq l)$$

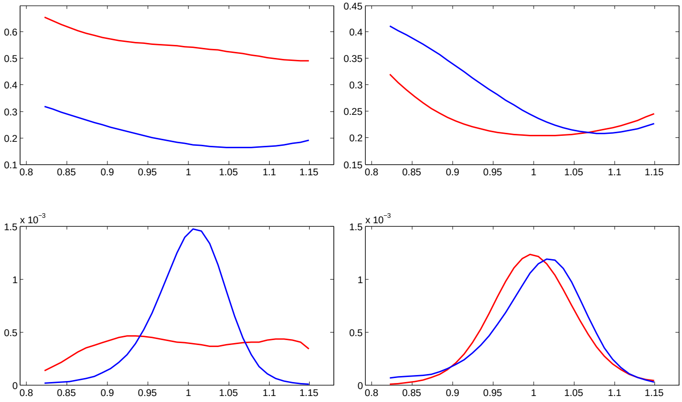

Figure 9: IV (above), RND (below) for variation in loading factor 1 (left) and 3 (right), $\tau = 20$ days

RND kurtosis and $\widehat{z}_1$

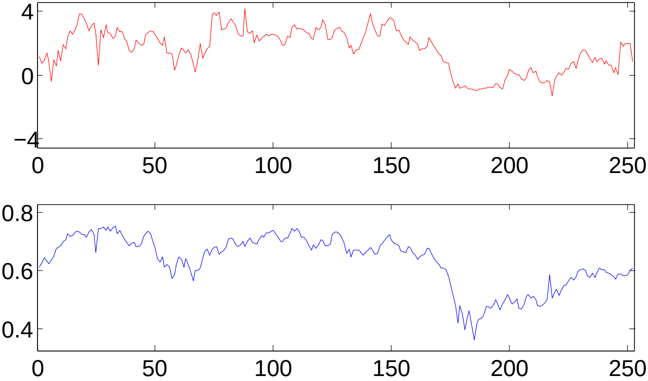

Figure 10: RND - excess kurtosis, $\tau = 18$ days (red), $\widehat{z}_1$ (blue)

RND kurtosis and $\widehat{z}_1$

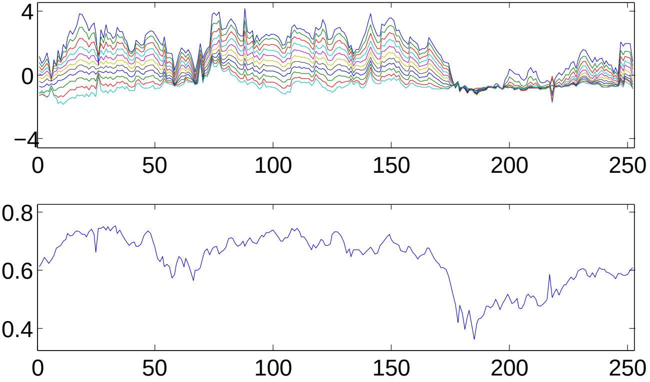

Figure 11: RND - excess kurtosis, $\tau = 18, (2), 40$ days (top), $\widehat{z}_1$ (bottom)

RND skewness and $\widehat{z}_3$

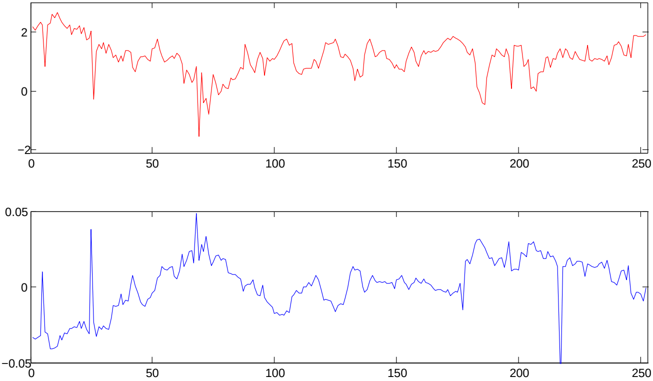

Figure 12: RND - skewness, $\tau = 40$ days (red), $\widehat{z}_3$ (blue)

RND skewness and $\widehat{z}_3$

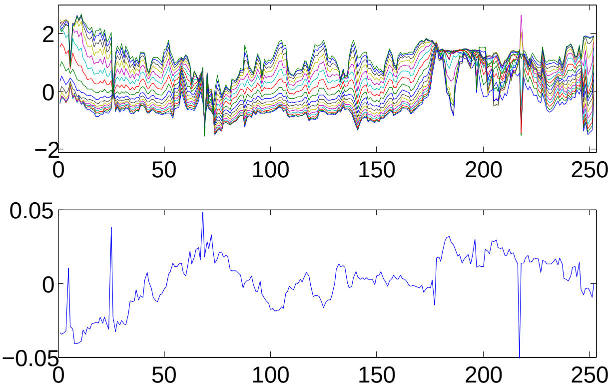

Figure 13: RND - skewness, $\tau = 18, (2), 50$ days (top), $\widehat{z}_3$ (bottom)

Pricing Kernels Dynamics

Dynamics from IV, RND and EPK described by DSFM loading factors

1. z_1: level (IVS), excess kurtosis (RND), "existence" of risk proclivity (EPK)
2. z_2: strike skewness (IVS), skewness (RND), "location" of risk proclivity (EPK)
3. z_3: term structure (IVS), term structure and skewness (RND), "location" (term structure) of risk proclivity (EPK)

Static Consumption Model with Extended Expected Utility Preferences

Each consumer $i = 1, \ldots, m$ has a **random endowment** $e_i(S_T)$ and

1. chooses among nonnegative random consumption $c(S_T)$ satisfying the **budget constraint**
$$E\left[c(S_T)\mathcal{K}_\pi(S_T)\right] \quad \leq \quad E\left[e_i(S_T)\mathcal{K}_\pi(S_T)\right]$$
2. has extended expected utility preferences
$$U^i\{c(S_T)\} \quad = \quad E\left[u^i\{S_T, c(S_T)\}\right]$$
where $u^i : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfies
$$u^i(\cdot, c) \text{ random variable for } c \geq 0$$
$$u^i(s_T, \cdot) \text{ strictly increasing and strictly concave for } s_T \geq 0.$$

Equilibrium

Contingent Arrow Debreu equilibrium $[(\overline{c}_1(S_T), ..., \overline{c}_m(S_T)); \mathcal{K}_\pi]$, in particular:

1. **individual optimization:** $\overline{c}_i(S_T)$ solves the optimization problem

$$\max \; U^i\{c(S_T)\}$$

   s.t. $c(S_T)$ satisfies individual budget constraint

2. **market clearing:** $\sum_{i=1}^{m} \overline{c}_i(S_T) = \sum_{i=1}^{m} e_i(S_T)$

Pareto optimality

There is no $(c_1(S_T), \ldots, c_m(S_T))$ satisfying

$$\begin{aligned} \sum_{i=1}^{m} c_i(S_T) &\leq \sum_{i=1}^{m} e_i(S_T) \\ U^i\{c_i(S_T)\} &\geq U^i\{\overline{c}_i(S_T)\} \text{ for every } i \\ U^{i_0}\{c_{i_0}(S_T)\} &\leq U^{i_0}\{\overline{c}_{i_0}(S_T)\} \text{ for some } i_0 \end{aligned}$$

Indirect Utilities of Representative Investor

Pareto optimality guarantees nonnegative weights $\alpha_1, ..., \alpha_m$ summing up to 1 s.t.

$$\begin{aligned}
\sum_{i=1}^{m} \alpha_i U^i\{\overline{c}_i(S_T)\} &= \max \left\{ \sum_{i=1}^{m} \alpha_i U^i\{c_i(S_T)\} \middle| \sum_{i=1}^{m} c_i(S_T) \leq \sum_{i=1}^{m} e_i(S_T) \right\} \\
&\stackrel{\mathrm{def}}{=} U_\alpha \left\{ \sum_{i=1}^{m} e_i(S_T) \right\}
\end{aligned}$$

Extended expected utility representation

$$U_\alpha \left\{ \sum_{i=1}^{m} e_i(S_T) \right\} = E \left[u_\alpha \left\{ S_T, \sum_{i=1}^{m} e_i(S_T) \right\} \right]$$

where for $s_T, e \geq 0$

$$u_\alpha (s_T, e) \stackrel{\text{def}}{=} \sup \left\{ \sum_{i=1}^{m} \alpha_i u^i(s_T, c_i) \middle| c_1, ..., c_m \geq 0, \sum_{i=1}^{m} c_i \leq e \right\}$$

and

$u_\alpha(\cdot, e)$ is random variable for $e \geq 0$

$u_\alpha(s_T, \cdot)$ is strictly increasing and strictly concave for $s_T \geq 0$.

Consumers' preferences and the pricing kernel

Theorem 1

Let $u^i(s_T,\cdot)|(0,\infty)$ be twice continuously differentiable satifying Inada conditions for $s_T \geq 0$.

For $s_T \geq 0$, and $\alpha_i > 0$, $u_\alpha(s_T,\cdot)|(0,\infty)$ is continuously differentiable and there exists $y_i > 0$ s.t. for any $\sum_{i=1}^{m} e_i(s_T) > 0$

$$\frac{du_\alpha(s_T,\cdot)}{de}\Big|_{e=\sum_{i=1}^{m} e_i(s_T)} = \alpha_i \frac{du^i(s_T,\cdot)}{dc}\Big|_{c=\bar{c}_i(s_T)} = \alpha_i \, y_i \, \mathcal{K}_\pi(s_T)$$

Classical risk averse expected utilities

Henceforth $\sum_{i=1}^{m} e_i(S_T) = S_T$

Corollary 2

Let $u^i(s_T, \cdot)$ be independent of s_T for $i = 1, ..., m$.

Then under assumptions of Theorem 1 there is some positive y such that

$$\frac{du_\alpha(s_T, \cdot)}{de}\Big|_{e=s_T} = \ y\ \mathcal{K}_\pi(s_T) \text{ for any positive } s_T,$$

in particular $\mathcal{K}_\pi|(0, \infty)$ has to be nonincreasing.

Hegemonial Representative Agents - a simple solution to the empirical pricing kernel paradoxon

Homogeneously switching utilities

Assume that there is some (measurable) subset $A \subseteq \mathbb{R}$ with

$$u^i(s_T, c) = 1_A(s_T) u_1^i(c) + 1_{\mathbb{R} \setminus A}(s_T) u_2^i(c) \text{ for } i = 1, \ldots, m,$$

where $u_1^i, u_2^i : [0, \infty) \to \mathbb{R}$ strictly increasing and strictly concave.

Hegemonial Representative Agents

Theorem 3

Under assumptions of Theorem 1 and homogeneously switching utilities

$$u_\alpha(s_T, e) = 1_A(s_T)u^1_\alpha(e) + 1_{\mathbb{R}\setminus A}(s_T)u^2_\alpha(e) \text{ for } s_T, e \geq 0,$$

where

$$u^j_\alpha(e) \stackrel{\text{def}}{=} \sup\left\{\sum_{i=1}^m \alpha_i u^i_j(c_i) \mid c_1, ..., c_m \geq 0, \sum_{i=1}^m c_i \leq e\right\} \text{ for } j = 1, 2.$$

A Simple Solution

Theorem 4

Let $u^i_j|(0,\infty)$ be twice continuously differentiable satifying Inada conditions for every $i \in \{1, ..., m\}$ and $j \in \{1, 2\}$.

Then $u^j_\alpha|(0,\infty)$ is continuously differentiable for $j \in \{1, 2\}$, and there is some positive y such that

$$1_A(s_T)\frac{du^1_\alpha}{de}\Big|_{e=s_T} + 1_{\mathbb{R}\setminus A}(s_T)\frac{du^2_\alpha}{de}\Big|_{e=s_T} = y\,\mathcal{K}_\pi(s_T) \text{ for any positive } s_T.$$

Suggested Solution:

$$u_\alpha^j(x) \stackrel{\text{def}}{=} \frac{x^{\gamma_j}}{\gamma_j} \; (j = 1, 2) \text{ with } 0 < \gamma_1 < \gamma_2 < 1, \;\; A \stackrel{\text{def}}{=} (1, \infty).$$

Theorem 4 implies for some $y > 0$

$$\begin{aligned} \mathcal{K}_\pi^t(s_T) &= \left.\frac{du_\alpha^1}{de}\right|_{e=s_T} \text{ for } s_T \le 1 \\ \mathcal{K}_\pi^t(s_T) &= \left.\frac{du_\alpha^2}{de}\right|_{e=s_T} \text{ for } s_T > 1 \end{aligned}$$

Interpretation

For states $s_T > 1$ the less risk averse representative agent is hegemonial, otherwise the more risk averse.

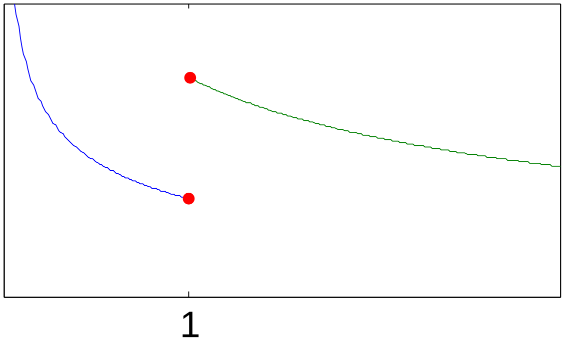

Figure 14: Suggested solution: $\mathcal{K}_t^\pi(s_T)$

Extension

1. hetereogeneously switching utilities

$$u^i(s_T, c) = 1_{(x_i,\infty)}(s_T)u_1^i(c) + 1_{(-\infty,x_i)}(s_T)u_2^i(c)$$

   for $i = 1, \ldots, m$, and $x_1 \leq x_2 \leq \ldots \leq x_m$.
2. switching behaviour of representative agent between several indirect utility indices, typically more than two.

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