Time Varying Lasso

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Motivation

Figure 1: Dynamic lasso

High dimensions

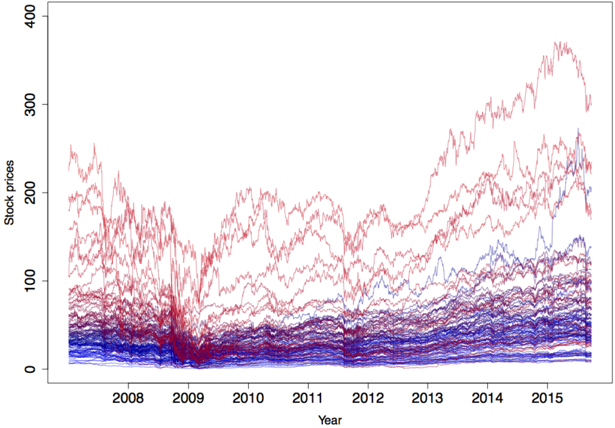


Figure 2: Daily stock prices (USD)

Systemic risk

"Risk of financial instability so widespread that it impairs the functioning of a financial system to the point where economic growth and welfare suffer materially."

ECB, Financial Network and Financial Stability, 2010.

- Adrian and Brunnermeier (2011): CoVaR as a measure of systemic risk
- Härdle et al. (2015): TENET - Tail-Event driven NETwork risk

Financial Risk Meter

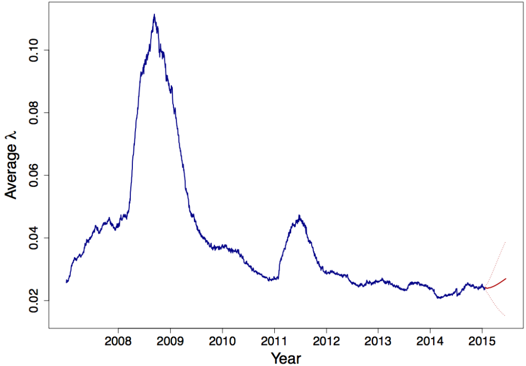


Figure 3: Time series of λ ($\tau = 0.05$) from financialriskmeter

Challenges

- Time series of Lasso penalty parameter λ
- Prediction of λ
  - Volatility
  - Multicollinearity
  - Significant explanatory variables
- Monitoring of systemic risk

Outline

1. Motivation ✓
2. Lasso in linear regression
   - Dynamics
3. Lasso in quantile regression
   - Dynamics
   - Simulation study
   - Real-data application

Lasso in linear regression

- Linear regression (LR) model

$$Y = X\beta + \varepsilon$$

with $Y = (Y_1, \ldots, Y_n)^\top$, $\beta = (\beta_1, \ldots, \beta_p)^\top$, $X_{(n\times p)}$, $\varepsilon_{(n\times 1)} \overset{iid}{\sim} (0, \sigma^2)$ ▸ Assumptions

- Tibshirani (1996): Estimation using a **penalty** function

$$\widehat{\beta} = \arg\min_{\beta} \left\{ \sum_{i=1}^{n} \left(Y_i - X_i^\top \beta \right)^2 + \lambda \sum_{j=1}^{p} |\beta_j| \right\}$$

with $X_i = (x_{i1}, \ldots, x_{ip})^\top$, tuning parameter $\lambda \geq 0$ ▸ Choosing λ

Penalty parameter λ

- Osborne et al. (2000): Duality interpretation ▸ Details

$$\lambda = \frac{(Y - X\widehat{\beta}(\lambda))^\top X\widehat{\beta}(\lambda)}{\| \widehat{\beta}(\lambda) \|_1} \tag{1}$$

- Penalty λ dependent on

  1. Residuals' size
  2. Condition number κ of $X^\top X$
  3. Cardinality of active set $q \stackrel{\text{def}}{=} \| \beta \|_0$

▸ Simulation study

λ dependent on volatility σ

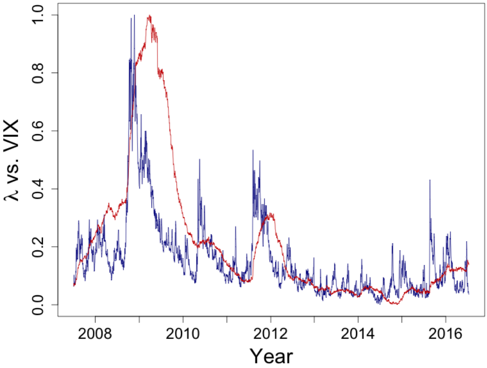


Figure 4: Implied volatility index (VIX) and λ for 20070103 - 20160718

XFGTVP_LambdaVIX

Lasso in quantile regression

- Quantile regression (QR) model

$$Y = \beta_0 + X\beta + \varepsilon$$

with $Y = (Y_1, \dots, Y_n)^\top$, $\beta = (\beta_1, \dots, \beta_p)^\top$, $X_{(n \times p)}$ and $\varepsilon_{(n \times 1)}$ *iid* such that $\mathrm{P}(\varepsilon_i \leq 0 | X_i = x) = \tau$ for almost every $x \in \mathbb{R}^p$, $\tau \in (0,1)$ conditional quantile of Y.

▸ Assumptions

Lasso in quantile regression

- Li and Zhu (2008): Estimation using a L_1-norm penalty function

$$\widehat{\beta} = \arg\min_{\beta_0,\beta} \left\{ \sum_{i=1}^{n} \rho_\tau \left(Y_i - \beta_0 - X_i^\top \beta \right) + \lambda \sum_{j=1}^{p} |\beta_j| \right\}$$

with $X_i = (x_{i1}, \ldots, x_{ip})^\top$, tuning parameter $\lambda \geq 0$ ▸ Choosing λ

- Koenker and Bassett (1978): Check function

$$\rho_\tau(x) = \begin{cases} \tau \cdot x & \text{if } x > 0; \\ -(1-\tau) \cdot x & \text{otherwise.} \end{cases}$$

Penalty parameter λ

- Li and Zhu (2008): Lagrangian primal function ▸ Details

$$\lambda = \frac{(\alpha - \gamma)^\top X \widehat{\beta}(\lambda)}{||\widehat{\beta}(\lambda)||_1}, \tag{2}$$

with $\alpha = (\alpha_1, \ldots, \alpha_n)^\top$, $\gamma = (\gamma_1, \ldots, \gamma_n)^\top$

- For $f(\beta_0, \beta) = \beta_0 + X_i^\top \beta$ it holds

$$(\alpha_i - \gamma_i) = \begin{cases} \tau & \text{if } Y_i - f(\widehat{\beta}_0(\lambda), \widehat{\beta}(\lambda)) > 0; \\ -(1-\tau) & \text{if } Y_i - f(\widehat{\beta}_0(\lambda), \widehat{\beta}(\lambda)) < 0; \\ \in (-(1-\tau), \tau) & \text{if } Y_i - f(\widehat{\beta}_0(\lambda), \widehat{\beta}(\lambda)) = 0. \end{cases}$$

Penalty parameter λ

- Belloni and Chernozhukov (2011): Random variable

$$\Lambda = \sup_{\tau \in (0,1)} \max_{1 \leq j \leq p} \left| \sum_{i=1}^{n} \frac{x_{ij}(\tau - \mathbf{I}\{u_i \leq \tau\})}{\widehat{\sigma}_j \sqrt{\tau(1-\tau)}} \right|$$

with $u_i \overset{iid}{\sim} U(0,1)$, $i = 1, \ldots, n$, and $\widehat{\sigma}_j^2 = n^{-1} \sum_{i=1}^{n} x_{ij}^2$

- Penalty parameter

$$\lambda = c \cdot \Lambda(1-\alpha|X), \tag{3}$$

with $c > 1$ and $\Lambda(1-\alpha|X)$ a $(1-\alpha)$-quantile of Λ conditional on X

Factors influencing λ

- From (2) λ directly depends on
  1. Cardinality of active set q
- Indirect impact of (cf. (1))
  2. Correlation structure of design X
  3. Variance of the error term (financialriskmeter evidence)

How does λ change

- 50 scenarios
- Design matrix $X_{(n\times p)}$

$$\{X_i\}_{i=1}^n \sim N_p(0, \Sigma),$$

  $n = 600,\ p = 100$

- Covariance matrix $\Sigma_{(p\times p)}$

$$\sigma_{ij} = 0.5^{|i-j|}$$

  $i, j = 1, \ldots, p$

- Quantile level $\tau = 0.05$

How does λ change

1. Volatility change ($\Delta\sigma$):

$$\beta_{(100\times 1)} = (1,1,1,1,1,0,\ldots,0)^\top$$

$$\varepsilon_t \sim \begin{cases} ALD(0,1,0.05), & \text{if } t \leq 300 \\ ALD(0,2,0.05), & t > 300 \end{cases}$$

2. β change ($\Delta\beta$):

$$\beta_t = \begin{cases} (1,1,1,1,1,1,1,1,1,1,0,\ldots,0)^\top, & \text{if } t \leq 300 \\ (1,0.9,0.8,\ldots,0.2,0.1,0,\ldots,0)^\top, & t > 300 \end{cases}$$

$$\varepsilon_t \sim ALD(0,1,0.05)$$

How does λ change

3. Active set q change (Δq):

$$\beta_t = \begin{cases} (1,1,1,1,1,1,1,1,1,1,0,\dots,0)^\top, & \text{if } t \le 300 \\ (1,1,1,1,1,0,\dots,0)^\top, & t > 300 \end{cases}$$

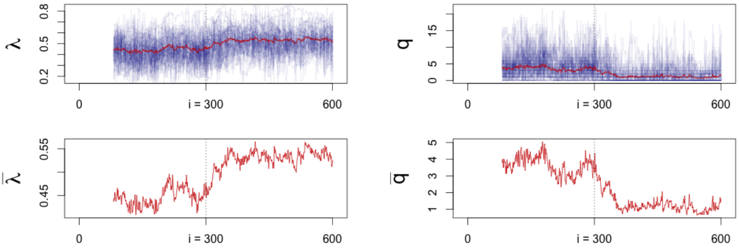
$$\varepsilon_t \sim ALD(0,1,0.05)$$

4. Condition number $\kappa(X^\top X)$ change ($\Delta \kappa$):

$$\sigma_{ijt} = \begin{cases} 0, & \text{if } t \le 300 \\ 0.5^{|i-j|}, & t > 300 \end{cases}$$

$$\varepsilon_t \sim ALD(0,1,0.05),\ \beta_{(100\times 1)} = (1,1,1,1,1,0,\dots,0)^\top$$

Simulation results ($\Delta\sigma$)



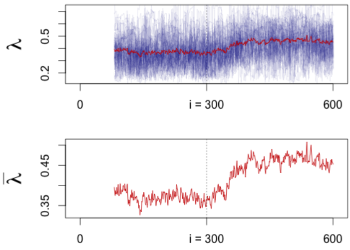
(a) Penalty parameter λ (b) Cardinality of active set q

Figure 5: Time series of λ and q with change of Var(ε_t) after $t = 300$, moving windows of length 80.

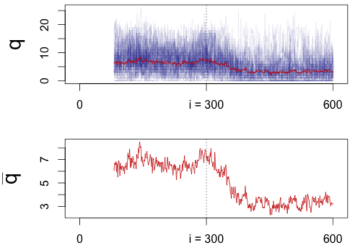
XFGTVP_LambdaSim

▸ More details

Simulation results ($\Delta\beta$)



(a) Penalty parameter λ



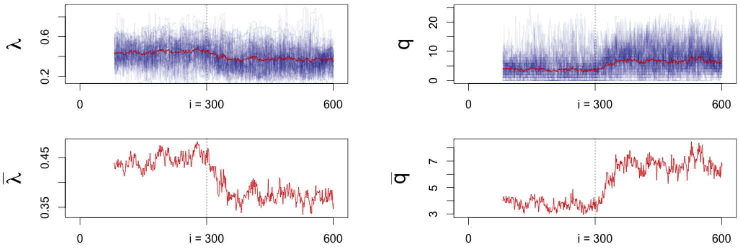
(b) Cardinality of active set q

Figure 6: Time series of λ and q with change of β_t after $t = 300$, moving windows of length 80.

XFGTVP_BetaChange

▸ More details

Simulation results (Δq)



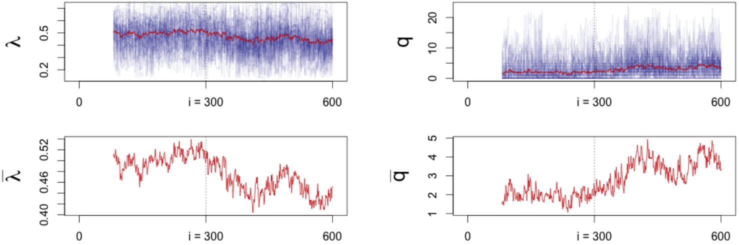
(a) Penalty parameter λ (b) Cardinality of active set q

Figure 7: Time series of λ and q with change of q after $t = 300$, moving windows of length 80.

XFGTVP_LambdaSim

▸ More details

Simulation results ($\Delta\kappa$)



(a) Penalty parameter λ

(b) L_1-norm of $\widehat{\beta}$

Figure 8: Time series of λ and L_1-norm of $\widehat{\beta}$ with change of $\kappa(X^\top X)$ after $t = 300$, moving windows of length 80.

XFGTVP_LambdaSim

More details

Simulations summary

	$\sigma^2 \nearrow$			$\sigma^2 \rightarrow$			$\sigma^2 \searrow$		
	$\kappa \nearrow$	$\kappa \rightarrow$	$\kappa \searrow$	$\kappa \nearrow$	$\kappa \rightarrow$	$\kappa \searrow$	$\kappa \nearrow$	$\kappa \rightarrow$	$\kappa \searrow$
$q \nearrow$	0.88	1.10	1.31	0.78	0.84	1.00	0.66	0.71	0.84
$q \rightarrow$	0.99	1.20	1.43	0.86	1.00	1.19	0.72	0.84	0.99
$q \searrow$	1.16	1.40	1.56	1.00	1.17	1.30	0.76	0.89	1.13

Table 1: Reactions of λ to changes in model.

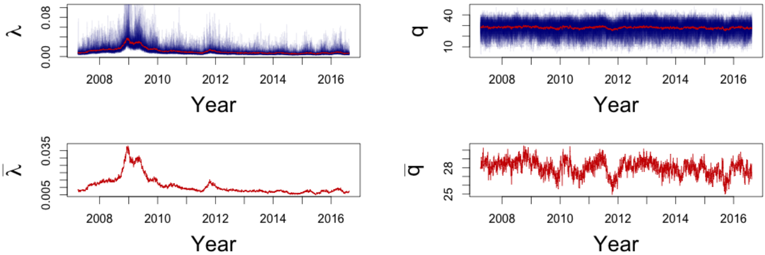
Data description

- NASDAQ: 100 US financial companies ▸ Details
- Log returns of daily adjusted stock close prices
- 6 macroprudential variables ▸ Details
- Time interval 20070103 - 20160817
- Source: Datastream, Yahoo Finance

Financial Risk Meter construction

- Each company regressed on the others and on the macroprudential variables
- Li and Zhu (2008) algorithm
- Moving windows of size $w = 63$
- λ chosen by BIC criterion

FRM visualization

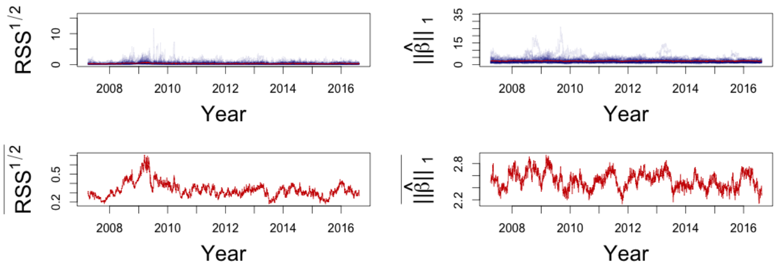


(a) Penalty parameter λ (b) Cardinality of active set q

Figure 9: Time series of λ and q, $w = 63$, $\tau = 0.05$.

XFGTVP_FRM

FRM visualization

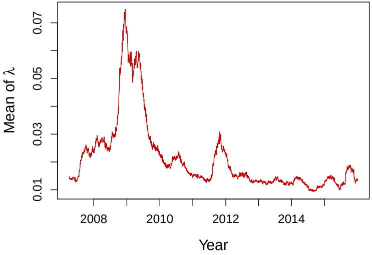


(a) L_2-norm of residuals (b) L_1-norm of $\widehat{\beta}$

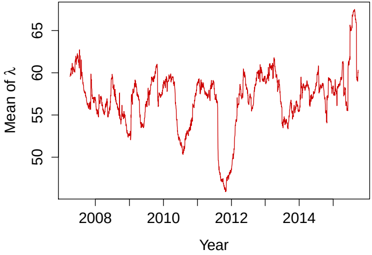
Figure 10: Time series of L_2-norm of residuals and L_1-norm of $\widehat{\beta}$, $w = 63$, $\tau = 0.05$.

XFGTVP_FRM

Methods comparison



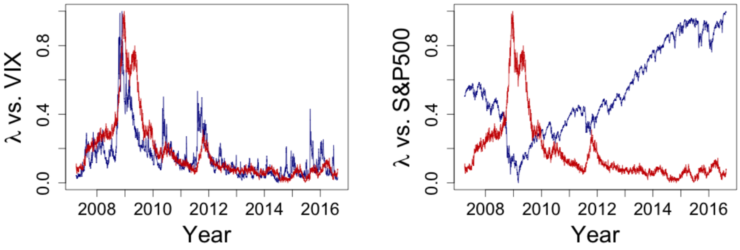
(a) Li and Zhu (2008)



(b) Belloni and Chernozhukov (2011)

Figure 11: Time series of average λ for 20070103 - 20150925 using different approaches, $w = 63$, $\tau = 0.05$.

Parameter λ and systemic risk measures



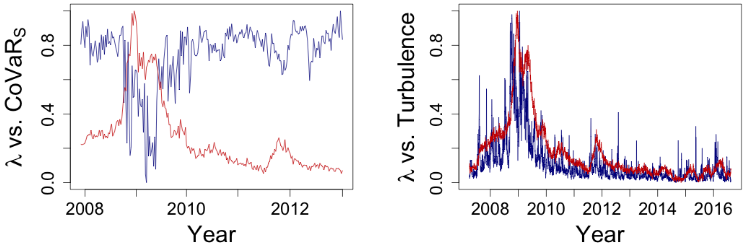
(a) λ and VIX

(b) λ and S&P500

Figure 12: Time series of average λ vs. VIX and S&P500 for 20070103 - 20160817.

XFGTVP_LambdaSysRisk

Parameter λ and systemic risk measures

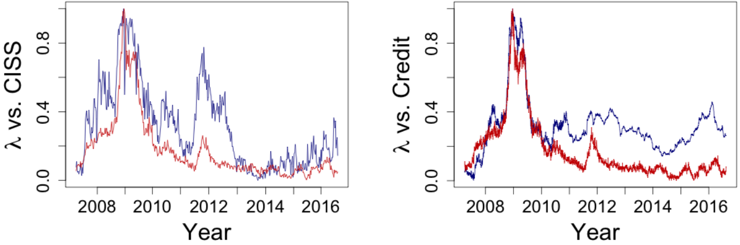


(a) λ and CoVaR (SIM) (b) λ and Turbulence

Figure 13: Time series of average λ vs. CoVaR (SIM) and Turbulence for 20070103 - 20160817.

XFGTVP_LambdaSysRisk

Parameter λ and systemic risk measures



(a) λ and CISS (b) λ and Credit spread

Figure 14: Time series of average λ vs. CISS and Credit Spread for 20070103 - 20160817.

XFGTVP_LambdaSysRisk

λ as a systemic risk measure

	H_0	Test statistic	10 %	5 %
VIX	$r \leq 1$	4.80	7.52	9.24
	$r = 0$	87.43	13.75	15.67
S&P500	$r \leq 1$	7.59	10.49	12.25
	$r = 0$	9.20	16.85	18.96
CoVaR (SIM)	$r \leq 1$	4.52	10.49	12.25
	$r = 0$	50.58	16.85	18.96
CoVaR (Linear Lasso)	$r \leq 1$	4.59	10.49	12.25
	$r = 0$	57.15	16.85	18.96
Turbulence	$r \leq 1$	8.94	10.49	12.25
	$r = 0$	212.24	16.85	18.96

Table 2: Cointegration of λ with systemic risk measures, Johansen (1991) procedure, r is number of cointegration relations.

λ as a systemic risk measure

		Test statistic	10 %	5 %
CISS	$r \leq 1$	6.90	10.49	12.25
	$r = 0$	31.12	16.85	18.96
Volatility Connectedness	$r \leq 1$	9.48	10.49	12.25
	$r = 0$	10.51	16.85	18.96
Yield Slope	$r \leq 1$	7.20	10.49	12.25
	$r = 0$	13.63	16.85	18.96
Credit Spread	$r \leq 1$	5.45	10.49	12.25
	$r = 0$	42.29	16.85	18.96

Table 3: Cointegration of λ with systemic risk measures, Johansen (1991) procedure, r is number of cointegration relations.

Summary

- Time series of penalty parameter λ
- Dependency of λ on
    - Residuals
    - Active set
    - Design matrix
- Evidence of variation over time

Time Varying Lasso

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Journal of the American Statistical Association **109**(505): 216–229, 2014

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Journal of Computational and Graphical Statistics **17**(1): 1–23, 2008

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Journal of Computational and Graphical Statistics **9**(2): 319–337, 2000

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Regression Shrinkage and Selection via the Lasso
Journal of the Royal Statistical Society, Series B **58**(1): 267–288, 1996

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On the "Degrees of Freedom" of the Lasso
The Annals of Statistics **35**(5): 2173–2192, 2007

Assumptions on Lasso in linear regression

- Normalized columns of $X = (x_{ij})_{i=1,\ldots,n,\ j=1,\ldots,p}$

$$n^{-1}\sum_{i=1}^{n} x_{ij} = 0, \quad n^{-1}\sum_{i=1}^{n} x_{ij}^2 = 1$$

Back to "Lasso in linear regression"

Back to "Lasso in quantile regression"

Selection of λ in Lasso

- Efron et al. (2004)
  - Least Angle Regression (LAR) algorithm
- Zou et al. (2007)
  - Bayesian Information Criterion (BIC)

$$\mathrm{BIC}(\lambda) = \frac{\| Y - X\widehat{\beta}(\lambda) \|^2}{n\widehat{\sigma}^2} + \frac{\log(n)}{n} \widehat{\mathrm{df}}(\lambda)$$

    with error variance estimator $\widehat{\sigma}^2 = n^{-1} \| Y - X\widehat{\beta}^{OLS} \|^2$ and $\widehat{\lambda}^{BIC} = \arg\min_{\lambda} \mathrm{BIC}(\lambda)$

  - Estimator of number of effective parameters

$$\widehat{\mathrm{df}}(\lambda) = ||\widehat{\beta}(\lambda)||_0$$

Back to "Lasso in linear regression"

Lasso duality interpretation I

- Alternative representation

$$\widehat{\beta} = \arg\min_{\beta} \frac{1}{2} \left\{ \sum_{i=1}^{n} \left(Y_i - X_i^\top \beta \right)^2 \right\}, \text{ s. t. } \sum_{j=1}^{p} |\beta_j| \leq s \quad (4)$$

with tuning parameter $s \geq 0$

- Notation

$$f(\beta) = \tfrac{1}{2} \sum_{i=1}^{n} \left(Y_i - X_i^\top \beta \right)^2$$
$$g(\beta) = \sum_{j=1}^{p} |\beta_j| - s$$

Back to "Penalty parameter λ"

Lasso duality interpretation II

- For (4) as a convex programming problem the Lagrangian is

$$L(\beta, \lambda) = f(\beta) + \lambda g(\beta)$$

- Primal-dual relationship

$$\underset{\beta}{\text{minimize}} \sup_{\lambda \geq 0} L(\beta, \lambda) \geq \underset{\lambda \geq 0}{\text{maximize}} \inf_{\beta} L(\beta, \lambda)$$

- The dual function is $\inf_{\beta} L(\beta, \lambda)$ with

$$\lambda = \frac{(Y - X\widehat{\beta})^\top X\widehat{\beta}}{\| \widehat{\beta} \|_1}$$

▸ Back to "Penalty parameter λ"

How does λ change

- 100 scenarios
- Design matrix $X_{(n \times p)}$

$$\{X_i\}_{i=1}^{n} \sim N_p(0, \Sigma),$$

$n = 1000,\ p = 100$

- Covariance matrix $\Sigma_{(p \times p)}$

$$\sigma_{ij} = 0.5^{|i-j|}$$

$i, j = 1, \ldots, p$

▸ Back to "Penalty parameter λ"

How does λ change

1. Volatility change ($\Delta\sigma$):

$$\beta_{(100\times 1)} = (1,1,1,1,1,0,\dots,0)^\top$$

$$\varepsilon_t \sim \begin{cases} N(0,1), & \text{if } t \leq 500 \\ N(0,1.21), & t > 500 \end{cases}$$

2. β change ($\Delta\beta$):

$$\beta_t = \begin{cases} (1,1,1,1,1,1,1,1,1,1,0,\dots,0)^\top, & \text{if } t \leq 500 \\ (1,0.9,0.8,\dots,0.2,0.1,0,\dots,0)^\top, & t > 500 \end{cases}$$

$$\varepsilon_t \sim N(0,1)$$

Back to "Penalty parameter λ"

How does λ change

3. Active set q change (Δq):

$$\beta_t = \begin{cases} (1,1,1,1,1,1,1,1,1,1,0,\dots,0)^\top, & \text{if } t \le 500 \\ (1,1,1,1,1,0,\dots,0)^\top, & t > 500 \end{cases}$$

$$\varepsilon_t \sim N(0,1)$$

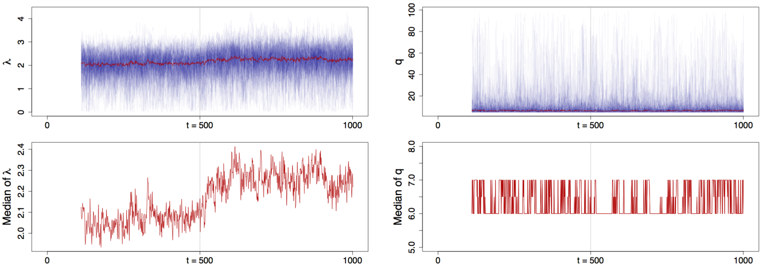
4. Condition number $\kappa(X^\top X)$ change ($\Delta\kappa$):

$$\kappa_t \approx \begin{cases} 12000, (\sigma_{ij} = 0.5^{|i-j|}), & \text{if } t \le 500 \\ 100000, (\sigma_{ij} = 0.9^{|i-j|}), & t > 500 \end{cases}$$

$$\varepsilon_t \sim N(0,1),\ \beta_{(100\times 1)} = (1,1,1,1,1,0,\dots,0)^\top$$

Back to "Penalty parameter λ"

Simulation results for Lasso in LR ($\Delta\sigma$)



(a) Penalty parameter λ (b) Cardinality of active set q

Figure 15: Time series of λ and q with change of $\text{Var}(\varepsilon_t)$ after $t = 500$, moving windows of length 110.

Back to "Penalty parameter λ"

Simulation results for Lasso in LR ($\Delta\sigma$)

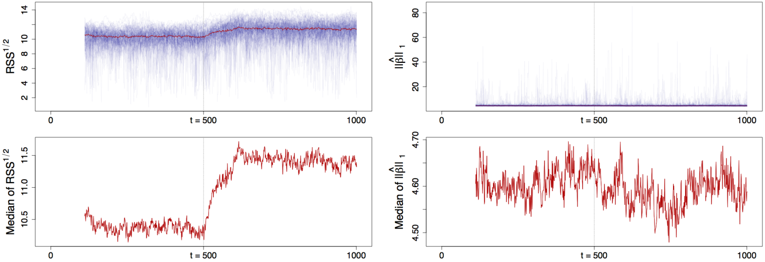
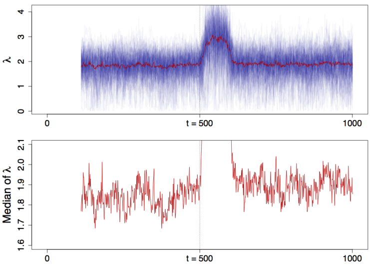


Figure 16: Time series of L_2-norm of residuals and L_1-norm of $\widehat{\beta}$ with change of $\mathrm{Var}(\varepsilon_t)$ after $t = 500$, moving windows of length 110.

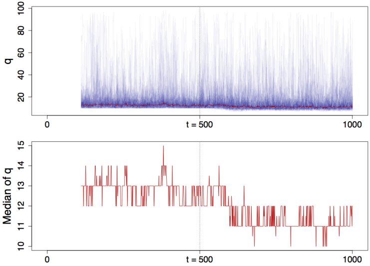
▸ Back to "Penalty parameter λ"

TVPvariance

Simulation results for Lasso in LR ($\Delta\beta$)



(a) Penalty parameter λ



(b) Cardinality of active set q

Figure 17: Time series of λ and q with change of β_t after $t = 500$, moving windows of length 110.

Back to "Penalty parameter λ"

TVPbetanorm

Simulation results for Lasso in LR ($\Delta\beta$)

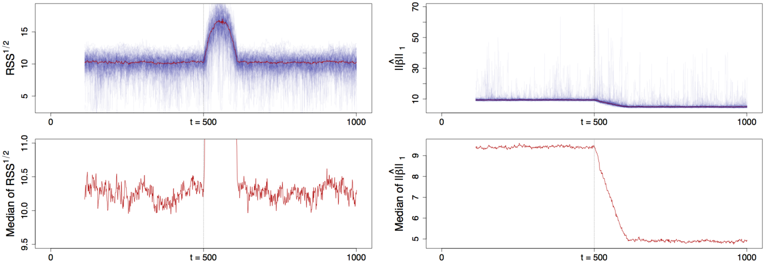
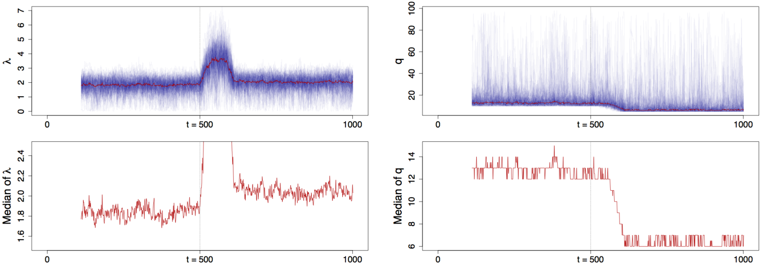


Figure 18: Time series of L_2-norm of residuals and L_1-norm of $\widehat{\beta}$ with change of β_t after $t = 500$, moving windows of length 110.

▸ Back to "Penalty parameter λ"

TVPbetanorm

Simulation results for Lasso in LR (Δq)



(a) Penalty parameter λ (b) Cardinality of active set q

Figure 19: Time series of λ and q with change of q after $t = 500$, moving windows of length 110.

Back to "Penalty parameter λ"

Simulation results for Lasso in LR (Δq)

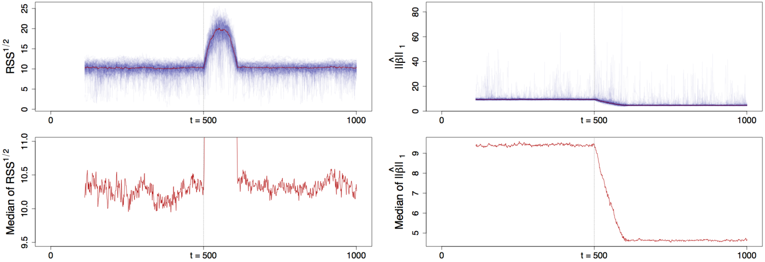
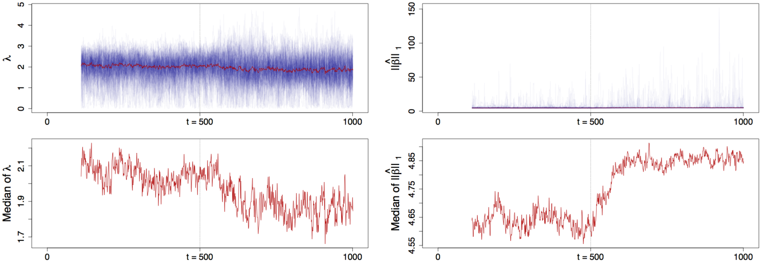


Figure 20: Time series of L_2-norm of residuals, L_1-norm of $\widehat{\beta}$ with change of q after $t = 500$, moving windows of length 110.

Back to "Penalty parameter λ"

TVPactiveset

Simulation results for Lasso in LR ($\Delta\kappa$)



(a) Penalty parameter λ (b) L_1-norm of $\widehat{\beta}$

Figure 21: Time series of λ and L_1-norm of $\widehat{\beta}$ with change of $\kappa(X^\top X)$ after $t = 500$, moving windows of length 110.

Back to "Penalty parameter λ"

Simulation results for Lasso in LR ($\Delta\kappa$)

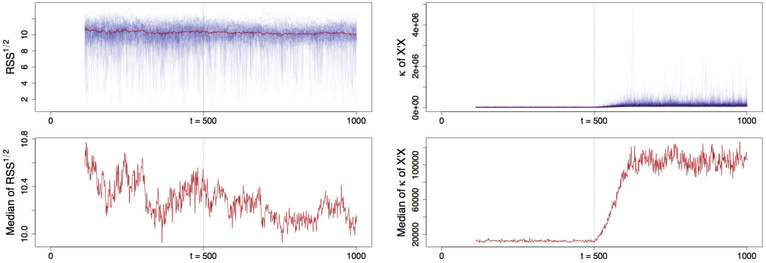


Figure 22: Time series of L_2-norm of residuals, $\kappa(X^\top X)$ with change of $\kappa(X^\top X)$ after $t = 500$, moving windows of length 110.

Back to "Penalty parameter λ"

TVPdesign

Simulations summary for Lasso in LR

	$\sigma^2 \nearrow$			$\sigma^2 \rightarrow$			$\sigma^2 \searrow$		
	$\kappa \nearrow$	$\kappa \rightarrow$	$\kappa \searrow$	$\kappa \nearrow$	$\kappa \rightarrow$	$\kappa \searrow$	$\kappa \nearrow$	$\kappa \rightarrow$	$\kappa \searrow$
$q \nearrow$	0.93	0.97	1.05	0.87	0.90	0.98	0.77	0.80	0.87
$q \rightarrow$	0.99	1.09	1.18	0.94	1.01	1.10	0.82	0.90	0.97
$q \searrow$	1.12	1.24	1.26	1.05	1.13	1.17	0.92	1.03	1.04

Table 4: Reactions of λ to changes in model.

Back to "Penalty parameter λ"

Selection of λ in quantile regression Lasso

- Li and Zhu (2008)
  - Quantile Lasso regression algorithm
- Lee et al. (2014): BIC for QR models

$$\mathrm{BIC}(\lambda) = \log\left[n^{-1}\sum_{i=1}^{n} \rho_\tau\{Y_i - \widehat{\beta}_0(\lambda) - X_i^\top \widehat{\beta}(\lambda)\}\right] + \frac{\log(n)}{2n}\,\widehat{\mathrm{df}}(\lambda),$$

$$\widehat{\lambda}^{BIC} = \arg\min_{\lambda} \mathrm{BIC}(\lambda)$$

  - Estimator of number of effective parameters

$$\widehat{\mathrm{df}}(\lambda) = ||\widehat{\beta}(\lambda)||_0$$

Back to "Lasso in quantile regression"

Quantile Lasso problem representation I

- Optimization problem from Li and Zhu (2008)

$$
\begin{aligned}
\min_{\beta_0,\beta} \quad & \tau \sum_{i=1}^{n} \xi_i + (1-\tau) \sum_{i=1}^{n} \zeta_i, \\
\text{subject to} \quad & \sum_{j=1}^{p} |\beta_j| \le s, \\
& -\zeta_i \le Y_i - f(\beta_0, \beta) \le \xi_i \\
& \zeta_i, \xi_i \ge 0, \quad i = 1, \dots, n
\end{aligned} \tag{5}
$$

with tuning parameter $s \geq 0$

Back to "Penalty parameter λ"

Quantile Lasso problem representation II

- Lagrangian primal function associated with problem (5)

$$
\begin{aligned}
L(\beta,\lambda) = \quad & \tau \sum_{i=1}^{n} \xi_i + (1-\tau) \sum_{i=1}^{n} \zeta_i + \lambda \left(\sum_{j=1}^{p} |\beta|_j - s \right) \\
& + \sum_{i=1}^{n} \alpha_i (Y_i - f(\beta_0,\beta) - \xi_i) - \sum_{i=1}^{n} \gamma_i (Y_i - f(\beta_0,\beta) + \zeta_i) \\
& - \sum_{i=1}^{n} \kappa_i \xi_i - \sum_{i=1}^{n} \eta_i \zeta_i,
\end{aligned} \tag{6}
$$

where $\lambda, \alpha_i, \gamma_i, \kappa_i, \eta_i \geq 0$ are Lagrangian multipliers

Back to "Penalty parameter λ"

Quantile Lasso problem representation III

- Derivatives of (6) set to 0 imply

$$
\begin{aligned}
\frac{\partial L(\beta,\lambda)}{\partial \beta} &= \lambda \cdot \operatorname{sign}(\beta_j) - \sum_{i=1}^{n} (\alpha_i - \gamma_i) x_{ij} \stackrel{!}{=} 0, \text{ with } \beta_j \neq 0,\\
\frac{\partial L(\beta,\lambda)}{\partial \beta_0} &= \sum_{i=1}^{n} (\alpha_i - \gamma_i) \stackrel{!}{=} 0,\\
\frac{\partial L(\beta,\lambda)}{\partial \xi_i} &= \tau \stackrel{!}{=} \alpha_i + \kappa_i,\\
\frac{\partial L(\beta,\lambda)}{\partial \zeta_i} &= 1 - \tau \stackrel{!}{=} \gamma_i + \eta_i.
\end{aligned}
$$

Back to "Penalty parameter λ"

Quantile Lasso problem representation IV

- Karush-Kuhn-Tucker (KKT) conditions of (5) are

$$
\begin{aligned}
\alpha_i(Y_i - f(\beta_0, \beta) - \xi_i) &= 0,\\
\gamma_i(Y_i - f(\beta_0, \beta) - \zeta_i) &= 0,\\
\kappa_i \xi_i &= 0,\\
\eta_i \zeta_i &= 0.
\end{aligned}
$$

- Solving for λ arrive at (2)

Back to "Penalty parameter λ"

Simulation results for Lasso in QR ($\Delta\sigma$)

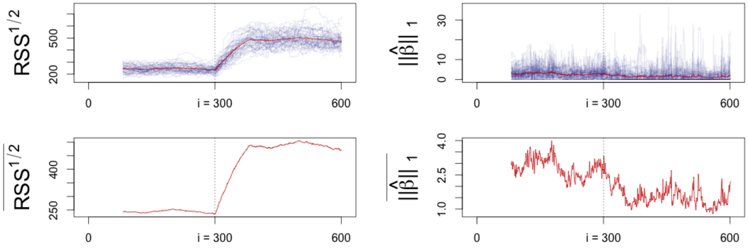


Figure 23: Time series of L_2-norm of residuals and L_1-norm of $\widehat{\beta}$ with change of $\mathrm{Var}(\varepsilon_t)$ after $t = 300$, moving windows of length 80.

XFGTVP_LambdaSim

Back to "Simulation results ($\Delta\sigma$)"

Simulation results for Lasso in QR ($\Delta\beta$)

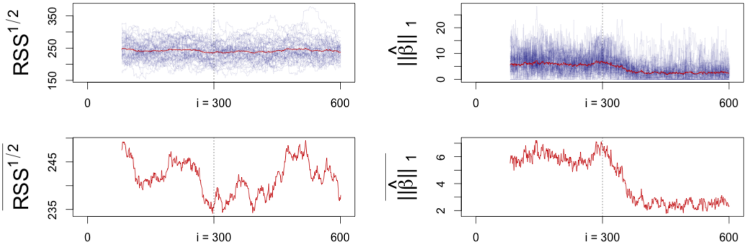


Figure 24: Time series of L_2-norm of residuals and L_1-norm of $\widehat{\beta}$ with change of β_t after $t = 300$, moving windows of length 80.

XFGTVP_BetaChange

Back to "Simulation results ($\Delta\beta$)"

Simulation results for Lasso in QR (Δq)

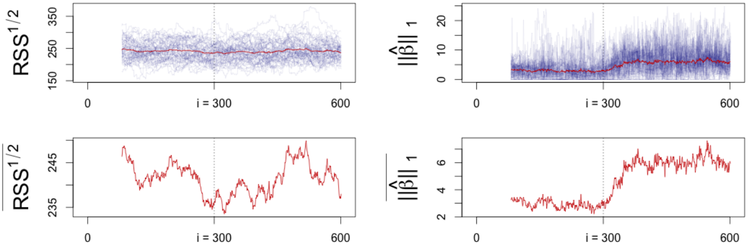


Figure 25: Time series of L_2-norm of residuals, L_1-norm of $\widehat{\beta}$ with change of q after $t = 300$, moving windows of length 80.

XFGTVP_LambdaSim

Back to "Simulation results (Δq)"

Simulation results for Lasso in QR ($\Delta\kappa$)

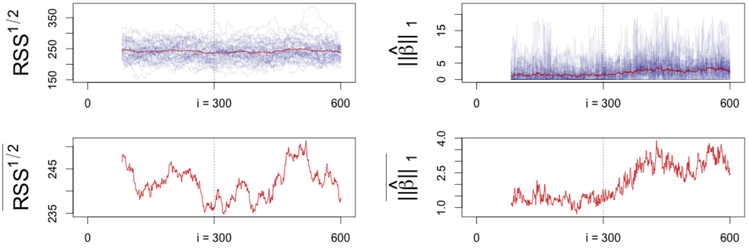


Figure 26: Time series of L_2-norm of residuals, L_1-norm of $\widehat{\beta}$ with change of $\kappa(X^\top X)$ after $t = 300$, moving windows of length 80.

XFGTVP_LambdaSim

Back to "Simulation results ($\Delta\kappa$)"

Financial companies

Back to "Data description"

WFC	Wells Fargo & Company	ALL	Allstate Corporation (The)
JPM	JP Morgan Chase & Co.	BEN	Franklin Resources, Inc.
BAC	Bank of America Corporation	STI	SunTrust Banks, Inc.
C	Citigroup Inc.	MCO	Moody's Corporation
AIG	American International Group, Inc.	PGR	Progressive Corporation (The)
GS	Goldman Sachs Group, Inc. (The)	AMP	AMERIPRISE FINANCIAL SERVICES, INC.
USB	U.S. Bancorp	AMTD	TD Ameritrade Holding Corporation
AXP	American Express Company	HIG	Hartford Financial Services Group, Inc. (The)
MS	Morgan Stanley	TROW	T. Rowe Price Group, Inc.
BLK	BlackRock, Inc.	NTRS	Northern Trust Corporation
MET	MetLife, Inc.	MTB	M&T Bank Corporation
PNC	PNC Financial Services Group, Inc. (The)	FITB	Fifth Third Bancorp
BK	Bank Of New York Mellon Corporation (The)	IVZ	Invesco Plc
SCHW	The Charles Schwab Corporation	L	Loews Corporation
COF	Capital One Financial Corporation	EFX	Equifax, Inc.
PRU	Prudential Financial, Inc.	PFG	Principal Financial Group Inc
TRV	The Travelers Companies, Inc.	RF	Regions Financial Corporation
CME	CME Group Inc.	MKL	Markel Corporation
CB	Chubb Corporation (The)	LNC	Lincoln National Corporation
MMC	Marsh & McLennan Companies, Inc.	CBG	CBRE Group, Inc.
BBT	BB&T Corporation	KEY	KeyCorp
ICE	Intercontinental Exchange Inc.	NDAQ	The NASDAQ OMX Group, Inc.
STT	State Street Corporation	CINF	Cincinnati Financial Corporation
AFL	Aflac Incorporated	CNA	CNA Financial Corporation
AON	Aon plc	HBAN	Huntington Bancshares Incorporated

Financial companies

Back to "Data description"

SEIC	SEI Investments Company	ERIE	Erie Indemnity Company
ETFC	E*TRADE Financial Corporation	OZRK	Bank of the Ozarks
AMG	Affiliated Managers Group, Inc.	WTM	White Mountains Insurance Group, Ltd.
RJF	Raymond James Financial, Inc.	SNV	Synovus Financial Corp.
UNM	Unum Group	ISBC	Investors Bancorp, Inc.
NYCB	New York Community Bancorp, Inc.	MKTX	MarketAxess Holdings, Inc.
Y	Alleghany Corporation	LM	Legg Mason, Inc.
SBNY	Signature Bank	CBSH	Commerce Bancshares, Inc.
CMA	Comerica Incorporated	BOKF	BOK Financial Corporation
AJG	Arthur J. Gallagher & Co.	EEFT	Euronet Worldwide, Inc.
JLL	Jones Lang LaSalle Incorporated	DNB	Dun & Bradstreet Corporation (The)
TMK	Torchmark Corporation	WAL	Western Alliance Bancorporation
WRB	W.R. Berkley Corporation	EV	Eaton Vance Corporation
AFG	American Financial Group, Inc.	CFR	Cullen/Frost Bankers, Inc.
SIVB	SVB Financial Group	MORN	Morningstar, Inc.
EWBC	East West Bancorp, Inc.	THG	The Hanover Insurance Group, Inc.
ROL	Rollins, Inc.	UMPQ	Umpqua Holdings Corporation
ZION	Zions Bancorporation	CNO	CNO Financial Group, Inc.
AIZ	Assurant, Inc.	FHN	First Horizon National Corporation
PACW	PacWest Bancorp	WBS	Webster Financial Corporation
AFSI	AmTrust Financial Services, Inc.	PB	Prosperity Bancshares, Inc.
ORI	Old Republic International Corporation	PVTB	PrivateBancorp, Inc.
PBCT	People's United Financial, Inc.	SEB	Seaboard Corporation
CACC	Credit Acceptance Corporation	FCNCA	First Citizens BancShares, Inc.
BRO	Brown & Brown, Inc.	MTG	MGIC Investment Corporation

Macroprudential variables

1.	VIX
2.	Daily change in the 3-month Treasury maturities
3.	Change in the slope of the yield curve
4.	Change in the credit spread
5.	Daily Dow Jones U.S. Real Estate index returns
6.	Daily S&P500 index returns

Table 5: Macro state variables. Source: Adrian and Brunnermeier (2016), Datastream.

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