ICARE - Localising Conditional AutoRegressive Expectiles

Wolfgang Karl Härdle
Andrija Mihoci
Xiu Xu

Ladislaus von Bortkiewicz Chair of Statistics
C.A.S.E. – Center for Applied Statistics and Economics
Humboldt–Universität zu Berlin
lvb.wiwi.hu-berlin.de
case.hu-berlin.de
irtg1792.hu-berlin.de

Motivation

(i) Risk Exposure
  - Measure tail event
  - Conditional autoregressive expectile (CARE) model

(ii) Time-varying parameter
  - Time-varying parameters in CARE Parameter Dynamics
  - Interval length reflects the structural changes in economy

Objectives

(i) Localising CARE Models

- Local parametric approach (LPA)
- Balance between modelling bias and parameter variability

(ii) Tail Risk Dynamics

- Estimation windows with varying lengths
- Time-varying expectile parameters

Economics and Risk Management

Economics

- Parameter dynamics and structural changes
- Interval length and Economic variables

Risk Management

- Modelling bias vs. parameter variability
- Measuring tail risk

Risk Exposure

An investor observes daily S&P 500 returns from 20050103 to 20141231 and estimates the underlying risk exposure via expectiles (UBS, e.g., 1% and 5%) over a one-year time horizon.

Modelling strategies - performance comparison

(a) Data windows fixed on an ad hoc basis

(b) Adaptively selected data intervals: time-varying parameters

Portfolio Insurance

A fund holds a stock market portfolio and sets a minimum return level. By calculating the excess risk (on top of the risk-less assets) it aims to maximize the trading profits.

Portfolio margin choice - P&L comparison

(a) Constant

(b) Adaptive selection: market conditions

Research Questions

How to account for time-varying parameters in tail event?

What are the typical data interval lengths assessing risk more accurately, i.e., striking a balance between bias and variability?

What are the benefits of the ICARE model in practice?

Outline

1. Motivation ✓
2. Conditional Autoregressive Expectile (CARE)
3. Local Parametric Approach (LPA)
4. Empirical Results
5. Conclusions

Conditional Autoregressive Expectile

- Taylor (2008), Kuan et al. (2009)
- Random variable Y with independent sample y_t, $t = 1, ..., n$
- CARE specification, $\mathcal{F}_t$ - information set up to t

$$y_t = e_{t,\tau} + \varepsilon_{t,\tau}$$

▸ $\varepsilon_\tau \sim \mathrm{AG}\left(0, \sigma^2_{\varepsilon,\tau}, \tau\right)$

$$e_{t,\tau} = \alpha_{0,\tau} + \alpha_{1,\tau} y_{t-1} + \alpha_{2,\tau} \left(y^+_{t-1}\right)^2 + \alpha_{3,\tau} \left(y^-_{t-1}\right)^2$$

- Expectile $e_{t,\tau}$ at $\tau \in (0,1)$, $\theta_\tau = \left\{\alpha_{0,\tau}, \alpha_{1,\tau}, \alpha_{2,\tau}, \alpha_{3,\tau}, \sigma^2_{\varepsilon,\tau}\right\}^\top$
- Returns: $y^+_{t-1} = \max\{y_{t-1}, 0\}$, $y^-_{t-1} = \min\{y_{t-1}, 0\}$

Parameter Estimation

- Data calibration with time-varying intervals
- Observed returns $\mathcal{Y} = \{y_1, \ldots, y_n\}$
- Quasi maximum likelihood estimate (QMLE)

$$\widetilde{\theta}_{I,\tau} = \arg\max_{\theta_\tau \in \Theta} \ell_I(\mathcal{Y}; \theta_\tau)$$

$\ell_I(\cdot)$

- $I = [t_0 - m, t_0]$ - interval of $(m+1)$ observations at t_0
- $\ell_I(\cdot)$ - quasi log likelihood

Estimation Quality

- Mercurio and Spokoiny (2004), Spokoiny (2009)
- Quality of estimating *true* parameter vector θ^*_τ by QMLE $\widetilde{\theta}_{I,\tau}$ in terms of Kullback-Leibler divergence; $\mathcal{R}_r(\theta^*_\tau)$ - risk bound

$$\mathsf{E}_{\theta^*_\tau} \left| \ell_I(\mathcal{Y}; \widetilde{\theta}_{I,\tau}) - \ell_I(\mathcal{Y}; \theta^*_\tau) \right|^r \leq \mathcal{R}_r(\theta^*_\tau)$$

▸ Gaussian Regression

- 'Modest' risk, $r = 0.5$ (shorter intervals of homogeneity)
- 'Conservative' risk, $r = 1$ (longer intervals of homogeneity)

Solomon Kullback and Richard A. Leibler on BBI:

Estimated Risk Bound

	$\tau = 0.05$			$\tau = 0.01$		
	Low	Mid	High	Low	Mid	High
$r = 0.5$	0.24	0.33	0.25	0.38	0.38	0.15
$r = 1.0$	2.40	4.62	2.75	5.90	5.81	1.15

Table 1: $\mathcal{R}_r\left(\theta^*_\tau\right)$, with expectile levels $\tau = 0.05$ and $\tau = 0.01$, for corresponding parameter group scenarios Parameter Scenarios

Local Parametric Approach (LPA)

- LPA, Spokoiny (1998, 2009)
  - Time series parameters can be locally approximated
  - Finding the (longest) *interval of homogeneity* ▸ $\widehat{k}$
  - Balance between modelling bias and parameter variability

- Time series literature
  - GARCH$(1, 1)$ models - Čížek et al. (2009)
  - Realized volatility - Chen et al. (2010)
  - Multiplicative Error Models - Härdle et al. (2014)

Interval Selection

- $(K+1)$ nested intervals with length $n_k = |I_k|$

$$\begin{array}{ccccccccc} I_0 & \subset & I_1 & \subset \cdots \subset & I_k & \subset \cdots \subset & I_K \\ \widetilde{\boldsymbol{\theta}}_0 & & \widetilde{\boldsymbol{\theta}}_1 & & \widetilde{\boldsymbol{\theta}}_k & & \widetilde{\boldsymbol{\theta}}_K \end{array}$$

Example: Daily index returns

Fix i_0, $I_k = [i_0 - n_k, i_0]$, $n_k = \left[n_0 c^k\right]$, $c > 1$

$\{n_k\}_{k=0}^{11} = \{20 \text{ days}, 25 \text{ days}, \ldots, 250 \text{ days}\}$, $c = 1.25$

Local Change Point Detection

- Fix t_0, sequential test $(k = 1, \ldots, K)$

H_0 : parameter homogeneity within I_k vs. H_1 : $\exists$ change point within J_k

$$T_{k,\tau} = \sup_{s \in J_k} \left\{ \ell_{A_{k,s}} \left(\mathcal{Y}, \widetilde{\theta}_{A_{k,s},\tau} \right) + \ell_{B_{k,s}} \left(\mathcal{Y}, \widetilde{\theta}_{B_{k,s},\tau} \right) - \ell_{I_{k+1}} \left(\mathcal{Y}, \widetilde{\theta}_{I_{k+1},\tau} \right) \right\},$$

with $J_k = I_k \setminus I_{k-1}$, $A_{k,s} = [t_0 - n_{k+1}, s]$ and $B_{k,s} = (s, t_0]$

Critical Values, $\mathfrak{z}_{k,\tau}$

- Simulate $\mathfrak{z}_k$ - homogeneity of the interval sequence $I_0, \ldots, I_k$
- 'Propagation' condition

$$\mathsf{E}_{\theta^*_\tau} \left| \ell_{I_k}\left(\mathcal{Y}; \widetilde{\theta}_{I_k,\tau}\right) - \ell_{I_k}\left(\mathcal{Y}; \widehat{\theta}_\tau\right) \right|^r \leq \rho_k \mathcal{R}_r\left(\theta^*_\tau\right)$$

$\rho_k = \dfrac{\rho k}{K}$ for a given significance level ρ $\widehat{\theta}_\tau$ - adaptive estimate

- Check $\mathfrak{z}_{k,\tau}$ for (six) different θ^*_τ Parameter Scenarios

Critical Values, $\mathfrak{z}_{k,\tau}$

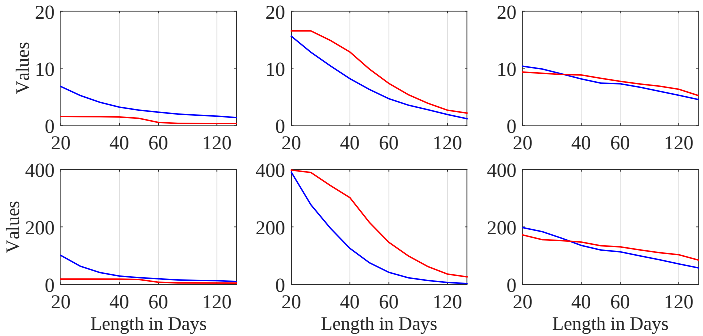

Figure 1: Simulated critical values across different parameter constellations ‣ Parameter Scenarios for the modest (upper panel, $r = 0.5$) and conservative (lower panel, $r = 1$) risk cases, $\tau = 0.05$ and $\tau = 0.01$.

Adaptive Estimation ▸ LPA ▸ $\mathfrak{z}_{k,\tau}$ - Critical Values

- Compare $T_{k,\tau}$ at every step with $\mathfrak{z}_{k,\tau}$
- Data window index of the *interval of homogeneity* - $\widehat{k}$
- Adaptive estimate

$$\widehat{\theta}_{\tau} = \widetilde{\theta}_{I_{\hat{k}},\tau}, \quad \widehat{k} = \max_{k \leq K} \{k : T_{\ell,\tau} \leq \mathfrak{z}_{\ell,\tau}, \ell \leq k\}$$

Data

- Series
  - DAX30, FTSE100 and S&P500 returns, 20050103-20141231 (2608 days)
  - Research Data Center (RDC) - Datastream
- Setup
  - Expectile levels: $\tau = 0.05$ and $\tau = 0.01$

Parameter Dynamics ▸ Motivation

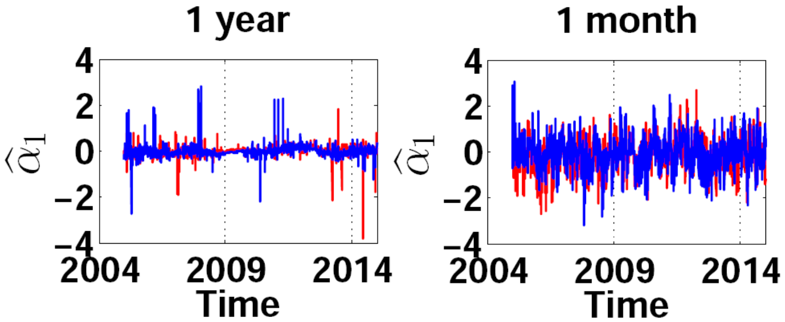

Figure 2: Estimated $\alpha_{1,0.05}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations ▸ more parameters

Parameter Dynamics

▸ Motivation

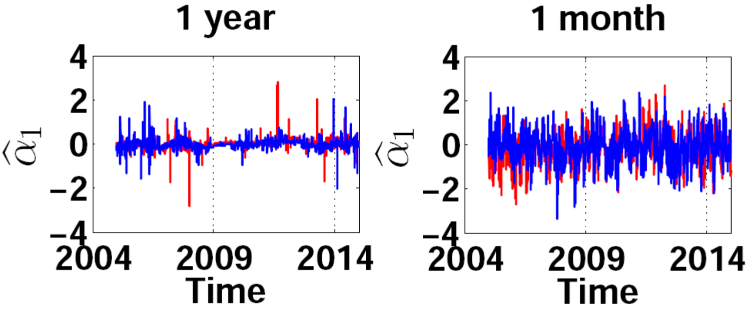

Figure 3: Estimated $\alpha_{1,0.01}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations ▸ more parameters

Parameter Distributions

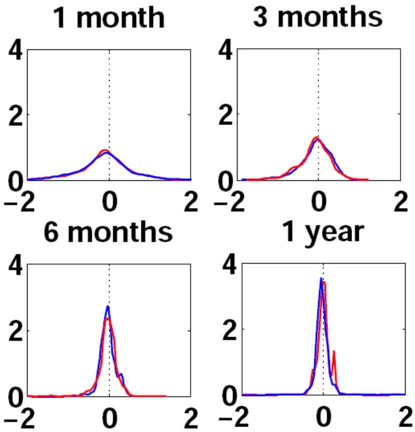

Figure 4: Kernel density estimates of $\alpha_{1,0.05}$ for DAX and FTSE100 using 20, 60, 125 or 250 observations

Parameter Distributions

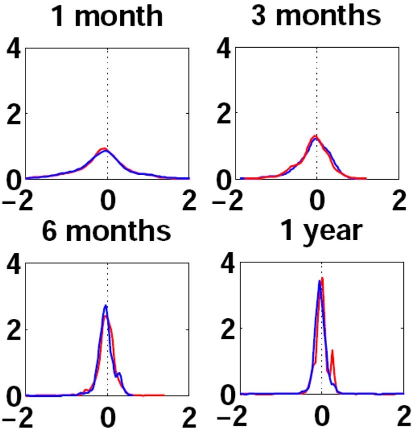

Figure 5: Kernel density estimates of $\alpha_{1,0.01}$ for DAX and FTSE100 using 20, 60, 125 or 250 observations

Adaptive Estimation - Results

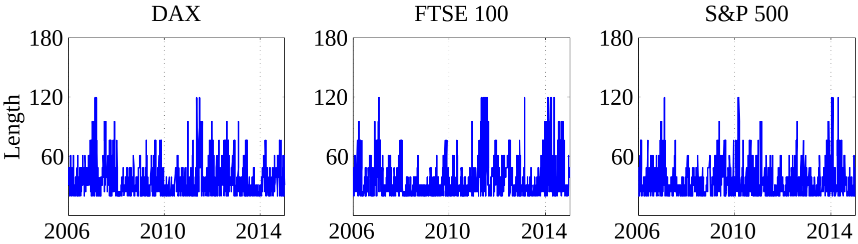

Figure 6: Estimated length $n_{\hat{k}}$ of *intervals of homogeneity* from 20060103-20141231 for the modest risk case $r = 0.5$, at expectile level $\tau = 0.05$.

Adaptive Estimation - Results

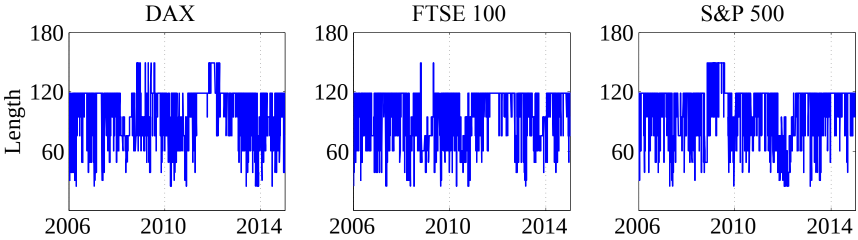

Figure 7: Estimated length $n_{\hat{k}}$ of *intervals of homogeneity* from 20060103-20141231 for the conservative risk case $r = 1$, at expectile level $\tau = 0.05$.

Adaptive Estimation - Results

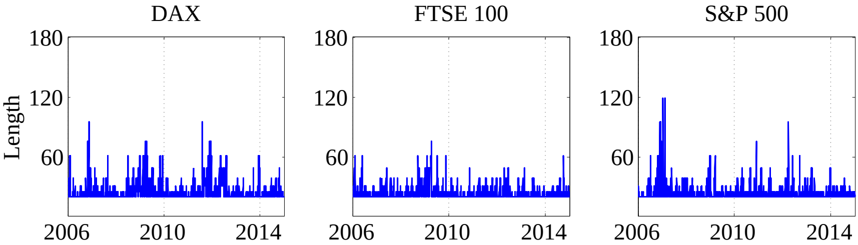

Figure 8: Estimated length $n_{\hat{k}}$ of *intervals of homogeneity* from 20060103-20141231 for the modest risk case $r = 0.5$, at expectile level $\tau = 0.01$.

Adaptive Estimation - Results

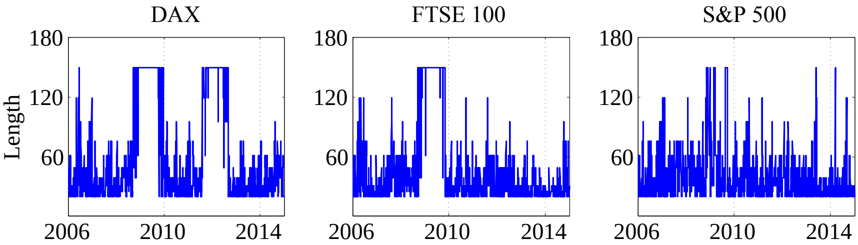

Figure 9: Estimated length $n_{\hat{k}}$ of *intervals of homogeneity* from 20060103-20141231 for the conservative risk case $r = 1$, at expectile level $\tau = 0.01$.

Conclusions

(i) Localising CARE Models

- Balance between modelling bias and parameter variability
- Parameter dynamics

(ii) Tail Risk Dynamics

- Varying distributional characteristics
- Expectile levels $\tau = 0.05$ and $\tau = 0.01$

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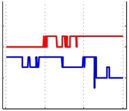

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Econometrics Journal **12**: 248–271, 2009

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Journal of Applied Econometrics, 2014

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The Annals of Statistics **26**(4): 1356–1378, 1998

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The Annals of Statistics **37**(3): 1405–1436, 2009

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Estimating Value at Risk and Expected Shortfall Using Expectiles
Journal of Financial Econometrics **6**(2): 231–252, 2008

Asymmetric Gaussian Distribution (AG)

CARE

- If $\varepsilon_\tau \sim \mathrm{AG}\left(\mu, \sigma^2_{\varepsilon,\tau}, \tau\right)$ with pdf, Gerlach et al. (2012)

$$f_\varepsilon(w) = \frac{2}{\sigma_{\varepsilon_\tau}} \left(\sqrt{\frac{\pi}{|\tau - 1|}} + \sqrt{\frac{\pi}{\tau}} \right)^{-1} \exp\left\{ -\rho_\tau \left(\frac{w - \mu}{\sigma_{\varepsilon_\tau}} \right) \right\}$$

  - Check function: $\rho_\tau(u) = |\tau - \mathbf{I}\{u \le 0\}|\, u^2$

Quasi Log Likelihood Function

▸ Parameter Estimation

- If $\varepsilon_\tau \sim \mathrm{AG}\left(\mu, \sigma_\varepsilon^2, \tau\right)$ with pdf $f_\varepsilon(\cdot)$ then $y \sim \mathrm{AG}\left(e_\tau + \mu, \sigma_\varepsilon^2, \tau\right)$
- Quasi log likelihood function for observed data $\mathcal{Y} = \{y_1, \ldots, y_n\}$ over a fixed interval I

$$\ell_I(\mathcal{Y}; \theta_\tau) = \sum_{t \in I} \log f_\varepsilon(y_t - e_{t,\tau})$$

Gaussian Regression ▸ Estimation Quality

$Y_i = f(X_i) + \varepsilon_i, i = 1, \ldots, n$, weights $W = \{w_i\}_{i=1}^n$

$L(W,\theta) = \sum_{i=1}^{n} \ell\{Y_i, f_\theta(X_i)\} w_i$, log-density $\ell(\cdot)$, $\widetilde{\theta} = \arg\max_{\theta \in \Theta} L(W,\theta)$

1. Local constant, $f(X_i) \approx \theta^*$, $\varepsilon_i \sim \mathrm{N}(0, \sigma^2)$

$$\mathrm{E}_{\theta^*} \left| L(W,\widetilde{\theta}) - L(W,\theta^*) \right|^r \leq \mathrm{E}\,|\xi|^{2r}, \quad \xi \sim \mathrm{N}(0,1)$$

2. Local linear, $f(X_i) \approx \theta^{*\top}\Psi_i$, $\varepsilon_i \sim \mathrm{N}(0, \sigma^2)$, basis functions $\Psi = \{\psi_1(X_1), \ldots, \psi_p(X_p)\}$, multivariate ξ

$$\mathrm{E}_{\theta^*} \left| L(W,\widetilde{\theta}) - L(W,\theta^*) \right|^r \leq \mathrm{E}\,|\xi|^{2r}, \quad \xi \sim \mathrm{N}(0,\mathcal{I}_p)$$

Parameter Scenarios

Risk Bound Critical Value

	$\tau = 0.05$			$\tau = 0.01$		
	Low	Mid	High	Low	Mid	High
$\widetilde{\alpha}_{0,\tau}$	-0.0003	0.0003	0.0007	-0.0003	0.0003	0.0007
$\widetilde{\alpha}_{1,\tau}$	-0.1058	-0.0306	0.0524	-0.1035	-0.0312	0.0547
$\widetilde{\alpha}_{2,\tau}$	-0.5800	-0.5288	0.2438	-0.5808	-0.5266	0.2089
$\widetilde{\alpha}_{3,\tau}$	0.5050	0.5852	2.1213	0.5134	0.5871	2.2066
$\widetilde{\sigma}^2_{\varepsilon,\tau}$	0.0001	0.0001	0.0002	0.0001	0.0001	0.0002

Table 2: Quartiles of estimated CARE parameters based on one-year estimation window, i.e., 250 observations, for the three stock market returns from 20050103-20141231 (2608 trading days)

Parameter Dynamics

▸ Parameter Dynamics

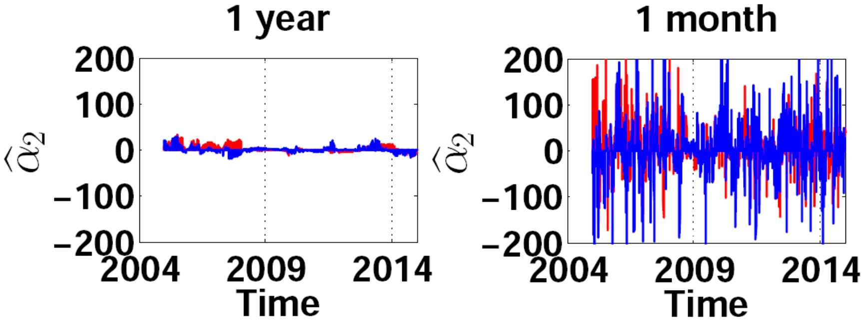

Figure 10: Estimated $\alpha_{2,0.05}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations

Parameter Dynamics

▸ Parameter Dynamics

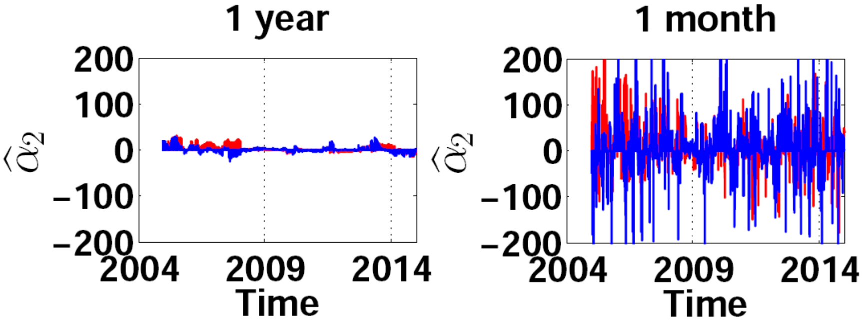

Figure 11: Estimated $\alpha_{2,0.01}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations

Parameter Dynamics

▸ Parameter Dynamics

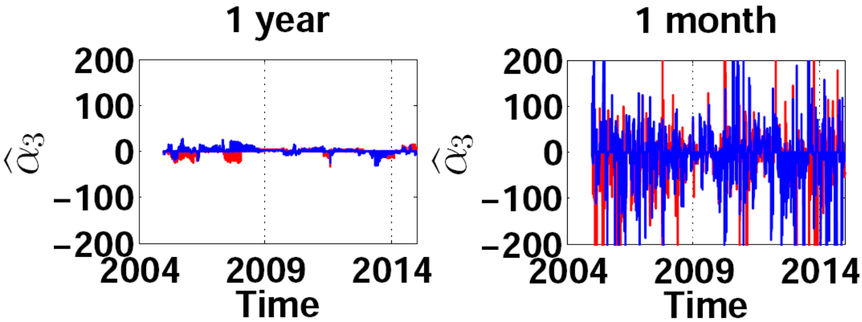

Figure 12: Estimated $\alpha_{3,0.05}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations

Parameter Dynamics

▸ Parameter Dynamics

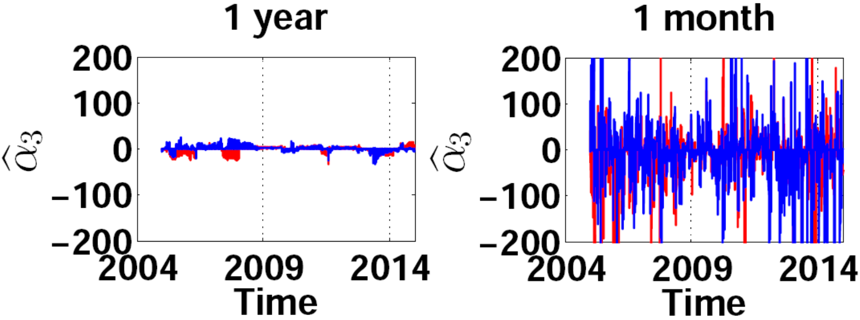

Figure 13: Estimated $\alpha_{3,0.01}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations