Skew Hedging

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Barrier options

Knock-out options are financial options that become worthless as soon as the underlying reaches a prespecified barrier.

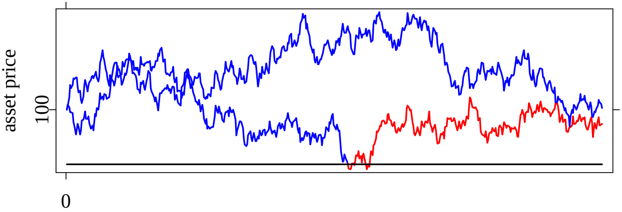
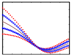


Figure 1: Example of two possible paths of asset's price. When the price hits the barrier (red) the option is no longer valid regardless further evolution of the price.



Barrier options

- In BS world prices of barrier options are given analytically, all greeks can be calculated directly.
- The price doesn't need to be a increasing function of the volatility parameter σ.
- Marking to BS model is precluded due to the σ choice
- BS is not a good choice for handling barrier options!!!

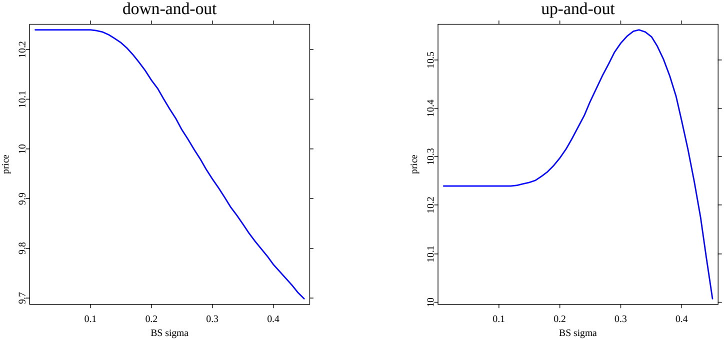


Figure 2: *Price of the call knock-out barrier options as a function of BS-σ. Asset value $S_0 = 90$, strike price $K = 80$ time to maturity $\tau = 0.1$ interest rate $r = 0.03$. Left panel: barrier $B = 80$. Right panel: barrier $B = 120$.*

Pricing Barrier Options

For pricing barrier options a local volatility (LV) model is employed. The asset price dynamics are governed by the stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma(S_t, t) dW_t \tag{1}$$

where W_t is a Brownian motion, μ the drift and $\sigma(S_t, t)$ the local volatility function which depends on the asset price and time only.

Pricing Barrier Options

Price depends on the **entire** implied volatility surface (IVS). From the IVS one can calculate $C_t(K, T)$.
Dupire formula:

$$\sigma^2(K,t) = 2\frac{\frac{\partial C_t(K,T)}{\partial T} + rK\frac{\partial C_t(K,T)}{\partial K}}{K^2\frac{\partial^2 C_t(K,T)}{\partial K^2}}$$

gives the LV surface $\sigma(S_t, t)$. For practical implementation see Andersen and Brotherthon-Ratcliffe (1997).

Dynamics of the IVS

The IVS reveals highly dynamic behavior, which influences the prices of the barrier options.

Example

Consider two one year knock-out put options with strike 110 and barrier 80, when the current spot level is 100. Take the IVS from 20000103 and 20010102. The prices of these options are respectively 1.91 and 2.37. This is a 25% difference.

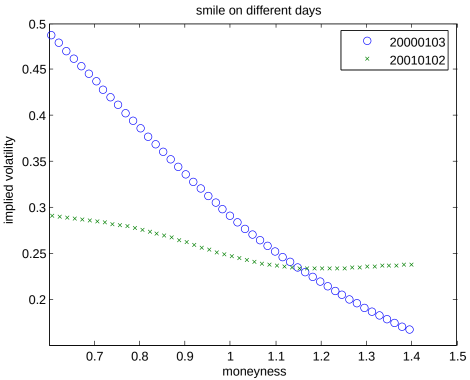
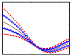


Figure 3: *Observed smile on 20000103 and 20010102 for the maturity 0.25.*



Vega Hedging

- In LV model the usual vega cannot be used because the whole IVS is an input
- The standard approach is to build vega hedging on the sensitivity of the "up-and-down" shifts.
- The skew changes, which may cause significant pricing differences, become unhedged.

DSFM

A complex dynamics of the IVS is explained in terms of a **dynamic semiparametric factor model (DSFM)** for the (log)-IVS $Y_{i,j}$ $(i = \text{day}, j = \text{intraday})$:

$$Y_{i,j} = m_0(X_{i,j}) + \sum_{l=1}^{L} \beta_{i,l} m_l(X_{i,j}) + \varepsilon_{i,j}. \tag{2}$$

Here $m_l(X_{i,j})$ are smooth factor functions and $\beta_{i,l}$ is a multivariate (loading) time-series.

Aims

- to apply DSFM for identification of key factors of the IVS dynamics
- to improve the vega hedge by hedging against most common changes of the IVS

Overview

DSFM

Consider DSFM for the log-IVS:

$$Y_{i,j} = m_0(X_{i,j}) + \sum_{l=1}^{L} \beta_{i,l} m_l(X_{i,j}) + \varepsilon_{i,j}, \tag{3}$$

$Y_{i,j}$ is log IV,i denotes the trading day $(i = 1, \ldots, I)$,
$j = 1, \ldots, J_i$ is an index of the traded options on day i.
$m_l(\cdot)$ for $l = 0, \ldots, L$ are basis functions in covariables $X_{i,j}$ (moneyness, time to maturity),
and β_i are time dependent factors.

DSFM estimation

Define estimates of $\widehat{m}_l$ and $\widehat{\beta}_{i,l}$ with $\widehat{\beta}_{i,0} \stackrel{\mathrm{def}}{=} 1$, as minimizers of:

$$\sum_{i=1}^{I} \sum_{j=1}^{J_i} \int \left\{ Y_{i,j} - \sum_{l=0}^{L} \widehat{\beta}_{i,l} \widehat{m}_l(u) \right\}^2 K_h(u - X_{i,j})\, du, \qquad (4)$$

where K_h denotes a two dimensional product kernel,
$K_h(u) = k_{h_1}(u_1) \times k_{h_2}(u_2), \quad h = (h_1, h_2)$ with a one-dimensional kernel $k_h(v) = h^{-1} k(h^{-1} v)$.
See Fengler et al. (2005), Fengler (2005).

Model parameters

We fit our model:

- $L = 3$ dynamic basis functions
- grid covering moneyness $\in [0.6, 1.3]$ and time to maturity $\in [0.05, 1]$
- fix bandwidths in moneyness direction and increasing bandwidths in maturity direction
- on the daily IVS data from 20000103 till 20011220

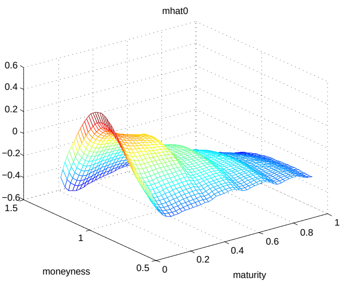


Figure 4: *Invariant basis function* $\widehat{m}_0$ *and dynamic basis function* $\widehat{m}_1$ *(level)*

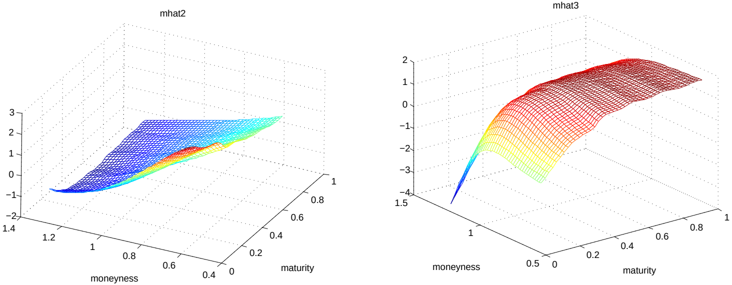


Figure 5: *Dynamic basis functions $\widehat{m}_2$ (skew) and $\widehat{m}_3$ (term structure)*

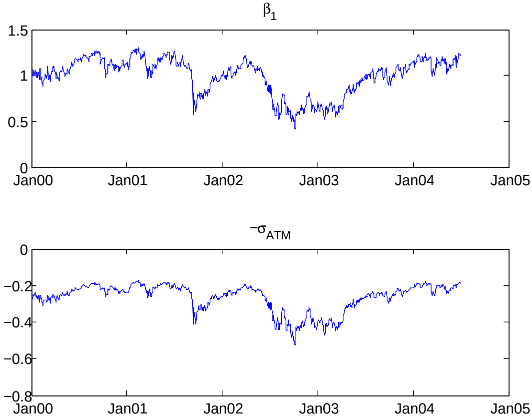


Figure 6: *time series of weights* $\widehat{\beta}_1$ *and ATM IVS for the fixed maturity 0.25.*

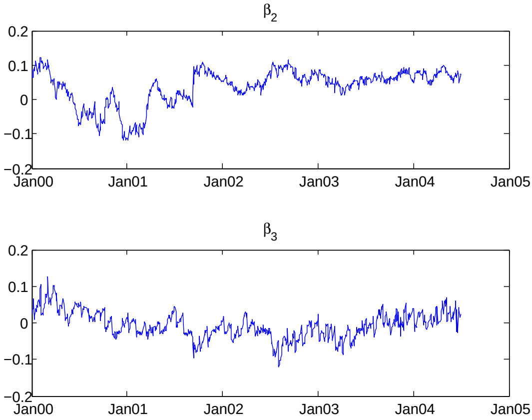


Figure 7: *Time series of weights* $\widehat{\beta}_2$ *and* $\widehat{\beta}_3$

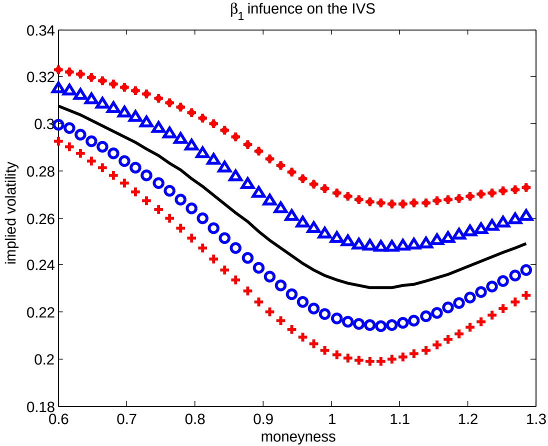


Figure 8: *Typical shape of the smile for different levels of $\widehat{\beta}_1$. Changes of the $\widehat{\beta}_1$ influence mainly the surface's level.*

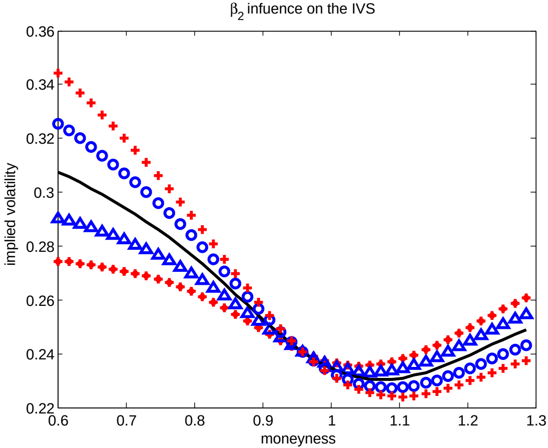


Figure 9: *Typical shape of the smile for different levels of $\widehat{\beta}_2$. Changes of the $\widehat{\beta}_2$ influence the smile's skew.*

Greeks

- In order to implement β-hedging one has to calculate β-greeks.
- They are obtained by shifting the IVS in the $\widehat{m}$ direction.

$$\frac{\partial option}{\partial \widehat{\beta}} \approx \frac{option(IVSe^{\Delta \widehat{\beta} \widehat{m}}) - option(IVSe^{-\Delta \widehat{\beta} \widehat{m}})}{2\Delta \widehat{\beta}} \tag{5}$$

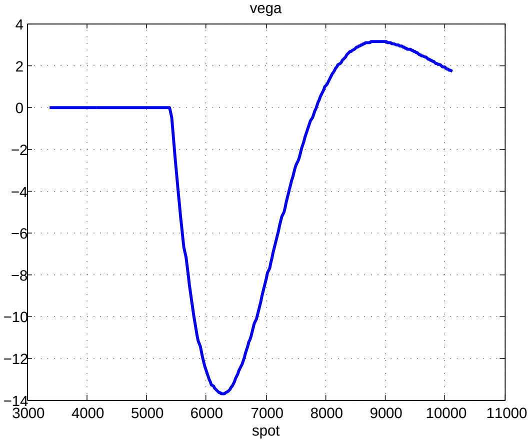


Figure 10: *vega "greek" for down-and-out put option with barrier* 5400 *and strike* 7425 *as a function of spot*

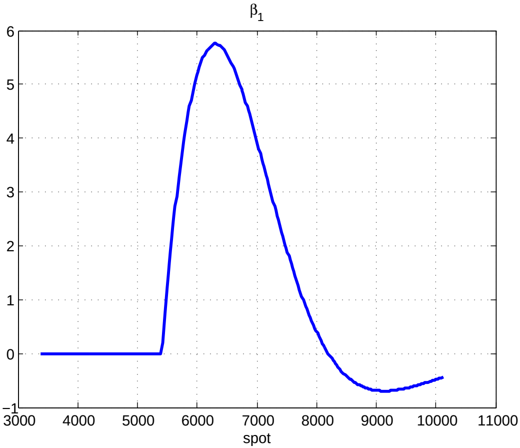


Figure 11: $\widehat{\beta}_1$ *"greek" for down-and-out put option with barrier* 5400 *and strike* 7425 *as a function of spot*

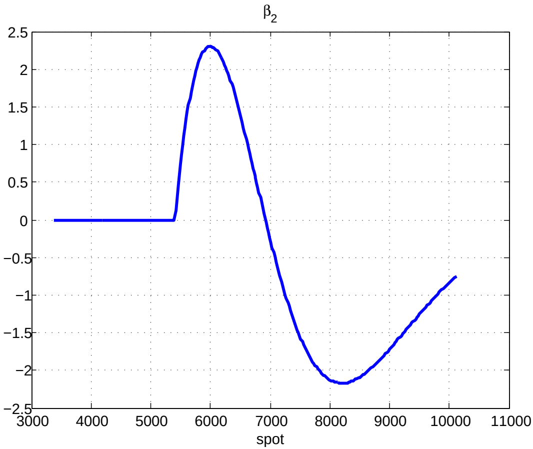


Figure 12: $\widehat{\beta}_2$ *"greek" for down-and-out put option with barrier* 5400 *and strike* 7425 *as a function of spot*

Example

In the BS model the hedge portfolio (HP) for hedging plain vanilla options consists of a stocks - $HP = aS$. The hedge ratio a (delta) is obtained from:

$$\frac{dHP}{dS} = a = \frac{\partial option}{\partial S}.$$

The hedge is financed by buying/selling bonds.

How to compute the hedge ratios

Take two hedge portfolios HP_1 and HP_2.
Compute the sensitivities of the hedge portfolios and the up-and-out call option (C^{KO}) with respect to $\widehat{\beta}_1$ and $\widehat{\beta}_2$.
Solve

$$\begin{pmatrix} \frac{\partial HP_1}{\partial \widehat{\beta}_1} & \frac{\partial HP_2}{\partial \widehat{\beta}_1} \\ \frac{\partial HP_1}{\partial \widehat{\beta}_2} & \frac{\partial HP_2}{\partial \widehat{\beta}_2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial C^{KO}}{\partial \widehat{\beta}_1} \\ \frac{\partial C^{KO}}{\partial \widehat{\beta}_2} \end{pmatrix} \begin{pmatrix} \textit{vega} \\ \textit{skew} \end{pmatrix}$$

for the hedge ratios a_1, a_2. For the down-and-out put option (P^{KO}) the procedure is analogous.

Choice of the hedge portfolio

Idea:
choose HP_1 and HP_2 with *maximum exposure* to $\widehat{\beta}_1$ and $\widehat{\beta}_2$, respectively:

> HP_1 should be most sensitive to up-and-down shifts:
> use a portfolio of at-the-money plain vanilla options;
>
> HP_2 should be most sensitive to slope changes:
> use a portfolio of vega-neutral risk reversals.

Then $\frac{\partial HP_1}{\partial \widehat{\beta}_2} \approx 0$ and $\frac{\partial HP_2}{\partial \widehat{\beta}_1} \approx 0$.

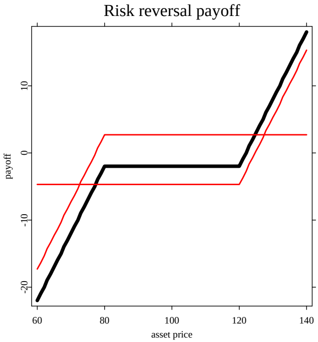
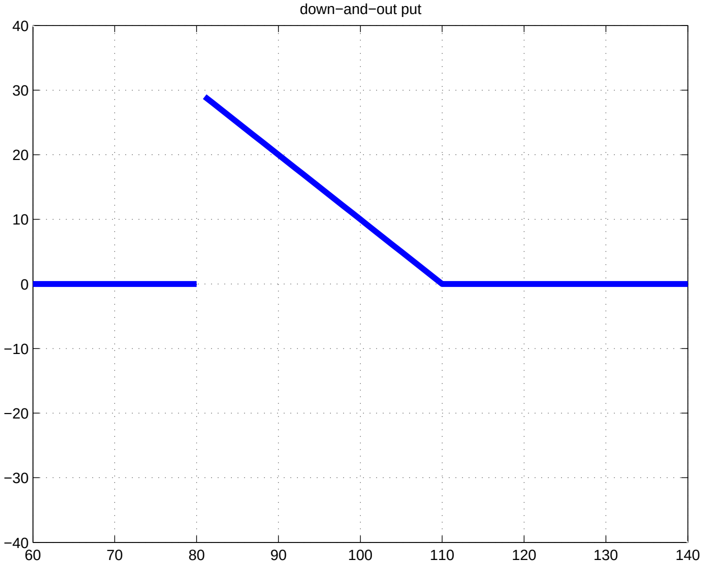


Figure 13: *The payoff of the risk reversal. It is composed from a long call with strike* $K_1 = 120$ *and a short put with strike* $K_2 = 80$.

As in standard vega hedging we apply final delta hedge. In our case we apply delta hedge to $C^{KO} + a_1 HP_1 + a_2 HP_2$ by calculating the number of underlying as:

$$\frac{\partial (C^{KO} + a_1 HP_1 + a_2 HP_2)}{\partial S}$$


down−and−out put
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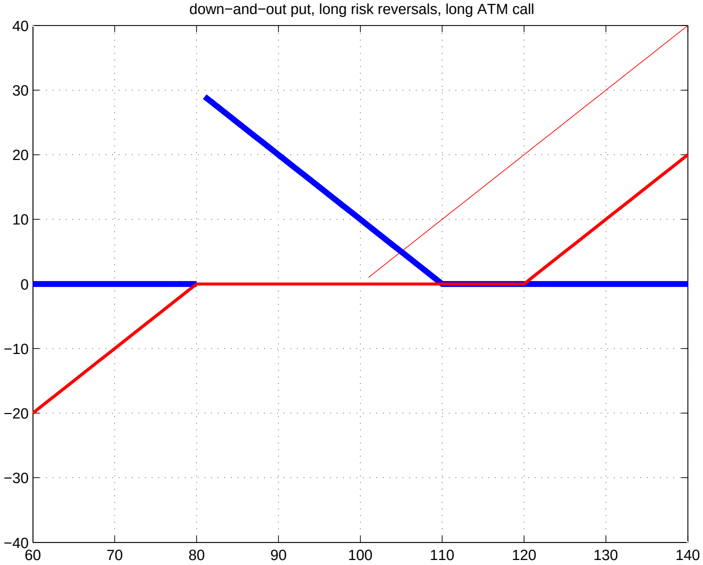

down−and−out put, long risk reversals, long ATM call
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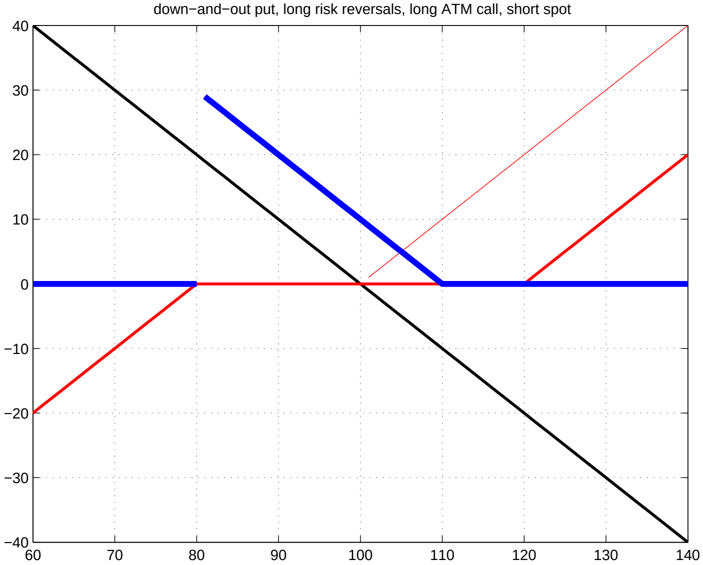
down−and−out put, long risk reversals, long ATM call, short spot
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Empirical Study

For each of 885 days (20000103-20030707) we start one long position in one year C^{KO} and P^{KO}.

Option	barrier	strike	maturity	knock-outs	in-the-money
C^{KO}	140 %	80 %	1 year	10 %	39 %
P^{KO}	80 %	110 %	1 year	81 %	5 %

Table 1: *barrier and strike are given as a percentage of the spot at the starting day*

We keep the position until maturity or knock-out.

Empirical Study

We compare the $\widehat{\beta}_1\widehat{\beta}_2$ (skew) hedging approach with:

- $\widehat{\beta}_1$ hedging - no risk reversal $(a_2 = 0)$ and $a_1 = \frac{-\partial C^{KO}}{\partial \widehat{\beta}_1} / \frac{\partial HP_1}{\partial \widehat{\beta}_1}$
- vega hedging - no risk reversal $(a_2 = 0)$ and $a_1 = \frac{-\partial C^{KO}}{\partial \sigma} / \frac{\partial HP_1}{\partial \sigma}$

Aims of Hedging

- We define the profit and loss of the strategy at the maturity as a portfolio's value divided by notional at the starting day.

$$\frac{C_T^{KO} + HP_T + money_T}{S_0}$$

- The aim of the hedging is possibly large reduction of the profit and loss variation around zero.

Results

Profit and loss of the strategy at the maturity.

C^{KO}	min	max	mean	median	std	med. abs.
vega	-0.1038	0.5813	-0.0165	-0.0175	0.0209	0.0413
β_1	-0.0752	0.5768	-0.0118	-0.0136	0.0183	0.0387
$\beta_1\beta_2$	-0.0830	0.5684	-0.0066	-0.0119	0.0137	0.0345

Table 2: *all values as percentage of the underlying*

C^{KO}	days	min	max	mean	median	std	med. abs.
vega	0	-0.1038	0.5813	-0.0165	-0.0175	0.0209	0.0413
	1	-0.1038	0.1710	-0.0186	-0.0171	0.0183	0.0276
	10	-0.0833	0.0710	-0.0184	-0.0164	0.0172	0.0241
	25	-0.0797	0.0590	-0.0191	-0.0151	0.0150	0.0207
β_1	0	-0.0752	0.5768	-0.0118	-0.0136	0.0183	0.0387
	1	-0.0751	0.1459	-0.0139	-0.0130	0.0157	0.0240
	10	-0.0766	0.0702	-0.0143	-0.0130	0.0154	0.0210
	25	-0.0731	0.0508	-0.0150	-0.0116	0.0130	0.0175
$\beta_1\beta_2$	0	-0.0830	0.5684	-0.0066	-0.0119	0.0137	0.0345
	1	-0.0829	0.1220	-0.0088	-0.0120	0.0112	0.0184
	10	-0.0375	0.0831	-0.0095	-0.0119	0.0106	0.0149
	25	-0.0360	0.0499	-0.0104	-0.0123	0.0082	0.0114

Table 3: Descriptive statistics for the hedging strategies 0, 1, 10 and 25 days before the knock-out or expiration - delta hedging effect (gap risk).

Results

Profit and loss of the strategy at the maturity.

P^{KO}	min	max	mean	median	std	med. abs.
vega	-0.0264	0.2799	0.0058	-0.0004	0.0105	0.0213
β_1	-0.0210	0.2808	0.0080	0.0016	0.0107	0.0214
$\beta_1\beta_2$	-0.0332	0.2676	0.0065	0.0008	0.0092	0.0196

Table 4: Descriptive statistics for the hedging strategies of the down-and-out put

P^{KO}	days	min	max	mean	median	std	med. abs.
vega	0	-0.0264	0.2799	0.0058	-0.0004	0.0105	0.0213
	1	-0.0209	0.0344	-0.0040	-0.0048	0.0042	0.0064
	10	-0.0161	0.0231	-0.0024	-0.0027	0.0037	0.0056
	25	-0.0142	0.0189	-0.0018	-0.0014	0.0033	0.0046
β_1	0	-0.0210	0.2808	0.0080	0.0016	0.0107	0.0214
	1	-0.0157	0.0350	-0.0017	-0.0030	0.0038	0.0060
	10	-0.0106	0.0276	-0.0002	-0.0009	0.0031	0.0053
	25	-0.0109	0.0202	-0.0001	-0.0002	0.0027	0.0041
$\beta_1\beta_2$	0	-0.0332	0.2676	0.0065	0.0008	0.0092	0.0196
	1	-0.0249	0.0270	-0.0032	-0.0032	0.0030	0.0044
	10	-0.0110	0.0200	-0.0017	-0.0016	0.0027	0.0038
	25	-0.0092	0.0200	-0.0011	-0.0007	0.0023	0.0034

Table 5: Descriptive statistics for the hedging strategies 0, 1, 10 and 25 days before the knock-out or expiration - delta hedging effect (gap risk).

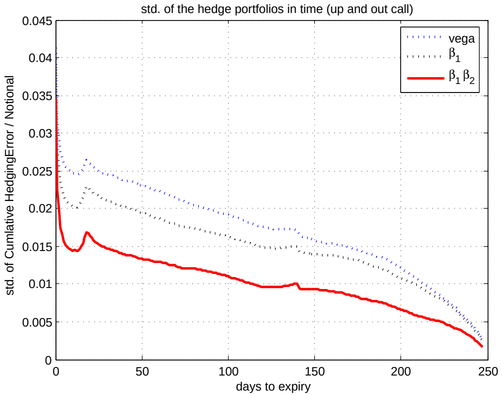
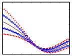


Figure 14: The standard deviation of the portfolios as a function of the days left to the maturity (call).



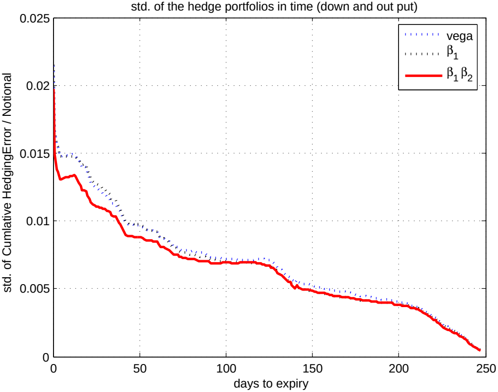


Figure 15: The standard deviation of the portfolios as a function of the days left to the maturity (put).

Conclusion

- the β hedge improves the hedging
- gap risk is still unhedged.
- better strategy might be to mix static and dynamic hedges

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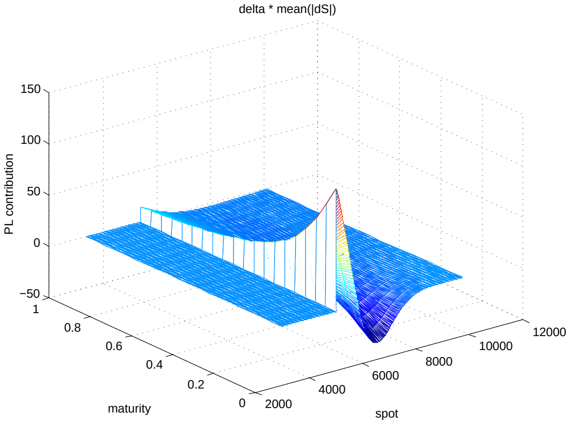

Figure 16: Profit-and-loss contribution of the delta for down-and-out put option with barrier 5400 and strike 7425. The mean value of absolute underlying changes $|\bar{dS}| = 66.01$.

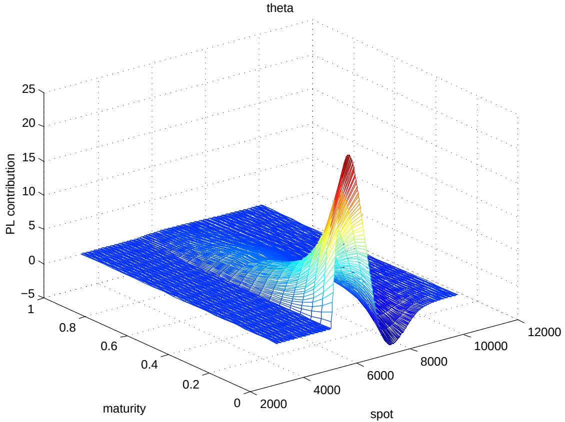

Figure 17: Profit-and-loss contribution of the theta for down-and-out put option with barrier 5400 and strike 7425.

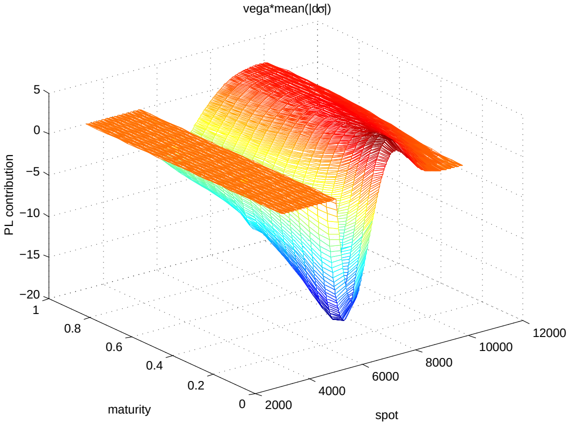

Figure 18: Profit-and-loss contribution of the vega for down-and-out put option with barrier 5400 and strike 7425. σ is taken as at-the-money IV for maturity 0.25. The mean value of absolute changes $|\bar{d\sigma}| = 0.67$ in vol. points.

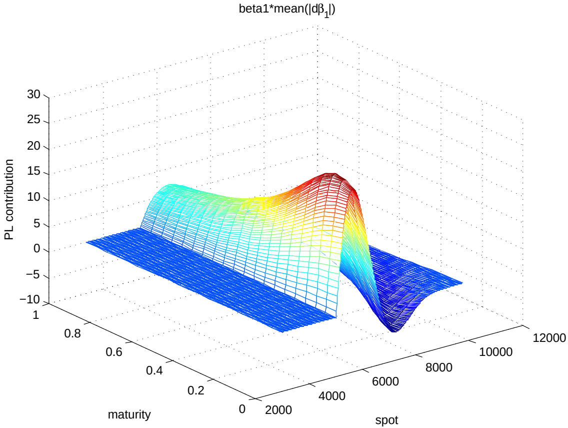

Figure 19: Profit-and-loss contribution of the β_1 for down-and-out put option with barrier 5400 and strike 7425.

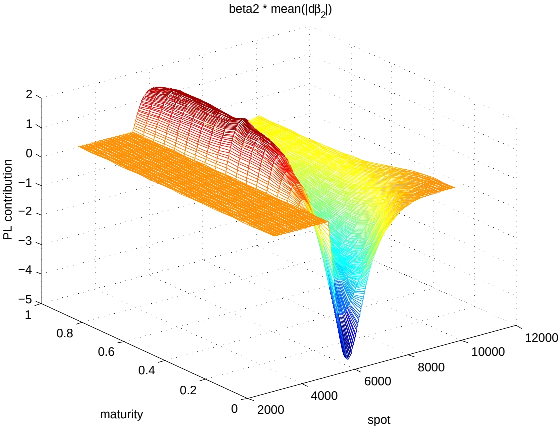

Figure 20: Profit-and-loss contribution of the β_2 for down-and-out put option with barrier 5400 and strike 7425.

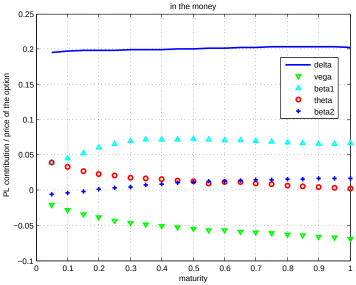

Figure 21: Profit-and-loss contribution divided by down-and-out put price for ITM option.

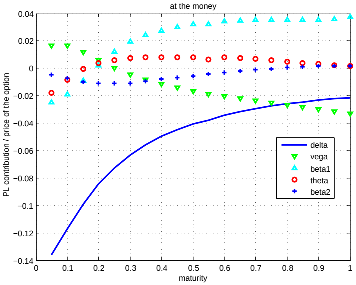

Figure 22: Profit-and-loss contribution divided by down-and-out put price for ATM option.

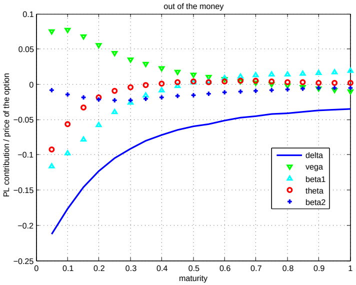

Figure 23: Profit-and-loss contribution divided by down-and-out put price for OTM option.

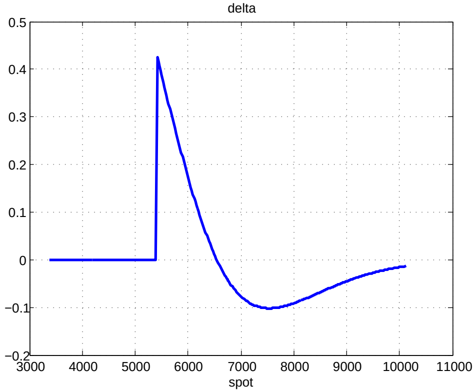

Figure 24: Delta as a function of the spot for the half year down-and-out put option with strike price 7425 and barrier 5400