Inflation Co-movement in Multi-maturity Term Structure: An Arbitrage-Free Approach

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Measure of Inflation

- Break-even inflation rate (BEIR) with maturity τ,

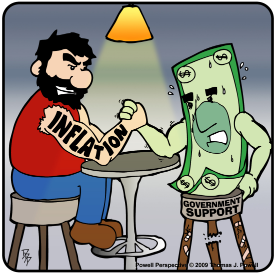
$$BEIR_t(\tau) = y_t^N(\tau) - y_t^R(\tau)$$

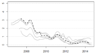
  - $y_t^N(\tau)$ - nominal yield
  - $y_t^R(\tau)$ - real yield

- Decompose BEIR,

$$BEIR_t(\tau) = \pi_t^e(\tau) + else$$

  - $\pi_t^e(\tau)$ is expected inflation





BEIR of European Countries

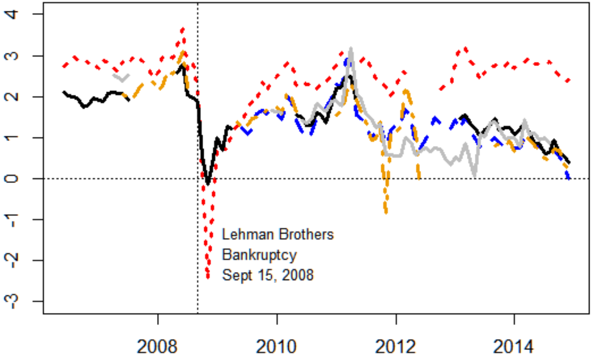
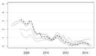


Figure 1: Observed BEI rates (percent) of UK, France, Italy, Sweden and Germany.

MTS_BEIR



Model Approach

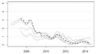
- Nelson and Siegel (1987): Classical NS model

$$y(\tau) = \beta_0 + \beta_1 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + \beta_2 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right)$$

- Diebold and Li (2006): Dynamic NS (DNS) model

$$y_t(\tau) = L_t + S_t \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + C_t \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right)$$

- Diebold, Li and Yue (2008): Global DNS model
- Christensen et.al (2011): Arbitrage-free DNS (AFNS) model



AFNS model

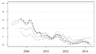
- The closest match to DNS yield is

$$y_t(\tau) = L_t + S_t\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + C_t\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) - \frac{A(\tau)}{\tau}$$

  - state variable $X_t^\top = (L_t, S_t, C_t)$
- Derived from affine AF model of Duffie & Kan (2002), the real-world P dynamics,

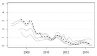
$$dX_t = K^P(\theta^P - X_t)dt + \Sigma dW_t^P$$

  - K^P and θ^P correspond to dynamics and drifts terms.
  - Σ is diagonal.



Challenge

- Joint yield curve modeling across multiple maturities
- BEIR decomposition
- Panel model of inflation expectations
- New estimation and forecast of inflation expectation within Europe



Outline

1. Motivation ✓
2. Multiple Yield Curve Modeling
3. BEIR decomposition
4. Dynamics of Inflation Expectation
5. Empirical Results
6. Conclusion

Joint AFNS model

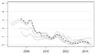
- The separate AFNS models of nominal and inflation-indexed type for a specific country i,

$$y_{it}^N(\tau) = L_{it}^N + S_{it}^N \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + C_{it}^N \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) - \frac{A_i^N(\tau)}{\tau}$$

$$y_{it}^R(\tau) = L_{it}^R + S_{it}^R \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + C_{it}^R \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) - \frac{A_i^R(\tau)}{\tau}$$

- We assume,

$$S_{it}^R = \alpha_i^S S_{it}^N$$
$$C_{it}^R = \alpha_i^C C_{it}^N$$

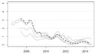


Joint AFNS model

- The joint AFNS yield curve for country i with maturity τ is

$$\begin{pmatrix} y_{it}^N(\tau) \\ y_{it}^R(\tau) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda_i \tau}}{\lambda_i \tau} & \frac{1-e^{-\lambda_i \tau}}{\lambda_i \tau} - e^{-\lambda_i \tau} & 0 \\ 0 & \alpha_i^S \frac{1-e^{-\lambda_i \tau}}{\lambda_i \tau} & \alpha_i^C \left(\frac{1-e^{-\lambda_i \tau}}{\lambda_i \tau} - e^{-\lambda_i \tau}\right) & 1 \end{pmatrix} \begin{pmatrix} L_{it}^N \\ S_{it}^N \\ C_{it}^N \\ L_{it}^R \end{pmatrix} + \begin{pmatrix} \varepsilon_{it}^N(\tau) \\ \varepsilon_{it}^R(\tau) \end{pmatrix} - \begin{pmatrix} \frac{A_i^N(\tau)}{\tau} \\ \frac{A_i^R(\tau)}{\tau} \end{pmatrix}$$

  - state variable $X_{it}^\top = (L_{it}^N, S_{it}^N, C_{it}^N, L_{it}^R)$ Dynamics of state variable
  - $\frac{A_i(\tau)}{\tau}$ is an unavoidable yield-adjustment term

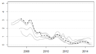


Multiple Yield Curve Modeling

- For country i with maturity $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ is

$$\begin{pmatrix} y_{it}^N(\tau_1) \\ y_{it}^R(\tau_1) \\ \vdots \\ y_{it}^R(\tau_n) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda_i \tau_1}}{\lambda_i \tau_1} & \frac{1-e^{-\lambda_i \tau_1}}{\lambda_i \tau_1} - e^{-\lambda_i \tau_1} & 0 \\ 0 & \alpha_i^S \frac{1-e^{-\lambda_i \tau_1}}{\lambda_i \tau_1} & \alpha_i^C \left(\frac{1-e^{-\lambda_i \tau_1}}{\lambda_i \tau_1} - e^{-\lambda_i \tau_1} \right) & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \alpha_i^S \frac{1-e^{-\lambda_i \tau_n}}{\lambda_i \tau_n} & \alpha_i^C \left(\frac{1-e^{-\lambda_i \tau_n}}{\lambda_i \tau_n} - e^{-\lambda_i \tau_n} \right) & 1 \end{pmatrix}$$

$$\begin{pmatrix} L_{it}^N \\ S_{it} \\ C_{it} \\ L_{it}^R \end{pmatrix} + \begin{pmatrix} \varepsilon_{it}^N(\tau_1) \\ \varepsilon_{it}^R(\tau_1) \\ \vdots \\ \varepsilon_{it}^R(\tau_n) \end{pmatrix} - \begin{pmatrix} \frac{A_i^N(\tau_1)}{\tau_1} \\ \vdots \\ \frac{A_i^R(\tau_n)}{\tau_n} \end{pmatrix}$$



BEIR decomposition

- Cochrane (2005), the price of the zero-coupon bond that pay one unit of consumption basket at time t,

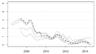
$$P_t^N(\tau) = \mathsf{E}_t\left(M_{t+1}^N M_{t+2}^N \cdots M_{t+\tau}^N\right)$$
$$P_t^R(\tau) = \mathsf{E}_t\left(M_{t+1}^R M_{t+2}^R \cdots M_{t+\tau}^R\right)$$

  - M_t^N and M_t^R are SDFs.

- Under assumption of no arbitrage,

$$\frac{M_t^N}{M_t^R} = \frac{Q_{t-1}}{Q_t}$$

  - Q_t is the overall price level of consumption basket.



BEIR decomposition

- Converting equations,

$$y_t(\tau) = -\frac{1}{\tau} \log P_t(\tau)$$

$$\pi_{t+1} = \log \frac{Q_{t+1}}{Q_t}$$

- The BEIR can be defined as follows,

$$\begin{aligned} y_t^N(\tau) - y_t^R(\tau) &= \frac{1}{\tau} \mathsf{E}_t \left(\log \frac{M_{t+1}^N}{M_{t+1}^R} \cdots \frac{M_{t+\tau}^N}{M_{t+\tau}^R} \right) - \frac{1}{2\tau} \mathsf{Var}_t \left(\log \frac{M_{t+1}^N}{M_{t+1}^R} \cdots \frac{M_{t+\tau}^N}{M_{t+\tau}^R} \right) \\ &+ \frac{1}{\tau} \mathsf{Cov}_t \left(\log \frac{M_{t+1}^N}{M_{t+1}^R} \cdots \frac{M_{t+\tau}^N}{M_{t+\tau}^R}, \log M_{t+1}^R \cdots M_{t+\tau}^R \right) \end{aligned}$$

BEIR decomposition

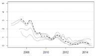
- This is,

$$BEIR_t(\tau) = y_t^N(\tau) - y_t^R(\tau) = \pi_t^e(\tau) + \eta_t(\tau) + \phi_t(\tau)$$

  - $\eta_t(\tau)$ is convexity effect
  - $\phi_t(\tau)$ is IRP
- We found out,

$$\pi_t^e(\tau) = -\frac{1}{\tau} \log \mathrm{E}_t^P \left[\exp \left\{ - \int_t^{t+\tau} (r_s^N - r_s^R) ds \right\} \right]$$

  - r_{it} is the instantaneous risk-free rate
  - real type: $r_{it}^R = L_{it}^R + \alpha_i^S S_{it}^N$, nominal type: $r_{it}^N = L_{it}^N + S_{it}^N$



Inflation Expectation Estimates

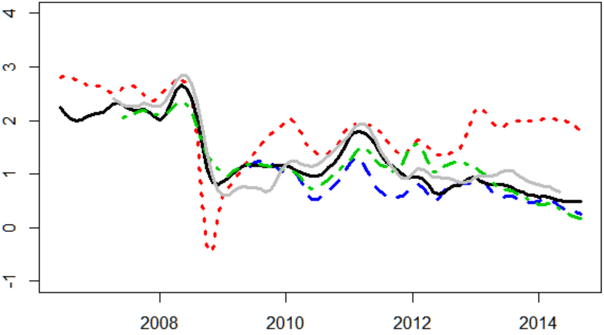


Figure 2: Model-implied inflation expectation for different countries - UK, France, Italy, Sweden and Germany. MTS_expinf

Inflation Expectation(IE) Dynamics

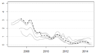
- The five-dimisional idiosyncratic factors to load on a common time-varying latent factor Π_t,

$$\hat{\pi}_{it}^{e} = m_i + n_i\Pi_t + \mu_{it}$$

- The dynamics of common factor,

$$\Pi_t = p + q\Pi_{t-1} + \nu_t$$

  - where m, n, p and q are unknown parameters
  - the errors μ_{it} and ν_{it} are both i.i.d white noises



IE Dynamics with Macroeconomic Factor

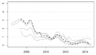
- Interaction between macro-economy and yield curve,

$$\hat{\pi}^e_{it} = m_i + n_i \Pi_t + l_i d_t + \mu_{it}$$

- The dynamics of common factor,

$$\Pi_t = p + q\Pi_{t-1} + \nu_t$$

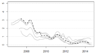
  - where m, n, p and q are unknown parameters
  - the errors μ_{it} and ν_{it} are both i.i.d white noises
  - d_t is the default proxy varying over time



Data

- Bloomberg: monthly zero-coupon government bond yield.
- Type: nominal y^N, inflation-indexed y^R.

Data	Span	Maturity
UK	30.06.2006-31.12.2014	3,4,5 years
France	30.06.2006-31.12.2014	3,5,10 years
Italy	29.06.2007-31.12.2014	3,5,10 years
Sweden	30.04.2007-29.08.2014	3,5,10 years
Germany	30.06.2009-31.12.2014	5,7,10 years

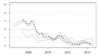


Model Residuals

MTS_multi_modelres

Estimated IE

MTS_expinf



3-year IE forecast

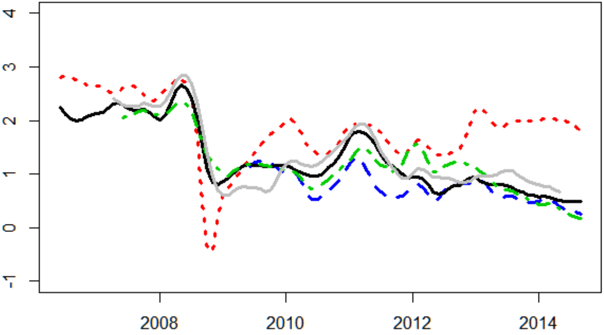
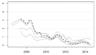


Figure 3: Model-implied inflation expectation for different countries - UK, France, Italy, Sweden and Germany. MTS_expinf



Common Effect

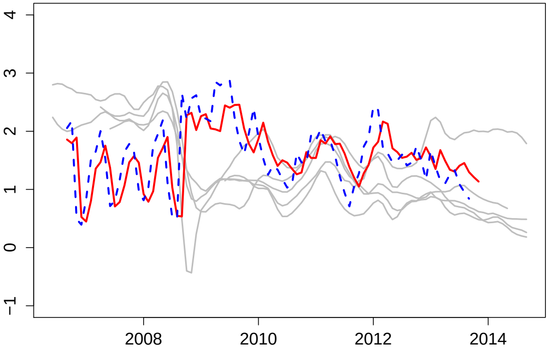
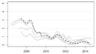


Figure 4: Common inflation factor - predicted Π_t, filtered Π_t.

MTS_comexpinf



Residuals of common effect

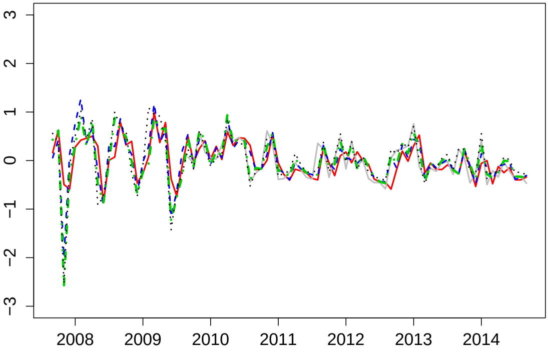
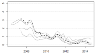


Figure 5: Model residual for IE dynamics without macroeconomic factor - UK, France, Italy, Sweden and Germany. MTS_comexpinf

Estimated Parameters

Country-specific equations		
UK	$\pi^e_{1t}(\tau) =$	$0.166 + 0.576\Pi_t$
France	$\pi^e_{2t}(\tau) =$	$-0.022 + 0.665\Pi_t$
Italy	$\pi^e_{3t}(\tau) =$	$-0.347 + 0.822\Pi_t$
Sweden	$\pi^e_{4t}(\tau) =$	$-0.057 + 0.665\Pi_t$
Germany	$\pi^e_{5t}(\tau) =$	$0.008 + 0.644\Pi_t$
Common Effect equation		
	$\Pi_t =$	$0.588 + 0.651\Pi_{t-1}$

Table 1: Estimates for the dynamics of IE.



Common effect with d_t

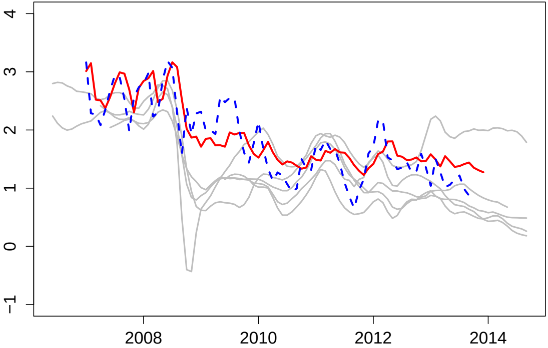


Figure 6: Common inflation factor - predicted Π_t, filtered Π_t.

MTS_comexpinf_cds

Residuals of common effect

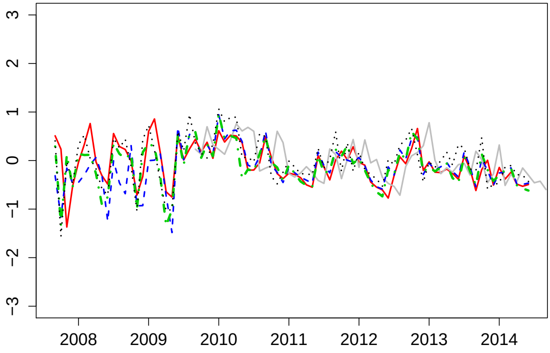


Figure 7: Model residual for IE dynamics without macroeconomic factor - UK, France, Italy, Sweden and Germany. MTS_comexpinf_cds

Estimates with d_t

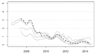
Country-specific equations		
UK	$\pi^e_{1t}(\tau) =$	$-0.358 d_t + 0.798 \Pi_t$
France	$\pi^e_{2t}(\tau) =$	$0.085 d_t + 0.714 \Pi_t$
Italy	$\pi^e_{3t}(\tau) =$	$1.078 d_t + 0.531 \Pi_t$
Sweden	$\pi^e_{4t}(\tau) =$	$-0.621 d_t + 0.805 \Pi_t$
Germany	$\pi^e_{5t}(\tau) =$	$0.045 d_t + 0.700 \Pi_t$
Common Effect equation		
	$\Pi_t =$	$0.382 + 0.976 \Pi_{t-1}$

Table 2: Estimates for the dynamics of IE.

Variance decomposition

- According to the joint model of IE dynamics, decompose the variation of IE - $\hat{\pi}^e_{it}$ into parts driven by,
  - common effect variation
  - country-specific variation
  - default-proxy variation

$$\text{Var}(\hat{\pi}^e_{it}) = n_i^2 \text{Var}\left(\Pi_t\right) + l_i^2 \text{Var}\left(d_t\right) + \text{Var}\left(\mu_{it}\right)$$



Joint IE dynamics without d_t

	U.K.	France	Italy	Sweden	Germany
Common effect	24.91	30.66	40.32	30.65	29.32
Country-specific effect	69.34	50.69	69.35	58.50	70.68

Table 3: Variations explained in percentage

Joint IE dynamics with d_t

	U.K.	France	Italy	Sweden	Germany
Common effect	36.08	33.59	11.54	31.87	32.84
Country-specific effect	56.66	65.88	40.92	49.17	67.02
Default risk effect	7.26	0.53	47.55	18.96	0.14

Table 4: Variations explained in percentage

Forecast

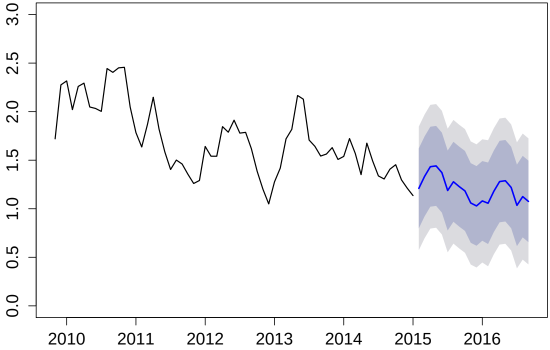
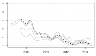


Figure 8: The forecast of common inflation factor derived from the joint model of IE dynamics with default proxy.



Comparison

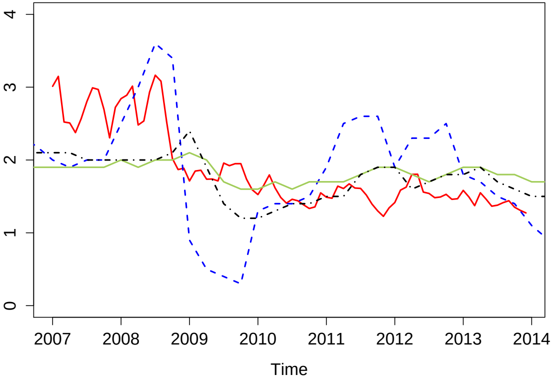
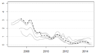
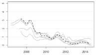


Figure 9: Comparison of model-implied level, the observed inflation level, 1-year SPF forecast and 2-year SPF forecast of inflation.



Conclusion

- Common inflation factor Π_t is an important drivers of country-specific inflation expectations.
- The model proposed will lead to a better forecast in benchmark level of inflation and give good implications in financial market.
- Important index for policy makers and financial investors.



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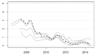

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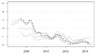

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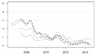

Dynamics of state variable

- Derived from affine AF model of Duffie&Kan(2002), the real world P-dynamics is

$$dX_{it} = K_i^P(t)[\theta_i^P(t) - X_{it}] + \Sigma_i(t)dW_{it}^P$$

  - $K_i^P(t)$, $\theta_i^P(t)$ can vary freely.
  - $\Sigma_i(t)$ is diagonal volatility matrix.
- Transition equation,

$$X_{it} = \Phi^0_{i,\Delta t} + \Phi^1_{i,\Delta t} X_{i,t-1} + \eta_{it}$$

with

$$\Phi^0_{i,\Delta t} = I - \exp(-K_i^P \Delta t)\theta_i^P$$

$$\Phi^1_{i,\Delta t} = \exp(-K_i^P \Delta t)$$

Return