Calibrating Weather Derivatives

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What are Weather derivatives (WD)?

Hedge weather related risk exposures:

- Payments based on weather-related measurements
- Underlying: temperature, rainfall, wind, snow, frost

Chicago Mercantile Exchange (CME):

- Monthly/seasonal temperature Future/Option contracts
- 18 US, 9 European and 2 Asian-Pacific cities
- From 2.2 billion USD in 2004 to 22 billion USD through September 2005

Weather Derivative

cme
Chicago Mercantile Exchange

Trade Date: Thursday, April 28, 2005

Select a Weather Contract

Select a Contract Type: CDD

Select a Contract Month: May 2005

CDD

May 2005

City	Last	High	Low	Bid Size	Bid Price	Ask Price	Ask Size	Volume
Atlanta	169.0	169	169	25	167.0	174.0	15	15
Chicago	37.0S	-	-	50	35.0	39.0	50	-
Cincinnati	61.0S	-	-	50	40.0	65.0	50	-
New York	55.0S	-	-	50	30.0	60.0	50	-
Dallas	280.0S	-	-	50	265.0	315.0	50	-
Philadelphia	70.0A	70A	70A	50	45.0	70.0	50	-
Portland	20.0S	-	-	50	14.0	25.0	50	-
Tuscon	347.0S	-	-	50	330.0	380.0	50	-
Des Moines	50.0S	-	-	50	45.0	75.0	50	-
Las Vegas	354.0S	-	-	50	325.0	375.0	50	-
Boston	30.0S	-	-	50	15.0	35.0	50	-
Houston	381.0B	381B	380	100	375.0	398.0	50	50
Kansas City	85.0S	-	-	50	60.0	90.0	50	-
Minneapolis	26.0S	-	-	75	20.0	33.0	25	-
Sacramento	75.0A	75A	75A	50	65.0	85.0	50	-

Figure 1: A WD table quoting prices of May 2005 contracts. Source: Chicago Mercantile Exchange's Weather-i

Pricing Methods

Price of a contingent claim F:

$$F = \exp\{-rT\}\,\mathrm{E}^{Q}\left[\psi(I)\right] \tag{1}$$

I: weather index, $\psi(I)$: payoff of the derivative at expiration, r: risk free interest rate, Q: risk neutral probability measure

- Burn analysis: $F = \exp\{-rT\} n^{-1} \sum_{t=1}^{n} \psi(I_t)$
- Stochastic Model/Daily simulation

Market ist Incomplete: need of a equivalent measure Q as a pricing measure

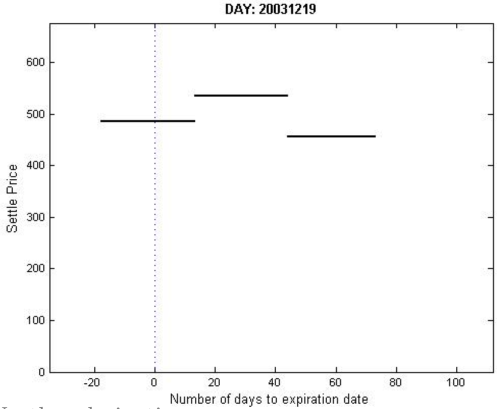
Stochastic Pricing Model for Temperature Derivatives

- Mean reversion model: Dornier et al. (2000), Alaton et al. (2002)
- Fractional Brownian Motion: Brody et al. (2002)
- ARMA model with seasonal ARCH innovations: Campbell and Diebold (2005)
- CAR model with seasonal volatility: Benth et al. (2007)

AIM: Calibrate WD from weather data and CME data

CME WD data

HDD-WD for Berlin 20031022-20060303 (289 days).
451 contracts: prices (0, 676.55), maturity (-35, 112)



Outline

1. Motivation ✓
2. Weather Derivatives Indices
3. Stochastic Pricing Model: CAR(p)
4. Application to Berlin data

Temperature Indices

Heating degree day (HDD): over a period $[\tau_1, \tau_2]$

$$\int_{\tau_1}^{\tau_2} \max(K - T_u, 0) du \tag{2}$$

Cooling degree day CDD): over a period $[\tau_1, \tau_2]$

$$\int_{\tau_1}^{\tau_2} \max(T_u - K, 0) du \tag{3}$$

K is the baseline temperature (typically 18°C or 65°F), T_u is the average temperature on day u.

Weather indices: temperature

Average of average temperature (AAT): measure the "*excess*" or deficit of temperature. The average of average temperatures over $[\tau_1, \tau_2]$ days is:

$$\frac{1}{\tau_1 - \tau_2} \int_{\tau_1}^{\tau_2} T_u du \tag{4}$$

Cumulative averages (CAT): The accumulated average temperature over $[\tau_1, \tau_2]$ days is:

$$\int_{\tau_1}^{\tau_2} T_u du \tag{5}$$

Weather indices: temperature

Event indices: number of times a certain meteorological event occurs in the contract period

- Frost days: temperature at 7:00-10.00 local time is less than or equal to $-3.5°$C

HDD-CDD parity:

$$CDD(\tau_1, \tau_2) - HDD(\tau_1, \tau_2) = CAT(\tau_1, \tau_2) - K$$

- Sufficient to analyse only CDD and CAT futures

Weather indices: temperature

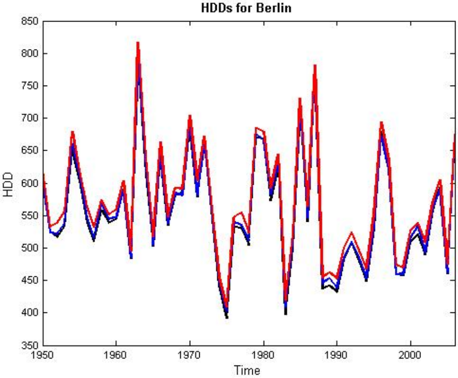


Figure 2: HDDs for Berlin over 57 years. Stations: Berlin Tempelhof (black line), Berlin Dahlem (blue line), Postdam (red line)

Stochastic Model for temperature

Define the vectorial Ornstein-Uhlenbleck process $\mathbf{X}_t \in \mathbb{R}^p$:

$$d\mathbf{X}_t = A\mathbf{X}_t dt + \mathbf{e}_{p t}\sigma_t dB_t$$

$\mathbf{e}_k$: k'th unit vector in $\mathbb{R}^p$ for $k = 1, ...p$, $\sigma_t > 0$: temperature volatility, A: $p \times p$-matrix, B_t: Wiener Process, α_k: constant

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & & 0 \\ 0 & 0 & 1 & \dots & & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \dots & \dots & & 0 & 1 \\ -\alpha_p & -\alpha_{p-1} & \dots & & & -\alpha_1 \end{pmatrix}$$

Solution of $\mathbf{X}_t = \mathbf{x} \in \mathbb{R}^p$:

$$\mathbf{X}_s = \exp\left(A(s-t)\right)\mathbf{x} + \int_t^s \exp\left(A(s-u)\right)\mathbf{e}_p \sigma_u dB_u$$

X_{1t} is a CAR (p) model

Resulting discrete-time dynamics:

For $p = 1$, then $\mathbf{X}_t = X_{1t}$:

$$dX_{1t} = -\alpha_1 X_{1t} dt + \sigma_t dB_t$$

For $p = 2$,

$$X_{1(t+2)} \approx (2 - \alpha_1) X_{1(t+1)} + (\alpha_1 - \alpha_2 - 1) X_{1(t)} + \sigma_t (B_{t-1} - B_t)$$

For $p = 3$,

$$\begin{aligned} X_{1(t+3)} &\approx (3 - \alpha_1) X_{1(t+2)} + (2\alpha_1 - \alpha_2 - 3) X_{1(t+1)} + (-\alpha_1 + \alpha_2 - \alpha_3 + 1) X_{1(t)} \\ &+ \sigma_t (B_{t-1} - B_t) \end{aligned}$$

Temperature Dynamics

Continuous-time AR(p) (CAR(p)) model for Temperature:

$$T_t = \Lambda_t + X_{1t} \tag{6}$$

X_q: q'th coordinate of vector $\mathbf{X}$ with $q = 1, .., p$
Λ_t: seasonality function
X_{1t} is a CAR (p) model
Stationarity holds when the variance matrix:

$$\int_o^t \sigma_{(t-s)}^2 \exp(A(s)) \mathbf{e}_p \mathbf{e}_p^\top \exp(A_s^\top) ds \tag{7}$$

converges as $t \to \infty$

Girsanov theorem: $\exists$ an equivalent probability measure Q^{θ} :

$$B_t^{\theta} = B_t - \int_0^t \theta_u du$$

is a Brownian montion for $t \leq \tau_{\max}$. θ_t: a real valued, bounded and piecewise continous function (market price of risk).
Under Q^{θ}:

$$d\mathbf{X}_t = (A\mathbf{X}_t + \mathbf{e}_p \sigma_t \theta_t)dt + \mathbf{e}_p \sigma_t dB_t^{\theta} \tag{8}$$

with explicit dynamics, for $s \geq t \geq 0$:

$$\begin{aligned} \mathbf{X}_s &= \exp(A(s-t))\mathbf{x} + \int_t^s \exp(A(s-u))\mathbf{e}_p \sigma_u \theta_u du \\ &\quad + \int_t^s \exp(A(s-u))\mathbf{e}_p \sigma_u dB_u^{\theta} \end{aligned} \tag{9}$$

Temperature futures price

Under the Q risk neutral probability:

$$0 = \exp\{-r(\tau_2 - t)\} \mathrm{E}^Q \left[Y - F_{(t,\tau_1,\tau_2)} | \mathcal{F}_t\right] \tag{10}$$

Under the Q^θ pricing probability:

$$F_{(t,\tau_1,\tau_2)} = \mathrm{E}^{Q^\theta} [Y | \mathcal{F}_t] \tag{11}$$

where Y may be equal to the payoff from the CAT/HDD/CDD future

CAT futures price

$$
\begin{aligned}
F_{CAT(t,\tau_1,\tau_2)} &= \mathrm{E}^{Q^\theta}\left[\int_{\tau_1}^{\tau_2} \max(T_s) ds | \mathcal{F}_t\right] \\
&= \int_{\tau_1}^{\tau_2} \Lambda_u du + \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{X}_t + \int_{\tau_1}^{\tau_2} \theta_u \sigma_u \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{e}_p du \\
&+ \int_{\tau_1}^{\tau_2} \theta_u \sigma_u \mathbf{e}_1^\top A^{-1}\left\{\exp\left(A(\tau_2 - u)\right) - \mathtt{I}_p\right\}\mathbf{e}_p du
\end{aligned}
$$

$\mathbf{a}_{t,\tau_1,\tau_2} = \mathbf{e}_1^\top A^{-1}\left\{\exp(A(\tau_2 - t)) - \exp(A(\tau_1 - t))\right\}$
$\mathtt{I}_p : p \times p$ identity matrix

Time Q^θ-dynamics of F_{CAT}: $dF_{CAT(t,\tau_1,\tau_2)} = \sigma_t \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{e}_p dB_t^\theta$

CAT call option

written on a CAT future during the period $[\tau_1, \tau_2]$ is:

$$\begin{aligned} C_{CAT(t,T,\tau_1,\tau_2)} &= \exp\{-r(T-t)\} \\ &\times \Big\{(F_{CAT(t,\tau_1,\tau_2)} - K)\Phi(d\,(t,T,\tau_1,\tau_2)) \\ &+ \int_t^T \Sigma^2_{CAT(s,\tau_1,\tau_2)} ds \phi(d\,(t,T,\tau_1,\tau_2))\Big\} \end{aligned} \tag{12}$$

where

$$d\,(t,T,\tau_1,\tau_2) = \frac{F_{CAT(t,\tau_1,\tau_2)-K}}{\sqrt{\int_t^T \Sigma^2_{CAT(s,\tau_1,\tau_2)} ds}}$$

and

$$\Sigma_{CAT(s,\tau_1,\tau_2)} = \sigma_t \mathbf{a}_{t,\tau_1,\tau_2} \mathbf{e}_p$$

and Φ denotes the standard normal cdf.

Hedging strategy for CAT call option

Delta of the call option:

$$\Phi(d\,(t, T, \tau_1, \tau_2)) = \frac{\partial C_{CAT(t,T,\tau_1,\tau_2)}}{\partial F_{CAT(t,\tau_1,\tau_2)}} \tag{13}$$

Hold: close to zero CAT futures when the option is far out of the money, otherwise close to 1.

CDD futures price

$$
\begin{aligned}
F_{CDD(t,\tau_1,\tau_2)} &= \mathrm{E}^{Q^\theta}\left[\int_{\tau_1}^{\tau_2} \max(T_u - K, 0)du|\mathcal{F}_t\right] \\
&= \int_{\tau_1}^{\tau_2} \upsilon_{t,s}\psi\left[\frac{m_{\{t,s,\mathbf{e}_1^\top \exp(A(s-t))\mathbf{X}_t\}}}{\upsilon_{t,s}}\right] ds
\end{aligned}
$$

where $m_{\{t,s,x\}} = \Lambda_s - c + \int_{\tau_1}^{s} \sigma_u\theta_u\mathbf{e}_1^\top \exp(A_{s-t})\mathbf{e}_p du + x$

$\upsilon_{t,s}^2 = \int_t^s \sigma_u^2 \left\{\mathbf{e}_1^\top \exp(A_{s-t})\mathbf{e}_p\right\}^2 du$

$\psi(x) = x\Phi(x) + \phi(x)$ with $x = \mathbf{e}_1^\top \exp(A(s-t))\mathbf{X}_t$

Φ is the standard normal cdf

CDD futures dynamics

$$
\begin{aligned}
dF_{CDD(t,\tau_1,\tau_2)} &= \sigma_t \int_{\tau_1}^{\tau_2} \left\{ \mathbf{e}_1^\top \exp(A(s-t))\mathbf{e}_p \right\} \\
&\quad \times \Phi \left[\frac{m\left\{t, s, e_1^\top \exp\left(A(s-t)\right)\mathbf{X}_t\right\}}{v_{t,s}} \right] ds dB_t^\theta
\end{aligned}
$$

CDD volatility $\Sigma_{CDD(s,\tau_1,\tau_2)}$ recovers CAT volatility

CDD call options

$$
\begin{aligned}
C_{CDD(t,T,\tau_1,\tau_2)} &= \exp\left\{-r(\tau-t)\right\} \times \\
& \mathrm{E}\left[\max\left(\int_{\tau_1}^{\tau_2} \upsilon_{\tau,s}\psi\left(\frac{m_{\text{index}}}{\upsilon_{\tau,s}}\right)ds - K, 0\right)\right]_{\mathbf{x}=\mathbf{X}_t} \quad (14)
\end{aligned}
$$

$$
\begin{aligned}
\text{index} &= \Bigg(\tau, s, \mathbf{e}_1^\top \exp\left(A(s-t)\right)\mathbf{x} + \int_t^\tau \mathbf{e}_1^\top \exp\left(A(s-u)\right)\mathbf{e}_p\sigma_u\theta_u du \\
&+ \Sigma_{s,t,\tau}Y\Bigg)
\end{aligned}
$$

Y is a std. normal variable,

$\Sigma^2_{s,t,T} = \int_t^T \left\{\mathbf{e}_1^\top \exp\left(A(s-u)\right)\mathbf{e}_p\right\}^2 \sigma_u^2 du$

Hedging strategies CDD call options

Let $C = \max(F_{CDD(\tau,\tau_1,\tau_2)} - K, 0)$ be the payoff of the option, its Clark Ocone representation is:

$$C = \mathrm{E}^{Q^\theta}[C] + \int_0^\tau \mathrm{E}^{Q^\theta}[D, C|\mathcal{F}_t] dB_t^\theta \tag{15}$$

Then, the hedging strategy in CDD-futures:

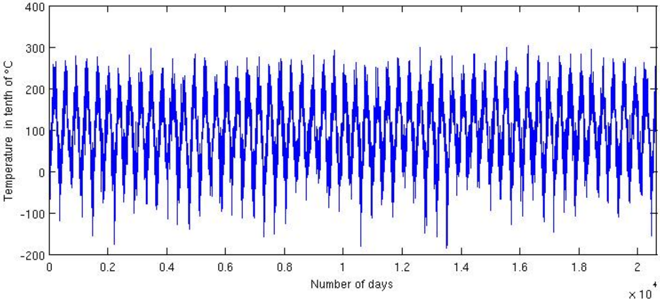
$$H_{CDD(t,\tau_1,\tau_2)} = \Sigma^{-1}_{CDD(t,\tau_1,\tau_2)} \mathrm{E}^{Q^\theta}[D, C|\mathcal{F}_t] \tag{16}$$

where D_t is the Malliavin derivative

Berlin temperature

Daily average temperatures: 1950/1/1-2006/7/24

- Station: BERLIN-TEMP.(FLUGWEWA)
- 29 February removed
- 20645 recordings

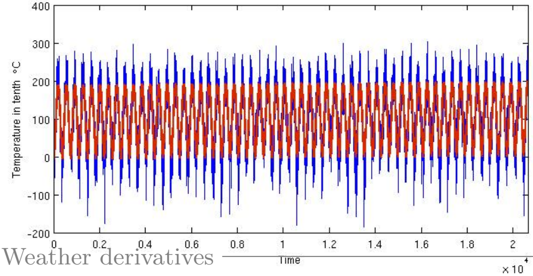


Seasonality

Suppose seasonal function with trend:

$$\Lambda_t = a_0 + a_1 t + a_2 cos\left(\frac{2\pi(t-a_3)}{365}\right)$$

Estimates: $\widehat{a}_0 = 91.52(90.47, 92.56), \widehat{a}_1 = 0.00(0.00, 0.00), \widehat{a}_2 = 97.96(97.22, 98.69), \widehat{a}_3 = -165.1(-165.5, -164.6)$ with 95% confidence bounds $RMSE = 38.2048, R^2 : 0.7672$



Seasonality

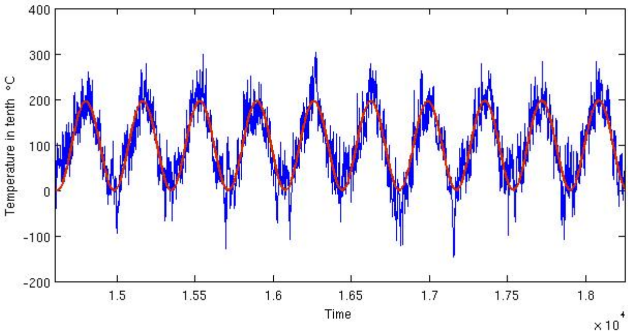


Figure 3: Temperature in Berlin 1990-2000

Temporal dependence

Remove seasonality: $Y_t = T_t - \Lambda_t$

ADF-Test:

$$(1-L)y = c_1 + c_2\text{trend} + \tau Ly + \alpha_1(1-L)Ly + \ldots \alpha_p(1-L)L^p y + u$$

- $\tau = -39.812$, with 1% critical value equal to -2.5659
- Reject H_0 ($\tau = 0$), hence Y_i is a stationary process $I(0)$

PACF

AR(3): $Y_{i+3} = 0.91Y_{i+2} - 0.20Y_{i+1} + 0.07Y_i + (510.63)^{\frac{1}{2}}\varepsilon_i$
CAR(3)-parameters: $\alpha_1 = 2.09, \alpha_2 = 1.38, \alpha_3 = 0.22$
Stationarity condition for the CAR(3) is fulfilled:
$\lambda_1 = -0.2317, \lambda_{2,3} = -0.9291 \pm 0.2934i$.

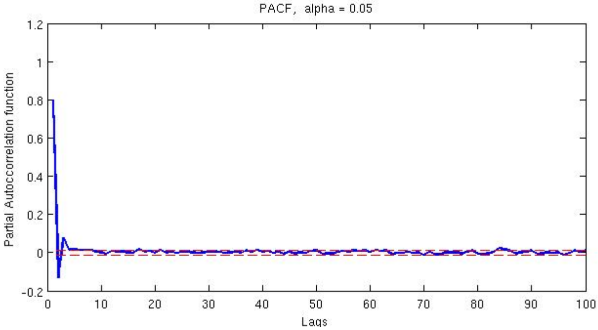


Figure 4: Partial autocorrelation function (PACF)

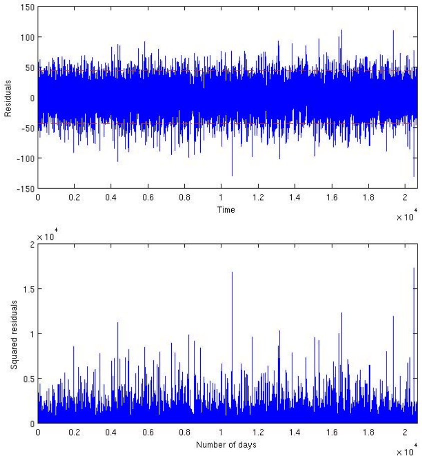


Figure 5: Residuals (up) and squared residuals (down) of the AR(3). Rejection of H_0 for zero-mean residuals at 1% significance level

Seasonal volatility

Close to zero ACF for residuals of AR(3) and according to Box-Ljung statistic the first few lags are insignificant.

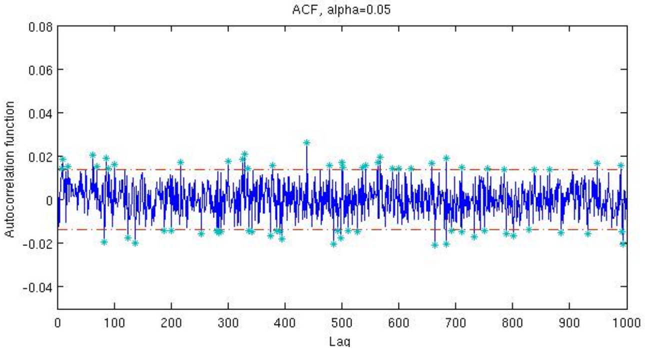


Figure 6: ACF for residuals AR(3)

Seasonal volatility

Highly seasonal ACF for squared residuals of AR(3)

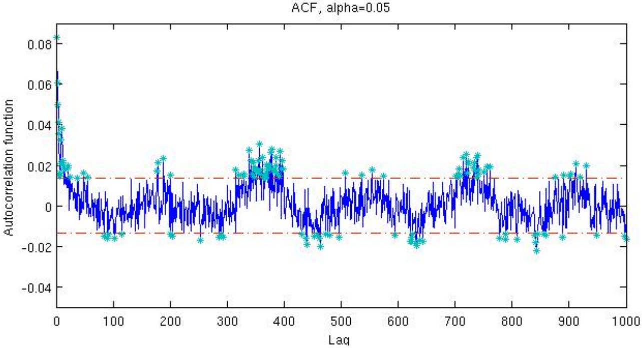


Figure 7: ACF for squared residuals AR(3)

Calibration of daily variances of residuals AR(3) for 56 years:

$$\sigma_t^2 = c_1 + \sum_{i=1}^{4} \left\{ c_{2k} \cos\left(\frac{2i\pi t}{365}\right) + c_{2i+1} \sin\left(\frac{2i\pi t}{365}\right) \right\}$$

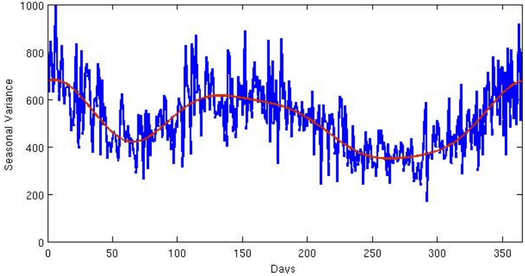


Figure 8: Seasonal variance: daily empirical variance (blue line), fitted squared volatility function (red line) at 10% significance level

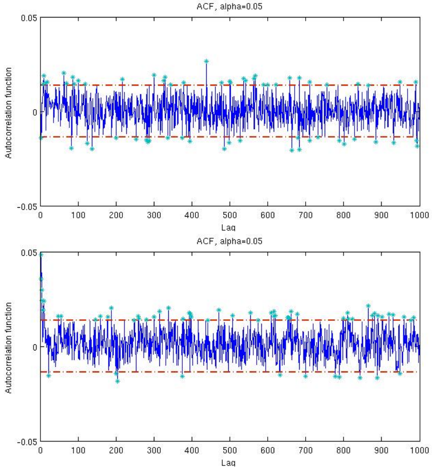


Figure 9: ACF for residuals (up) and squared residuals (down) after dividing out the seasonal volatility

Residuals become normal

T-test: Accept H_0 of normality with p= 0.9611, Skewness= -0.0765, Kurtosis=3.5527.

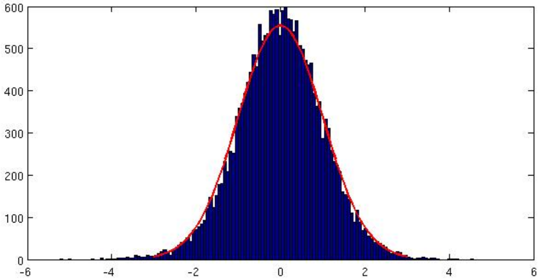


Figure 10: Left: pdf for residuals (black line) and a normal pdf (red line).

Samuelson Effect

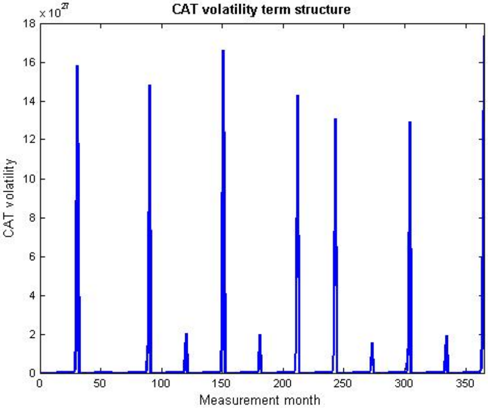


Figure 11: The CAT term structure of volatility

Samuelson and Autoregressive effect

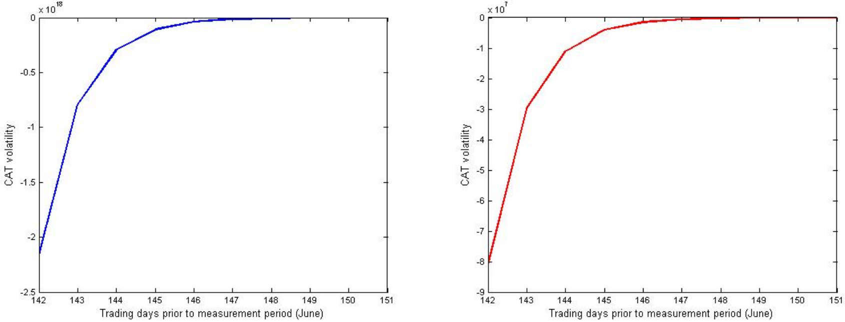
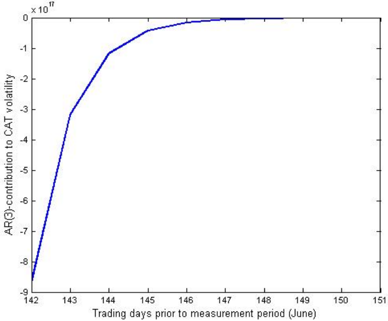


Figure 12: CAT volatility prior of 2 contracts in June: one with measurement period of 1 month (blue line) and the other of 1 week (red line)

AR(3)-contribution to CAT volatility



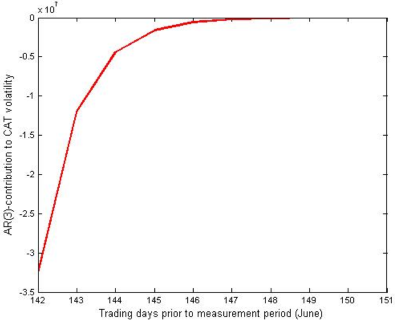


Figure 13: AR(3) contribution to the CAT volatility prior of 2 contracts in June.

to do..

Compute market price risk θ_u from WD data:

$$\begin{aligned} F_{CAT(t,\tau_1,\tau_2)} &= \mathrm{E}^{Q^\theta}\left[\int_{\tau_1}^{\tau_2} \max(T_s)ds|\mathcal{F}_t\right] \\ &= \int_{\tau_1}^{\tau_2} \Lambda_u du + \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{X}_t + \int_{\tau_1}^{\tau_2} \theta_u\sigma_u\mathbf{a}_{t,\tau_1,\tau_2}\mathbf{e}_p du \\ &+ \int_{\tau_1}^{\tau_2} \theta_u\sigma_u\mathbf{e}_1^\top A^{-1}\left\{\exp\left(A(\tau_2-u)\right)-\mathtt{I}_p\right\}\mathbf{e}_p du \end{aligned}$$

θ_u is a real valued piecewise linear function:

$$\theta(u) = \left\{ \begin{array}{l} \theta_1, u \in (u_1, u_2) \\ \theta_2, u \in (u_1, u_2) \end{array} \right\}$$

Questions

- Explicit prices/hedging strategies of WD traded at CME
- Spatial dependence in temperature dynamics: DSFM?
- Random internal climate/urbanisation variability
- Role of the strike value

Conclusion

- CAR(3) model for the temperature dynamics
- Samuelson effect and autoregressive effect observed in Berlin data

Reference

F.E. Benth (2004)
Option Theory with Stochastic Analysis: An Introduction to Mathematical Finance
Berlin: Springer.

F.E Benth and J.S. Benth and S. Koekebakker (2007)
Putting a price on temperature
Scandinavian Journal of Statistics

F.E. Benth and J.S. Benth (2005)
Stochastic Modelling of temperature variations with a view towards weather derivatives
Appl.Math. Finance.

K. Burnecki (2004)
Weather derivatives
Warsaw.

M. Cao, A. Li, J. Wei (2003)
Weather Derivatives: A new class of Financial Instruments
Working Paper, Schulich School of Business, York University, Canada, 2003

S. Campbell, F. Diebold (2005)
Weather Forecasting for weather derivatives
J.American Stat. Assoc.

J.C. Hull (2006)
Option, Futures and other Derivatives
6th ed. New Jersey: Prentice Hall International

- S. Jewson, A. Brix (2005)
  Weather Derivatives Valuation: The Meteorological, Statistical, Financial and Mathematical Foundations
  Cambridge: Cambridge University Press, 2005.

- M. Odening (2004)
  Analysis of Rainfall Derivatives using Daily precipitation models: opportunities and pitfalls

- C. Turvey (1999)
  The Essentials of Rainfall Derivatives and Insurance
  Working Paper WP99/06, Department of Agricultural Economics and Business, University of Guelph, Ontario.

Appendix

Residuals with and without seasonal volatility:

Lag	$\text{Qstat}_r es$	$\text{QSIG}_r es$	$\text{Qstat}_r es1$	$\text{QSIG}_r es1$
1	0.03	0.85	0.67	0.41
2	0.05	0.97	0.74	0.69
3	3.16	0.36	4.88	0.18
4	4.70	0.32	6.26	0.18
5	4.76	0.44	6.67	0.24
6	5.40	0.49	7.17	0.30
7	6.54	0.47	7.51	0.37
8	10.30	0.24	10.34	0.24
9	14.44	0.10	14.65	0.10
10	21.58	0.01	21.95	0.10

Table 1: Q test using Ljung-Box's for residuals with (res) and without seasonality in the variance (res1)

Appendix

Proof $CAR(3) \approx AR(3)$:
Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{pmatrix}$$

- Use $B_{t+1} - B_t = \epsilon_t$
- Substitute iteratively in X_1 dynamics:

$$\begin{aligned}
X_{1(t+1)} - X_{1(t)} &= X_{1(t)} dt + \sigma_t \epsilon_t \\
X_{2(t+1)} - X_{2(t)} &= X_{3(t)} dt + \sigma_t \epsilon_t \\
X_{3(t+1)} - X_{3(t)} &= -\alpha_3 X_{1(t)} dt - \alpha_2 X_{2(t)} dt - \alpha_1 X_{3(t)} dt + \sigma_t \epsilon_t \\
X_{1(t+2)} - X_{1(t+1)} &= X_{1(t+1)} dt + \sigma_{t+1} \epsilon_{t+1} \\
X_{2(t+2)} - X_{2(t+1)} &= X_{3(t+1)} dt + \sigma_{t+1} \epsilon_{t+1} \\
X_{3(t+2)} - X_{3(t+1)} &= -\alpha_3 X_{1(t+1)} dt - \alpha_2 X_{2(t+1)} dt - \alpha_1 X_{3(t+1)} dt + \sigma_{t+1} \epsilon_{t+1} \\
X_{1(t+3)} - X_{1(t+2)} &= X_{1(t+2)} dt + \sigma_{t+2} \epsilon_{t+2} \\
X_{2(t+3)} - X_{2(t+2)} &= X_{3(t+2)} dt + \sigma_{t+2} \epsilon_{t+2} \\
X_{3(t+3)} - X_{3(t+2)} &= -\alpha_3 X_{1(t+2)} dt - \alpha_2 X_{2(t+2)} dt - \alpha_1 X_{3(t+2)} dt + \sigma_{t+2} \epsilon_{t+2}
\end{aligned}$$