The Dynamics of Pricing Kernels

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Outline of the talk

1. Introduction and Theory ✓
2. The Model
3. Some Empirical Results
4. Conclusions

Arrow-Debreu Securities - Definition

- Starting point of all modern financial asset pricing theories
- Developed by Arrow (1964) and Debreu (1959)
- State-dependent contingent claims, entitle their holder to a payoff of 1$ in one specific state of the world, and 0 in all other states of the world
- Prices are always non negative and sum up to one, due to the dependence on the probability
- Priced differently in different states of the world, a one Euro loss does not always have the same value

Options as Arrow-Debreu Securities

- Option price is the discounted expectation of random payoffs received at maturity
- Payoff = Value of the claim at maturity, so the Value Process is a martingale
- If q_s is the price of an Arrow-Debreu security when $r = 0$ and Q denotes the risk neutral probability measure:

$$C_t = e^{-r(T-t)} \mathsf{E}_t^Q[\psi(S_T)] \stackrel{\text{def}}{=} e^{-r(T-t)} \sum_s q_s \psi_s(S_T) \qquad (1)$$

we discount the payoff to get the option price

- The continuous counterpart of the Arrow-Debreu state contingent claims is the *State Price Density* (SPD)

A Standard Dynamic Exchange Economy

- Described by Lucas (1978), Rubinstein (1976) and others.
- Complete markets for securities, one consumption good, no exogenous income, investors maximize state-dependent utility function
- A risky stock in the economy follows:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{2}$$

- A riskless bond described by

$$dB_t = rB_t dt \tag{3}$$

A Standard Dynamic Exchange Economy

- Defining corrected stock price $\widetilde{S}_t \stackrel{\text{def}}{=} e^{-(r+\delta)t} S_t$ and considering dividend yield $\delta = 0$:

$$\begin{aligned} d\widetilde{S}_t &= d(e^{-rt} S_t) \\ &= -re^{-rt} S_t dt + e^{-rt} dS_t \\ &= -re^{-rt} S_t dt + e^{-rt} [\mu S_t dt + \sigma S_t dW_t] \\ &= (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dW_t \\ &= \sigma \widetilde{S}_t d\overline{W}_t \end{aligned} \tag{4}$$

- $\overline{W}_t \stackrel{\text{def}}{=} W_t + \frac{\mu - r}{\sigma} t$ is a Brownian motion on the probability space corresponding to the risk-neutral measure Q. The term $\frac{\mu - r}{\sigma}$ measures the excess return per unit of risk borne by the investor and hence vanishes under Q

Representative Agent's Utility Function

Constantinides (1982) proved existence of a representative agent under certain conditions. Its utility function can depend on any variable in the state-contingent vector s_t:

$$U(s_t, s_{t+1}) = u(s_t) + \beta \, \mathsf{E}_t[u(s_{t+1})] \stackrel{\text{def}}{=} u(s_t) + \beta \sum_{s_{t+1}} u(s_{t+1}) p_t(s_{t+1}|s_t) \tag{5}$$

$p_t(s_{t+1}|s_t)$ – the subjective probability of the state of the world at time $t+1$ conditioned on information at time t, $u(s_t)$ – one-period utility at state s_t and β – subjective discount factor

Representative Agent's Optimization Problem

Agent can buy or sell freely an asset with payoff ψ_{t+1} at price P_t. Y_t – agent's wealth (endowment) at t, ξ – amount of asset he chooses to buy, c_t – his consumption at t:

$$\max_{\{\xi\}}\{u(c_t) + \mathsf{E}_t[\beta u(c_{t+1})]\}$$

subject to

$$\begin{aligned} c_t &= Y_t - P_t \cdot \xi \\ c_{t+1} &= Y_{t+1} + \psi_{t+1} \cdot \xi \end{aligned}$$

FOC:

$$P_t = \mathsf{E}_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \psi_{t+1}\right] \tag{6}$$

MRS = Pricing Kernel

- $\text{MRS}_t \stackrel{\text{def}}{=} \beta \cdot \frac{u'(c_{t+1})}{u'(c_t)}$ is the *Marginal Rate of Substitution* at t, the rate at which the investor is willing to substitute wealth at $t+1$ for wealth at t. c can be substituted with every variable in s.
- Lucas (1978), Merton (1973) and others generalized to multi-period models. Utility and wealth are all functions of stock price S_t (the only income), time to maturity is $\tau \stackrel{\text{def}}{=} T - t$:

$$P_t = e^{-r\tau} \int_0^{\infty} \psi(S_T) \lambda \frac{U'(S_T)}{U'(S_t)} p_t(S_T|S_t) dS_T \qquad (7)$$

where $\lambda e^{-r\tau} = \beta$, λ – constant independent of index level or interest rate

MRS = Pricing Kernel

- Aït-Sahalia and Lo (2000) show, that equation (7) can be rewritten using the risk-neutral probability measure as:

$$P_t = e^{-r\tau} \int_0^\infty \psi(S_T) q_t(S_T|S_t) dS_T = e^{-r\tau}\, \mathsf{E}_t^Q[\psi(S_T)] \quad (8)$$

  where $q_t(S_T|S_t)$ is the State Price Density

- Combining equations (7) and (8), the *pricing kernel* $M_t(S_T)$ is defined:

$$M_t(S_T) \stackrel{\text{def}}{=} \frac{q_t(S_T|S_t)}{p_t(S_T|S_t)} = \lambda \frac{U'(S_T)}{U'(S_t)} = \text{MRS}_t \quad (9)$$

Pricing Kernels and Risk Aversion

- Risk aversion is a measure of the curvature of the agent's utility function
- Arrow (1965) and Pratt(1964) defined representative agent's *coefficient of Relative Risk Aversion* (RRA) as

$$\rho_t(S_T) \stackrel{\text{def}}{=} -\frac{S_T u''(S_T)}{u'(S_T)} \tag{10}$$

- According to equation (9):

$$\begin{aligned} & M_t(S_T) = \lambda \frac{U'(S_T)}{U'(S_t)} \\ \Rightarrow \quad & M_t'(S_T) = \lambda \frac{U''(S_T)}{U'(S_t)} \end{aligned} \tag{11}$$

Pricing Kernels and Risk Aversion

Expressing equation (10) using equation (11):

$$\rho_t(S_T) = -\frac{S_T \lambda M_t'(S_T) U'(S_t)}{\lambda M_t(S_T) U'(S_t)} = -\frac{S_T M_t'(S_T)}{M_t(S_T)} \tag{12}$$

Using equation (9) the RRA is:

$$\begin{aligned}
\rho_t(S_T) &= -\frac{S_T[q_t(S_T|S_t)/p_t(S_T|S_t)]'}{q_t(S_T|S_t)/p_t(S_T|S_t)} \\
&= -S_T \frac{q_t'(S_T|S_t)p_t(S_T|S_t) - p_t'(S_T|S_t)q_t(S_T|S_t)}{q_t(S_T|S_t)p_t(S_T|S_t)} \\
&= S_T \left[\frac{p_t'(S_T|S_t)}{p_t(S_T|S_t)} - \frac{q_t'(S_T|S_t)}{q_t(S_T|S_t)} \right]
\end{aligned} \tag{13}$$

Power Utility of Consumption

- Rubinstein (1976) showed that for a power utility of consumption:

$$u(c_t) = \begin{cases} \frac{1}{1-\gamma} c_t^{1-\gamma} & \text{for } 0 < \gamma \neq 1 \\ \log(c_t) & \text{for } \gamma = 1 \end{cases} \quad (14)$$

  aggregate consumption $\propto$ aggregate wealth (wealth, stock price and consumption utility are interchangeable)

- It can be seen that $\lim_{\gamma \to 0} u(c_t) = c_t$

Constant Relative Risk Aversion (CRRA)

The risk aversion of an investor with a power utility can be calculated using equation (10), with consumption instead of wealth as an argument (as they are interchangeable):

$$\rho(c_t) = -c_t \frac{-\gamma(c_t)^{-\gamma-1}}{(c_t)^{-\gamma}} = \gamma \qquad (15)$$

This equation shows that the RRA turns out to be a constant, and for the logarithmic utility case, the risk aversion is 1.

P and Q under B&S

- Implied risk-neutral probability is log-normal:

$$q^{BS}(S_T|S_t) = \frac{1}{S_T\sqrt{2\pi\sigma^2\tau}} \cdot e^{-\frac{[\ln(S_T/S_t)-(r-0.5\sigma^2)\tau]^2}{2\sigma^2\tau}} \tag{16}$$

and the underlying asset price follows the stochastic process:

$$\frac{dS_t}{S_t} = r \cdot dt + \sigma \cdot dW_t$$

- The subjective probability is also log-normal:

$$p(S_T|S_t) = \frac{1}{S_T\sqrt{2\pi\sigma^2\tau}} \cdot e^{-\frac{[\ln(S_T/S_t)-(\mu-0.5\sigma^2)\tau]^2}{2\sigma^2\tau}} \tag{17}$$

and the underlying asset price follows the stochastic process:

$$\frac{dS_t}{S_t} = \mu \cdot dt + \sigma \cdot dW_t$$

Pricing Kernel and Risk Aversion under B&S

Plugging B&S Q and P into equation (9) yields a closed-form solution for the investor's pricing kernel:

$$M_t^{BS}(S_T) = \left(\frac{S_T}{S_t}\right)^{-\frac{\mu-r}{\sigma^2}} \cdot e^{\frac{(\mu-r)(\mu+r-\sigma^2)\tau}{2\sigma^2}} \tag{18}$$

The investor's utility function can be derived by solving the differential equation. It can be shown to be a power utility and the RRA is therefore constant:

$$\rho_t^{BS}(S_T) = \gamma = \frac{\mu - r}{\sigma^2} \tag{19}$$

Constant RRA utility function under B&S was shown by Rubinstein (1976), Breeden and Lietzenberger (1978) and many others

The Stock Price under B&S

Revisiting the stochastic process in equation (4), the Brownian motion defined on the probability space of the risk neutral measure under B&S with a constant RRA is:

$$\overline{W}_t = W_t + \frac{\mu - r}{\sigma} t = W_t + \sigma\gamma t \tag{20}$$

whereas the stochastic process can be expressed as:

$$d\widetilde{S}_t = \sigma \widetilde{S}_t d\overline{W}_t = \sigma \widetilde{S}_t dW_t + \sigma^2 \widetilde{S}_t \gamma dt \tag{21}$$

On Estimating the PK

- The assumptions of B&S don't hold in practice, the implied volatility experiences a smile and the SPD does not have a closed form solution
- A good estimation of the pricing kernel can be achieved by estimating the p and q empirically and derive the PK and RRA from them.
- The *Pricing Kernel Puzzle* of Jackwerth (2000)
- Doubt whether the ratio of two estimators equals the estimate of the ratio - beyond the scope of this work
- Rubinstein (1994) showed, that any two of the following imply the third: Utility function, P and SPD. We will estimate P and Q and calculate PK based on equation (9)

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Model Description

- Static (daily) model to estimate subjective density, SPD, pricing kernel and relative risk aversion on a daily basis
- Repeating this process for each trading day in the database (April 1999 - April 2002)
- Examining the time-series of the PK and RRA to draw conclusions on a changing investors behavior

Subjective Density Estimation

- Taking the preceding three months as historical data and estimating a GARCH (1,1) model

$$\begin{aligned} \varepsilon_t &= \sigma_t Z_t \\ \sigma_t^2 &= \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \end{aligned} \tag{22}$$

  where ω, α, β are estimated using QMLE.
- Simulating a GARCH (1,1) model with the estimated parameters to obtain the time-series ε_t.
- Creating the simulated DAX level as:

$$S_t = S_{t-1} e^{\varepsilon_t} \quad \forall t \in \{1, \dots, T\}, S_0 \text{ given} \tag{23}$$

- Estimating the density of the DAX at certain maturities

Daily Subjective Density Estimates

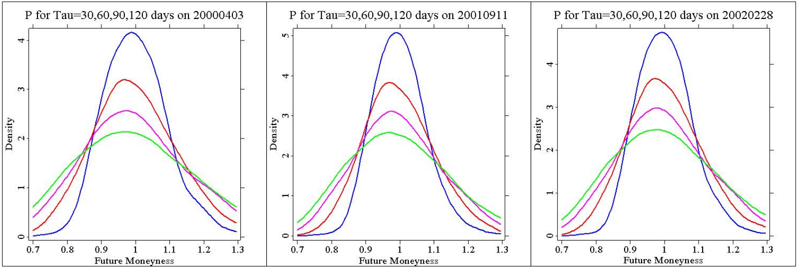

Figure 1: Subjective density for different maturities (30,60,90,120 days) on different trading days. EPKdailyprocess.xpl

SPD Estimation

- Estimating IVS from the trivariate data moneyness (κ), time to maturity (τ) and IV ($\sigma^{BS}(\kappa, \tau)$) using local polynomial regression (Rookley (1997)) from options data of a specific day
- Local polynomial regression yields the IVS and its first two derivatives with respect to moneyness and time to maturity in one step
- Starting from $\frac{\partial^2 \sigma}{\partial \kappa^2}$, it is possible to derive $\frac{\partial^2 C}{\partial K^2}$
- Breeden and Lietzenberger (1978):

$$e^{r\tau} \left. \frac{\partial^2 C(S_t, K, \tau)}{\partial K^2} \right|_{K=S_T} = q_t(S_T) = \text{SPD} \qquad (24)$$

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Daily SPD Estimates

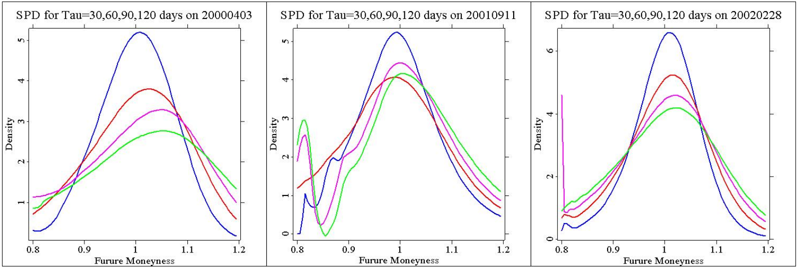

Figure 2: SPD for different maturities (30,60,90,120 days) on different trading days. EPKdailyprocess.xpl

Daily Pricing Kernel Estimates

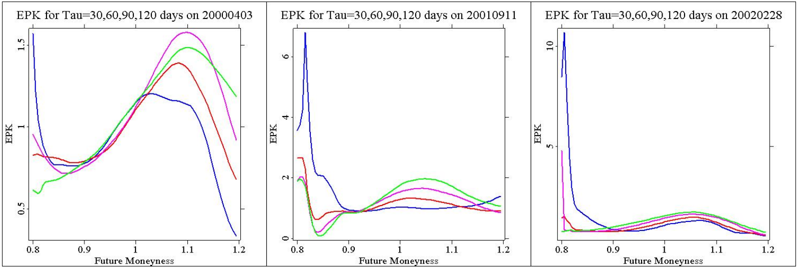

Figure 3: Pricing Kernel for different maturities (30,60,90,120 days) on different trading days. EPKdailyprocess.xpl

Daily Relative Risk Aversion Estimates

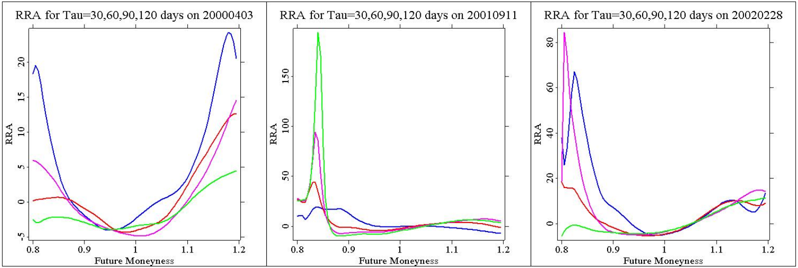

Figure 4: Relative Risk Aversion for different maturities (30,60,90,120 days) on different trading days. EPKdailyprocess.xpl

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The Database

- Intraday DAX and options data from MD*Base between January 4^{th}, 1999 and April 30^{th}, 2002, after a thorough preparation scheme.
- For each observation: $\widetilde{S}_t$(corrected for dividends), K, r, τ, P_t, type of option, κ_f, σ_t^{BS}.
- Observations with $\tau <= 1$ day, $IV > 0.7$, $\kappa_f > 1.22$ or $\kappa_f < 0.74$ are dropped.
- 2,719,640 observations on 843 trading days.
- Daily estimation begins in April 1999 to enable a three months window for GARCH estimation.
- The days on which the GARCH model doesn't fit the data or local polynomial estimation reveals negative volatilities are dropped.

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The Analyzed Moments

Considering 5 moments ($\widehat{f}_t$ – daily estimation of PK or RRA):

- Pricing Kernel and RRA at the money $\widehat{f}_t^{ATM}(\kappa = 1)$
- Expectation: $\mu_t = \int \kappa \widehat{f}_t d\kappa$
- Standard Deviation: $\sigma_t = \sqrt{\int (\kappa - \mu_t(\widehat{f}_t))^2 \widehat{f}_t d\kappa}$
- Skewness: $Skew_t = \frac{1}{\sigma_t^3} \int (\kappa - \mu_t(\widehat{f}_t))^3 \widehat{f}_t d\kappa$
- Kurtosis: $Kur_t = \frac{1}{\sigma_t^4} \int (\kappa - \mu_t(\widehat{f}_t))^4 \widehat{f}_t d\kappa$

Also: Differences and Log Differences of the moments.
Altogether: 5 moments $\times$ 3 time-series $\times$ 4 maturities $=$ 60 series for each, PK and RRA.

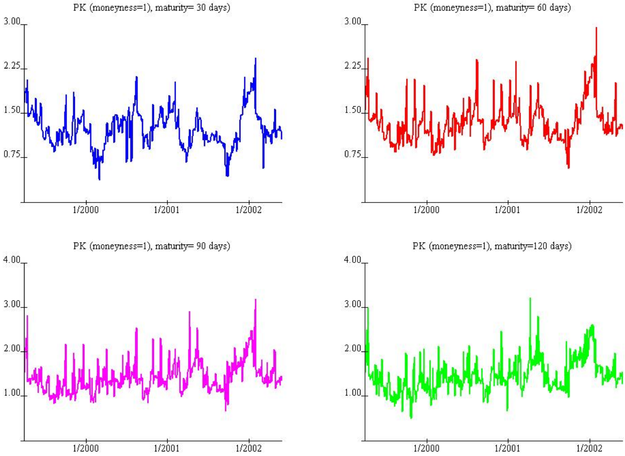


Figure 5: ATM Pricing Kernel for different maturities (30,60,90,120 days).

EPKtimeseries.xpl

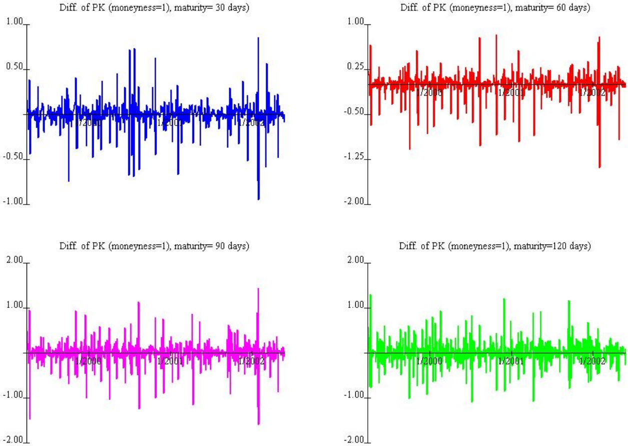


Figure 6: Differences of ATM Pricing Kernel for different maturities (30,60,90,120 days). EPKtimeseries.xpl

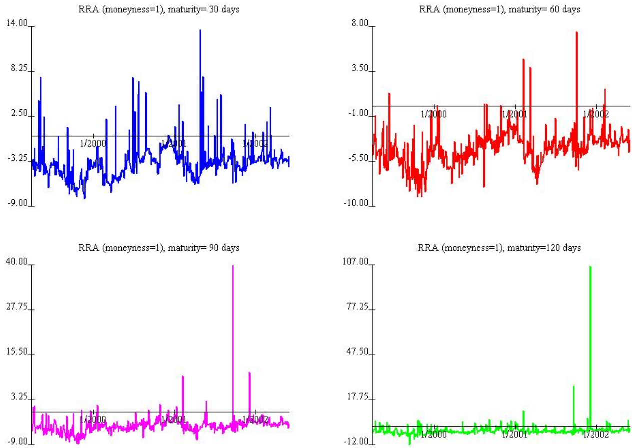


Figure 7: ATM Relative Risk Aversion for different maturities (30,60,90,120 days). EPKtimeseries.xpl

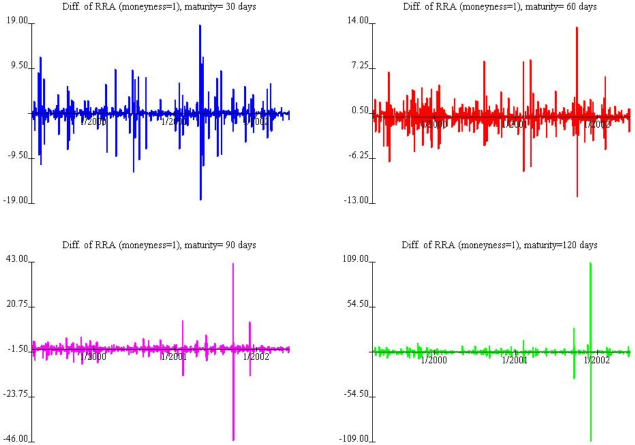


Figure 8: Differences of ATM Relative Risk Aversion for different maturities (30,60,90,120 days). EPKtimeseries.xpl

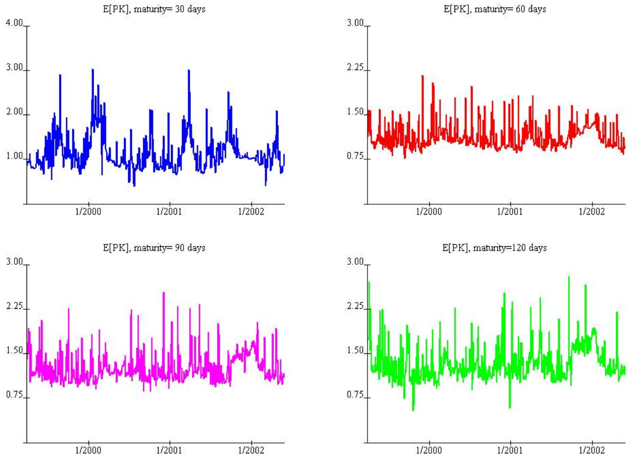


Figure 9: Expected Pricing Kernel for different maturities (30,60,90,120 days). EPKtimeseries.xpl

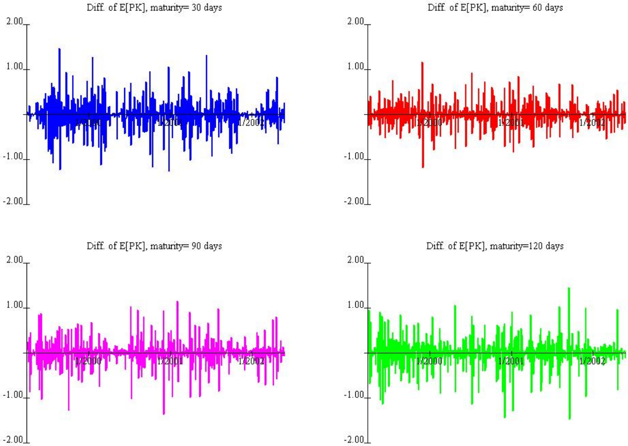


Figure 10: Differences of Expected Pricing Kernel for different maturities (30,60,90,120 days). EPKtimeseries.xpl

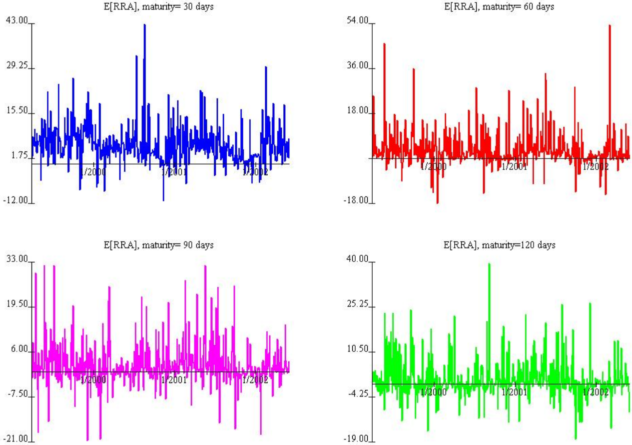


Figure 11: Expected Relative Risk Aversion for different maturities (30,60,90,120 days). EPKtimeseries.xpl

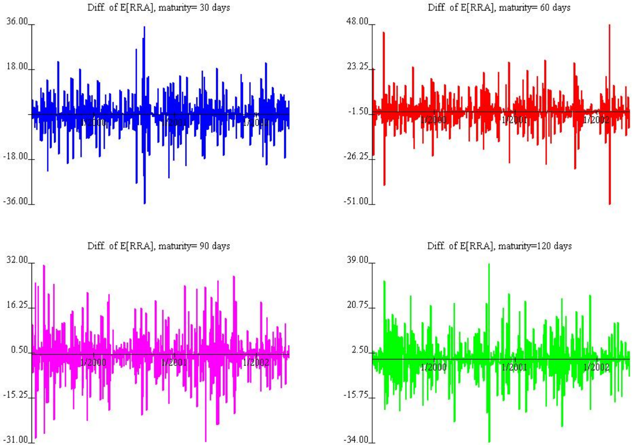


Figure 12: Differences of Expected Relative Risk Aversion for different maturities (30,60,90,120 days). EPKtimeseries.xpl

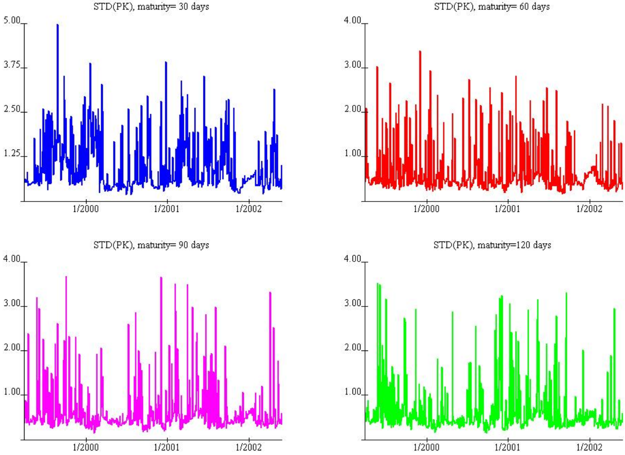


Figure 13: Standard Deviation of Pricing Kernel for different maturities (30,60,90,120 days). EPKtimeseries.xpl

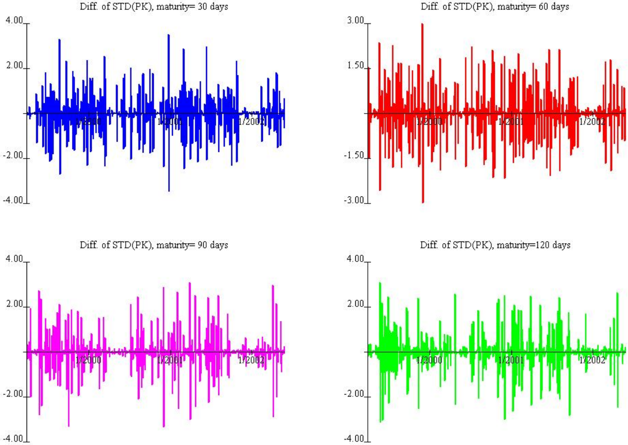


Figure 14: Differences of Standard Deviation of Pricing Kernel for different maturities (30,60,90,120 days). EPKtimeseries.xpl

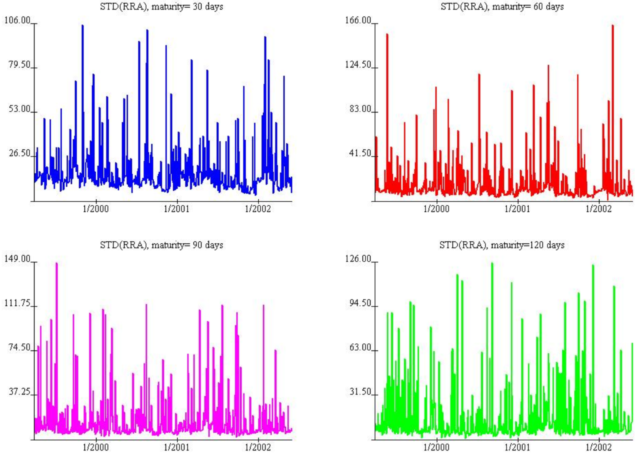


Figure 15: Standard Deviation of Relative Risk Aversion for different maturities (30,60,90,120 days). EPKtimeseries.xpl

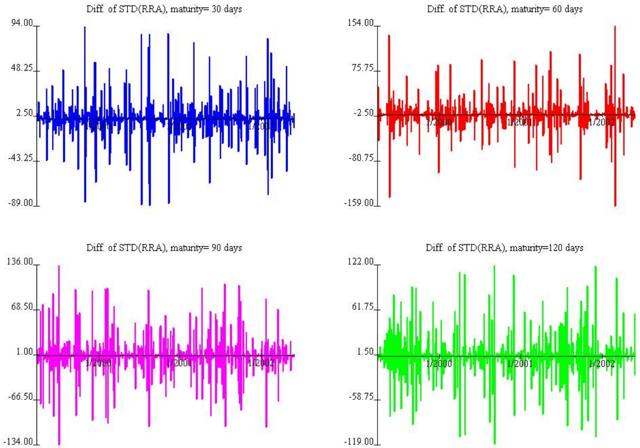


Figure 16: Differences of Standard Deviation of Relative Risk Aversion for different maturities (30,60,90,120 days). EPKtimeseries.xpl

KPSS Test for Stationarity

- Checking which of the 120 time-series are stationary
- Null hypothesis: series is stationary, $\phi = 0$

$$X_t = c + \nu t + \phi \sum_{i=1}^{t} \xi_i + \eta_t \tag{25}$$

  where ν is a linear time trend, η_t is stationary
- KPSS test is sensitive to existence of trend and to choice of the order of the autocovariance estimator, test conducted with various T and with/without trend

Example for KPSS Test

Maturity	No Linear Trend			With Linear Trend			Stationary
	T=0	T=7	T=21	T=0	T=7	T=21	
$\tau = 30$							
μ_t	0.883	0.135	0.078	0.579	0.089	0.052	For higher order
$\Delta\mu_t$	0.003	0.022	0.036	0.003	0.016	0.027	YES
$\Delta \log \mu_t$	0.003	0.022	0.037	0.003	0.022	0.028	YES
$\tau = 60$							
μ_t	0.281	0.107	0.067	0.224	0.086	0.054	Without trend
$\Delta\mu_t$	0.002	0.014	0.037	0.037	0.012	0.034	YES
$\Delta \log \mu_t$	0.002	0.014	0.038	0.001	0.013	0.035	YES

KPSS Results - PK Stationarity

Maturity	PK^{ATM}	μ_t	σ_t	$Skew_t$	Kur_t
$\tau = 30$	Large T	Large T	Large T	Large T	Large T
$\tau = 60$	Large T	No trend	With trend	Large T	Large T
$\tau = 90$	Large T no trend	Large T	Large T without trend	Large T	Large T
$\tau = 120$	Large T with trend	No trend	Large T	Large T with trend	N.A.
Δ	Always Stationary				
Δ log	Always Stationary			N.A.	Stationary

KPSS Results - RRA Stationarity

Maturity	RRA^{ATM}	μ_t	σ_t	$Skew_t$	Kur_t
$\tau = 30$	Large T with trend	Large T	Always Stationary	No trend	Always Stationary
$\tau = 60$	Large T with trend	Large T	Always Stationary	Always Stationary	Always Stationary
$\tau = 90$	Large T with trend	No trend	Always Stationary	Always Stationary	Always Stationary
$\tau = 120$	With trend	Large T	Always Stationary	With trend	N.A.
Δ		Always	Stationary		
Δ log	N.A.	N.A.	Stationary	N.A.	Stationary

Principal Components Analysis

Conducting PCA on the differences of the moments of PK and RRA:

$$\mathcal{X} = \begin{pmatrix} \Delta PK_2^{ATM} & \Delta\mu_2 & \Delta\sigma_2 & \Delta Skew_2 & \Delta Kur_2 \\ \Delta PK_3^{ATM} & \Delta\mu_3 & \Delta\sigma_3 & \Delta Skew_3 & \Delta Kur_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta PK_n^{ATM} & \Delta\mu_n & \Delta\sigma_n & \Delta Skew_n & \Delta Kur_n \end{pmatrix}$$

PC Analysis - RRA Differences

Principal Component	Eigenvalue	Explained Variance	Cum. Expl. Variance
$\tau = 30$			
1	1.87	37.48%	37.48%
2	1.63	32.55%	70.03%
3	1.00	19.95%	89.98%
4	0.36	7.23%	97.21%
5	0.14	2.79%	100.00%
$\tau = 60$			
1	1.88	37.58%	37.58%
2	1.77	35.45%	73.03%
3	0.98	19.53%	92.56%
4	0.23	4.58%	97.14%
5	0.14	2.86%	100.00%

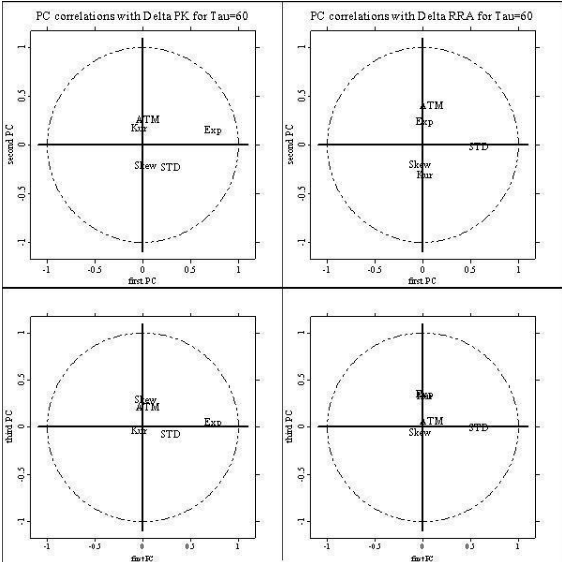
The Principal Components

ΔPK Principal Components:

1. Central mass movement factor: $\Delta\mu_t$ and $\Delta\sigma_t$
2. Change of tendency factor: $\Delta Skew_t$ and $\Delta\sigma_t$
3. The shifts in investors' beliefs at the money: ΔPK^{ATM}

ΔRRA Principal Components:

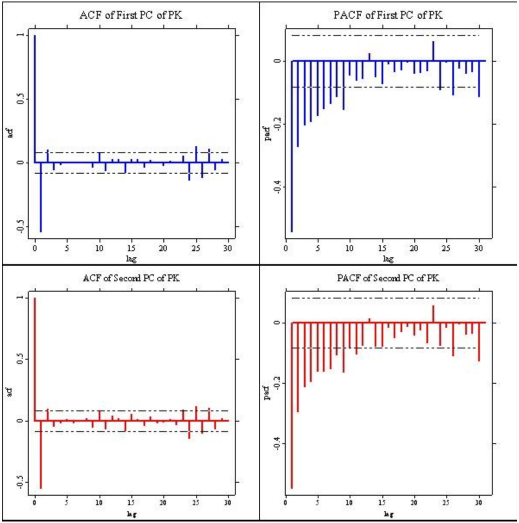
1. Dispersion change factor: $\Delta\sigma_t$
2. The shifts in investors' beliefs at the money: ΔRRA^{ATM}
3. Dispersion change factor: ΔKur_t


PC correlations with Delta PK for Tau=60
PC correlations with Delta RRA for Tau=60
ATM
Kur
Exp
Skew
STD
second PC
third PC
first PC
1
0.5
0
-0.5
-1

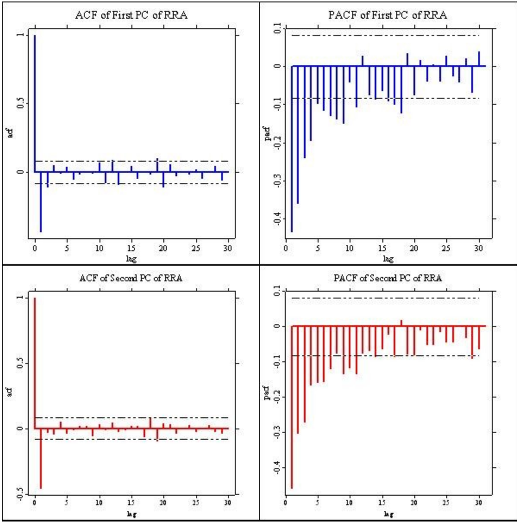
EPKPCA.xpl

Time-Series Analysis of PCs

- After defining proper PCs for the differences of PK and RRA, checking for autocorrelation of those PCs.
- Demonstrating on a maturity of 60 days, checking ACF and PACF functions, we detect a MA(1) structure..



EPKtimeseries.xpl



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Outline of the talk

1. Introduction and Theory
2. The Model
3. Some Empirical Results
4. Conclusions ✓

Conclusions and Further Research

- Changes in daily moments of PK and RRA are stationary.
- Variability of daily estimates of PK and RRA can be explained using three factors, behavior at the money is one of them.
- The principal components exhibits a MA(1) structure.

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