TEDAS - Tail Event Driven ASset allocation: τ-spine optimization

Wolfgang Karl Härdle

Sergey Nasekin

Alla Petukhina

Ladislaus von Bortkiewicz Chair of Statistics
C.A.S.E. – Center for Applied Statistics
and Economics
Humboldt–Universität zu Berlin
http://lvb.wiwi.hu-berlin.de
http://case.hu-berlin.de

TEDAS with Y= S&P 500

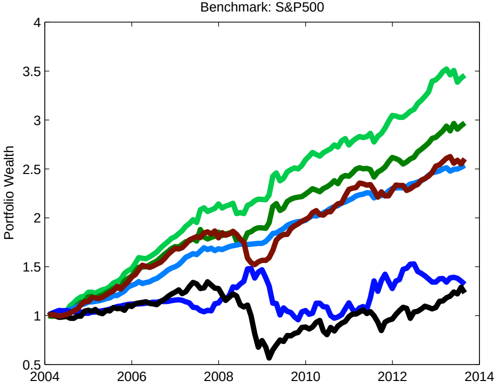


Figure 1: Cumulative portfolio wealth comparison: TEDAS Naïve, TEDAS Expert, TEDAS Advanced, RR, PESS, S&P 500 buy & hold; X = hedge funds' indices' returns matrix TEDAS_strategies2

- Härdle et al. (2014)
  - TEDAS applied to hedge funds' indices performs better than benchmark models
- Limitation of using hedge indices as portfolio assets

Core & Satellites

Mutual funds, SDAX, MDAX and TecDAX constituents

- diversification - reduction of the portfolio risk
- construction - a more diverse universe of assets
- allocation - a higher risk-adjusted return.

Objectives

- Application of TEDAS approach to Global mutual funds' data and German stock market
- Comparison of the TEDAS with more benchmark strategies
- TEDAS parameters optimisation
  - Choice of downside risk level
  - Multi-period model

Tail Risk

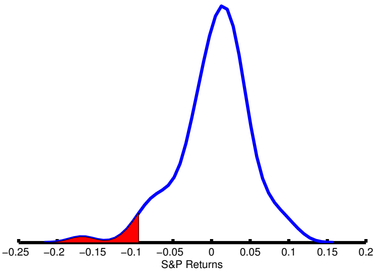


Figure 2: Estimated density of S&P 500 returns

Outline

1. Motivation ✓
2. TEDAS framework
3. Data
4. Empirical Results
5. Choice of τ-spine: dynamic optimization model
6. Outlook

Tail Events

- $Y \in \mathbb{R}^n$ core log-returns; $X \in \mathbb{R}^{n \times p}$ satellites' log-returns, $p > n$
-

$$q_\tau(x) \stackrel{\text{def}}{=} F^{-1}_{Y|x}(\tau) = x^\top \beta(\tau) = \arg\min_{\beta \in \mathbb{R}^p} \mathsf{E}_{Y|X=x}\, \rho_\tau\{Y - X^\top \beta\},$$

$$\rho_\tau(u) = u\{\tau - \mathbf{I}(u < 0)\}$$

- L_1 penalty $\lambda_n \|\hat{\omega}^\top \beta\|_1$ to nullify "excessive" coefficients; λ_n and $\hat{\omega}$ controlling penalization; constraining $\beta \leq 0$ yields ALQR ▸ Details

$$\hat{\beta}^{\text{adapt}}_{\tau,\lambda_n} = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} \rho_\tau(Y_i - X_i^\top \beta) + \lambda_n \|\hat{\omega}^\top \beta\|_1 \qquad (1)$$

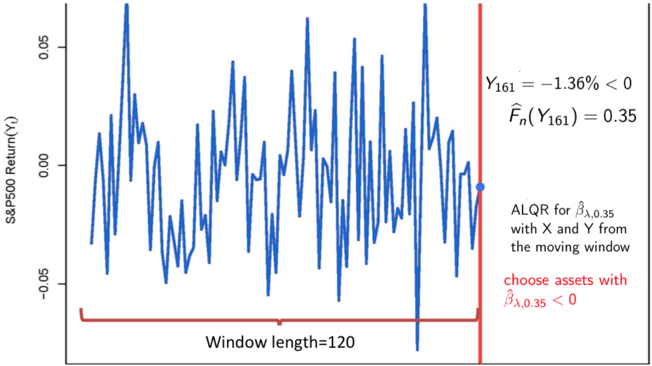
TEDAS Step 1

Initial wealth $W_0 = \$1$, $t = l, \dots, n$; $l = 120$ length of the moving window

- Portfolio constituents' selection

  1. determine core asset return Y_t, set $\tau_t = \widehat{F}_n(Y_t)$
     $\tau_{j=1,\dots,5} = (0.05, 0.15, 0.25, 0.35, 0.50)$ - τ-spine
  2. ALQR for $\hat{\beta}_{\tau_t,\lambda_n}$ using the observations $X \in \mathbb{R}^{t-l+1,\dots,t \times p}$, $Y \in \mathbb{R}^{t-l+1,\dots,t}$
  3. Select $\tau_{j,t}$ according to the right-side $\hat{q}_{\tau_{j,t}}$ in: $Y_t \leq \hat{q}_{\tau_{\mathbf{1},t}}$ or $\hat{q}_{\tau_{\mathbf{1},t}} < Y_t \leq \hat{q}_{\tau_{j,t}}$

TEDAS Step 1


S&P500 Return(Y_t)
0.05
0.00
−0.05
$Y_{161} = -1.36\% < 0$
$\widehat{F}_n(Y_{161}) = 0.35$
ALQR for $\hat{\beta}_{\lambda,0.35}$
with X and Y from
the moving window
choose assets with
$\hat{\beta}_{\lambda,0.35} < 0$
Window length=120

TEDAS Step 2

- Portfolio selection
    1. apply TEDAS Gestalt to X_j, obtain $\widehat{w}_t \in \mathbb{R}^k$
    2. determine the realized portfolio wealth for $t+1$, $\widehat{X}_{t+1} \stackrel{\text{def}}{=} (X_{t+1,1}, \ldots, X_{t+1,k})^\top$: $W_{t+1} = W_t(1 + \widehat{w}_t^\top \widehat{X}_t)$

Rebalancing of portfolio:

- one of inequalities in step 3 holds
  - sell the core portfolio and buy satellites (step 4) with estimated weights (step 5)
  - stay "in cash" if there are no adversely moving satellites (step 4)
- no one of inequalities holds: invest in the core portfolio
- period (t+1), if no one of inequalities (step 3) holds, we return to the core portfolio

TEDAS Example

1. Suppose $t = 161$ (May. 2011), accumulated wealth $W_{161} = \$2.301$, $Y_{161} = -1.36\% < 0$
2. $\widehat{F}_n(Y_{161}) = 0.35$, so estimate ALQR for $\hat{\beta}_{\lambda,0.35}$
3. ALQR on $X \in \mathbb{R}^{120\times 583}$, $Y \in \mathbb{R}^{120}$ yields $\hat{\beta}_{0.35} = (-1.12, -0.41)^\top$, *Blackrock Eurofund Class I, Pimco Funds Long Term United U.S. States Government Institutional Shares*
4. TEDAS CF-CVaR optimization $\widehat{w}_{161} = (0,1)^\top$; $\widehat{X}_{162} = (0.014, 0.026)^\top$, $W_{162} = W_{161}(1 + \widehat{w}_{161}^\top \widehat{X}_{161}) = \$2361$

TEDAS Gestalten

TEDAS gestalt	Dynamics modelling	Weights optimization
TEDAS Naïve	NO	Equal weights
TEDAS Hybrid	NO	Mean-variance optimization of weights ▸ Details
TEDAS Basic	DCC volatility ▸ Details	CF-VaR optimization ▸ Details

Small and mid caps German stocks

- MDAX
  - 50 medium-sized German public limited companies and foreign companies primarily active in Germany from traditional sectors
  - Ranks after the DAX30 based on market capitalisation and stock exchange turnover
- SDAX
  - The selection index for smaller companies from traditional sectors
  - 50 stocks from the Prime Standard
- TecDAX
  - Comprises the 30 largest technology stocks below the DAX

Size premium

- Banz (1981) and Reinganum (1981): the US small cap stocks outperformed large-cap stocks (in 1936-1975)
- Fama, French (1992, 1993): a size premium of 0.27% per month in the US over the period 1963-1991
- Results are robust:
  - for stock price momentum by Jegadeesh , Titman (1993) and Carhart (1997)
  - for liquidity by Pastor, Stambaugh (2003) and Ibbotson, Hu (2011)
  - for industry factors, high leverage, low liquidity by Menchero et al. (2008)

Why small and mid cap stocks?

- Strong absolute returns
- Diversification benefits (Eun, Huang, Lai (2006))
- High risk-adjusted returns

Strong absolute returns

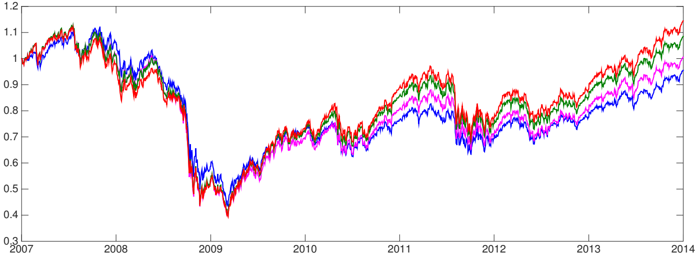


Figure 3: Cumulative index performance: MSCI World Large Cap, MSCI World Mid Cap, MSCI World Small Cap, MSCI World Small and Mid Cap

Diversification

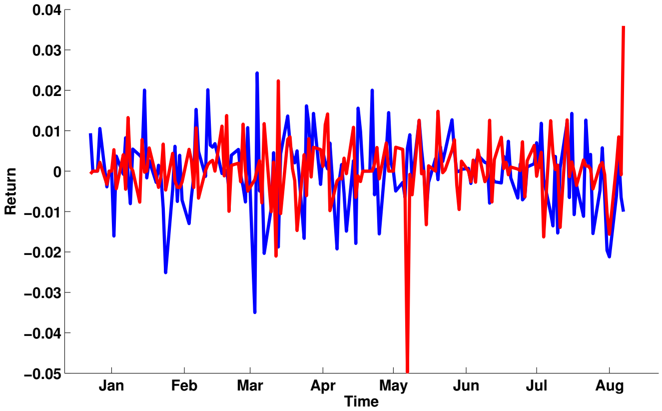
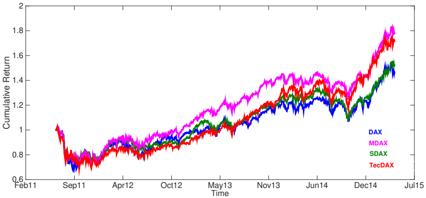


Figure 4: DAX and Hamborner REIT AG daily returns in 20131220-20140831

German equity

- Frankfurt Stock Exchange (Xetra), weekly data
  - 125 stocks - SDAX (48), MDAX (47) and TecDAX (50) as on 20140801
  - DAX index



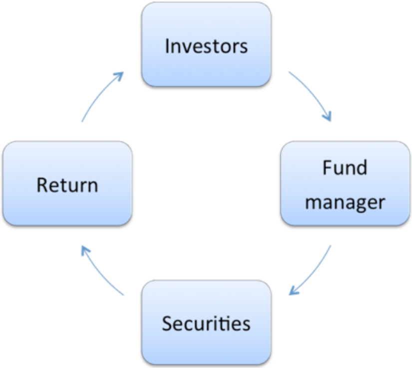
- Span: 20121221 - 20141127 (100 trading weeks)

- Source: Datastream

Mutual Fund

- Open-End: buy and sell the shares, meet the demand for customers
- Unit Investment Trust: exchange-traded fund (ETF), Fixed/ unmanaged Portfolio
- Closed-End: fixed number of shares, not redeemable by the fund, buy and sell on the exchange

Mutual fund flowchart



Why Mutual Funds?

- Importance of MF
  - $30 trillion worldwide, 15 trillion in U.S in 2013
  - 88% investment companies managed asset by holding MF
- Big data: 76 200 MFs worldwide in 2013
- Diversification

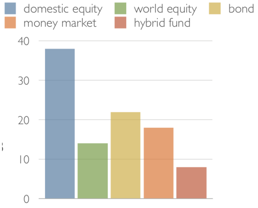


Figure 5: Structure of U.S. Mutual funds, by asset classes

Dynamics of Mutual funds investment

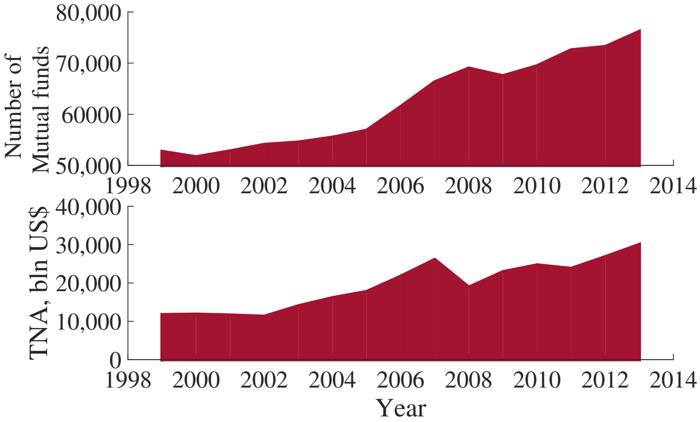
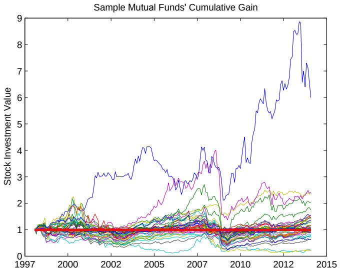


Figure 6: Worldwide Mutual Funds: total number and TNA

Mutual Funds

- Monthly data
  - Core asset (Y): S&P500
  - Satellite assets (X): 583 Mutual funds
- Span: 19980101 - 20131231 (192 months)
- Source: Datastream



Benchmark Strategies

1. RR: dynamic risk-return optimization ▸ Details
2. ERC: Risk-parity portfolio (equal risk contribution) ▸ Details
3. 60/40 portfolio ▸ Details

TEDAS approach:German stocks' results

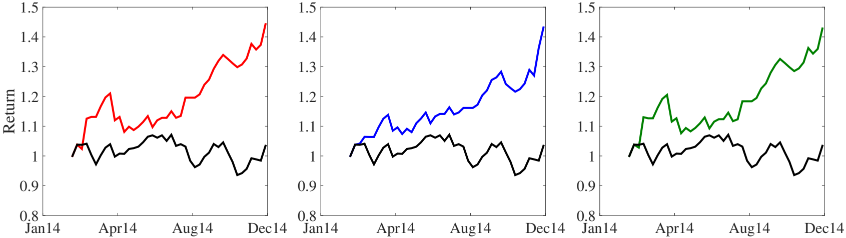


Figure 7: Strategies' cumulative returns' comparison: TEDAS Basic, TEDAS Naïve, TEDAS Hybrid, DAX30

TEDAS_gestalts

TEDAS approach:German stocks' results

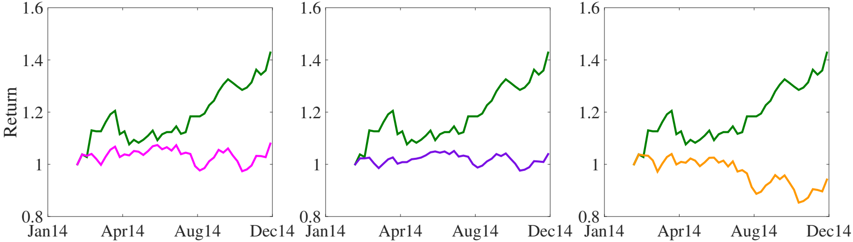


Figure 8: Strategies' cumulative returns' comparison (with transaction costs 1% of portfolio value): TEDAS Hybrid, 60/40, ERC, RR

TEDAS_gestalts

Strategies' performance: German stocks

Strategy	Cumulative return	Sharpe ratio	Maximum drawdown
TEDAS Basic	144%	0.3792	0.1069
TEDAS Naïve	143%	0.3184	0.0564
TEDAS Hybrid	143%	0.3079	0.1068
RR	108%	0.0687	0.0934
ERC	129%	-0.0693	0.1792
60/40	121%	0.0306	0.0718
DAX30	103%	0.0210	0.1264

TEDAS_perform

Selected Stocks

Table 1: The selected German Stocks for $\tau = 0.05$

Top 5 influential Stocks	Frequency	Index	Industry
Sartorius Aktiengesellschaft	12	TecDAX	Provision of laboratory and process technologies and equipment
XING AG	8	TecDAX	Online business communication services
Surteco SE	7	SDAX	Household Goods & Home Construction
Kabel Deutschland Holding AG	7	MDAX	Cable-based telecommunication services
Biotest AG	6	MDAX	Producing biological medications

-$\widehat{\beta}$ in each window, $\tau = 0.05$

Figure 9: Different $-\widehat{\beta}$ in application; $\tau = 0.05$ ▸ Selected Stocks

TEDAS approach:Mutual Funds results

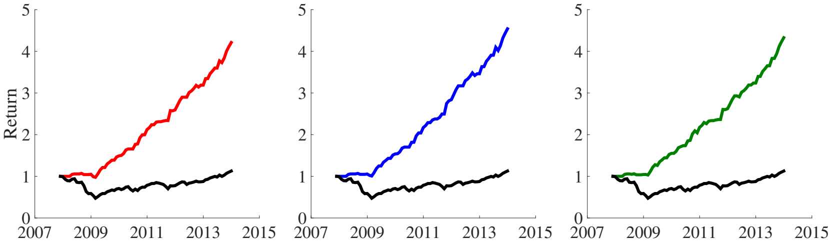


Figure 10: Strategies' cumulative returns' comparison: TEDAS Basic, TEDAS Naïve, TEDAS Hybrid, S&P500

TEDAS_gestalts

TEDAS approach:Mutual Funds results

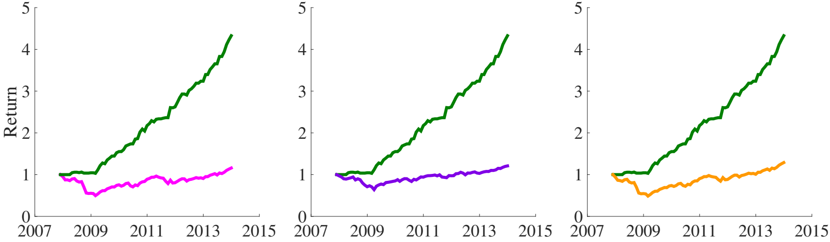


Figure 11: Strategies' cumulative returns' comparison (with transaction costs 1% of portfolio value): TEDAS Hybrid, 60/40, ERC, RR

TEDAS_gestalts

Strategies' performance: Mutual funds

Strategy	Cumulative return	Sharpe ratio	Maximum drawdown
TEDAS Basic	421%	0.6393	0.0855
TEDAS Naïve	454%	0.6974	0.0583
TEDAS Hybrid	433%	0.6740	0.0276
RR	116%	0.0214	0.4772
ERC	129%	0.0487	0.4899
60/40	121%	0.0252	0.3473
S&P500	113%	0.0132	0.5037

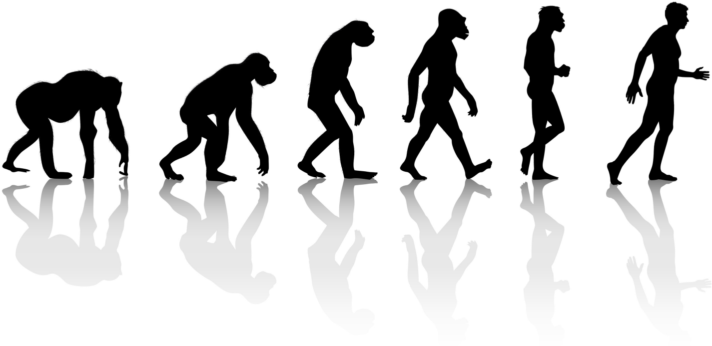
TEDAS_perform

Selected Mutual Funds

Table 2: The selected Mutual Funds for $\tau = 0.05$

Top 5 influential Stocks	Frequency	Market
Blackrock Eurofund Class I	12	U.S.
Pimco Funds Long Term United States Government Institutional Shares	8	U.S.
Prudential International Value Fund Class Z	4	U.S.
Artisan International Fund Investor Shares	3	U.S.
American Century 2OTH Century International Growth Investor Class	1	U.S.

How to choose optimal τ-spine?



TEDAS Naive: τ-spine vs KDE-τ

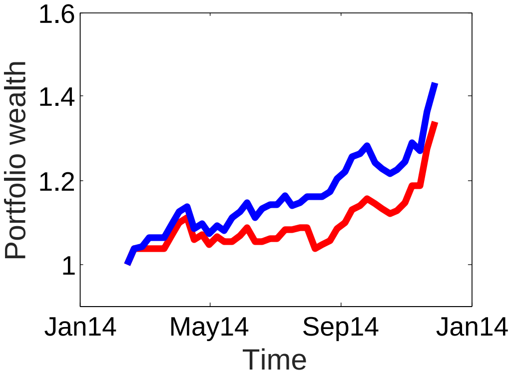


Figure 12: TEDAS Naïve cumulative returns'
$\tau_{j=1,\dots,5} = (0.05, 0.15, 0.25, 0.35, 0.5)$ (Härdle et. al. 2015b),
$\tau_t = \widehat{F}_n(Y_t)$ (Härdle et. al. 2015a) ▸ Notation

TEDAS Basic with different τ-spines

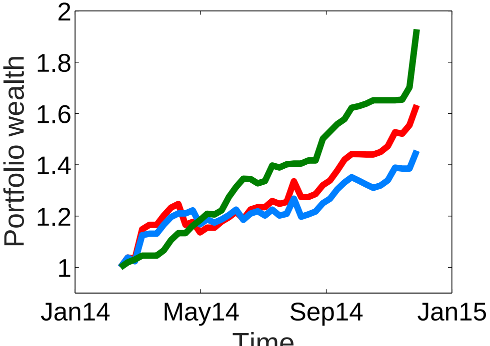


Figure 13: TEDAS Basic cumulative returns' for τ-spines:

$\tau_{j=1,\ldots,5} = (0, 0.002, 0.0233, 0.1311, 0.5)$,

$\tau_{j=1,\ldots,5} = (0.05, 0.15, 0.25, 0.35, 0.5)$,

$\tau_{j=1,\ldots,50} = (0.01, 0.02, 0.03 \ldots 0.49, 0.5)$

Generation of different τ-spines

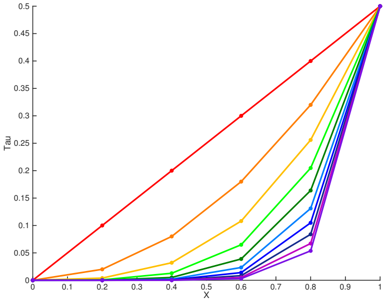


Figure 14: $\tau = 0.5X^n$ (5 grids)

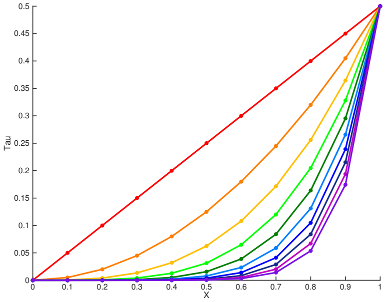


Figure 15: $\tau = 0.5X^n$ (10 grids)

Analysis of returns depending on $\tau - spine$

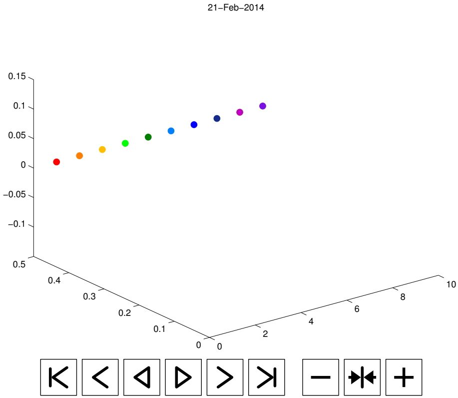


Figure 16: German market sample

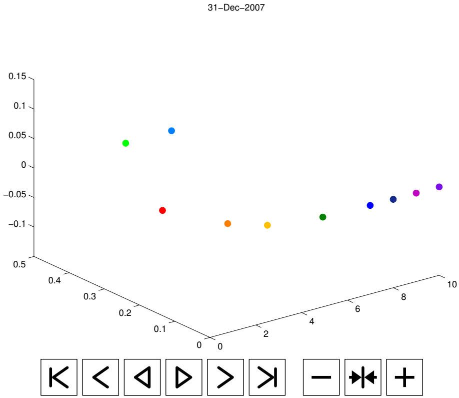


Figure 17: Mutual funds' sample

Cumulative portfolio wealth depending on τ-spines

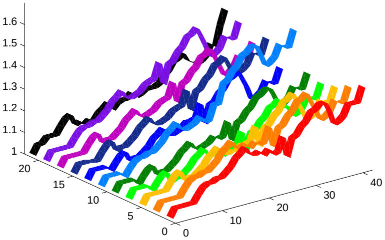


Figure 18: German market sample

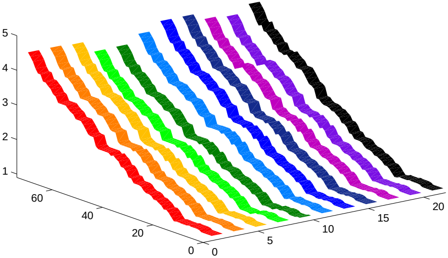
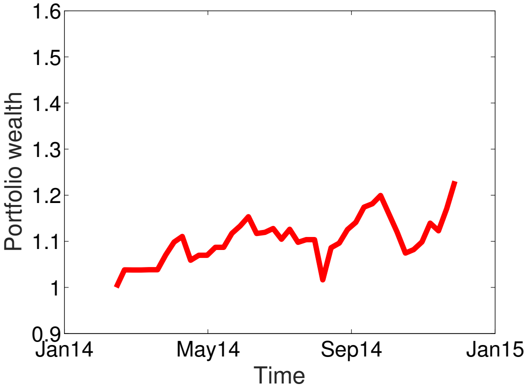
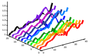


Figure 19: Mutual funds' sample

TEDAS Naïve with various 10τ-spines



TEDAS_strategies



τ- dynamic optimization

- Time horizon T - finite, e.g. one year (T=12)
- Solution - sequence of optimal TEDAS strategies
- State variable – W_t-wealth of portfolio in period t(cumulative return)
- TEDAS - strategies' set $A \stackrel{\text{def}}{=} \{a^1 \ldots a^{50}\}$ with only different parameter τ
  $\tau_{j=1,\ldots,50} = 0.01, 0.02, 0.03 \ldots 0.49, 0.5$ (can be extended, e.g. allocation rule (Markowitz, CF-VAR etc.)
- a_t - chosen strategy in period t, where $t = 0 \ldots T - 1$

τ- dynamic optimization: State space

- Transition equation

$$W_{t+1} = f_t(W_t, a_t, \xi_{t+1})$$

where ξ_{t+1} - randomness of returns

- $W_{t+1}|a_t \overset{\text{def}}{=} W_t(1 + R^a_{t+1}(\xi_{t+1}))$,
where R^a_{t+1} - random return of strategy a TEDAS framework

- $\widehat{F}_{r*,a}$ - Stationary Bootstraping (Politis and Romano, 1994)

- B - desired number of resampled monthly (weekly) returns $r^{*,a}$

τ- dynamic optimization: State transition

- State transition
  - $G_w \overset{\text{def}}{=} \{g^i | i = 1 \ldots I\}$, where i - level of wealth, $I \overset{\text{def}}{=} \{1 \ldots I\}$ - set of grid points (equally-spaced)
  - $P_a = (W_{t+1} = g^j | W_t = g^i, a_t)$
  - $P_a(g^j | g^i, a) \overset{\text{def}}{=}$ $B^{-1} \cdot \sum_{\omega=1}^{B} \mathbf{1} \left\{ g^i \cdot (1 + r^{*,a}) \in \left[\frac{g^{j-\mathbf{1}} + g^j}{2}; \frac{g^j + g^{j+\mathbf{1}}}{2} \right) \right\}$

- Terminal value function (Protection portfolio strategy):

$$V_T(W_T) = W_T - m[max(F - W_T, 0)]^2,$$

where F - floor value, m - multiplier

τ- dynamic optimization: Backward recursion

- Backward Recursion (Bellman, 1957)
  $V_t(W_t) = \max\{f_t(W_t, a_t, \xi_{t+1}) + \mathsf{E}_t[V_{t+1}(W_{t+1})]\}$
- To get optimal choice of $a^{*,i}_{T-1}$ solve:

$$V_{t-1}(W_{t-1}) =$$
$$\max\left[\left\{B^{-1}\sum_{\omega=1}^{B} V_T(W_{T-1}(1 + r^{*,a_{T-1},\omega}))\right\}\right]$$

- Optimal τ-spine

$$\pi^* = \{a_0^{*,i}(W_0^i), \ldots, a^{*,i}_{T-1}(W^i_{T-1})\}$$

Conclusions

- TEDAS approach performs better than traditional benchmark strategies
- TEDAS outperforms for
  - different regions (global and Germany),
  - various assets
  - alternative time periods (daily, weekly and monthly),
  - big data and small data
- Results for 3 gestalts of TEDAS are robust
- Choice of τ-spine - dynamic-optimization model

Lasso Shrinkage

Linear model: $Y = X\beta + \varepsilon$; $Y \in \mathbb{R}^n$, $X \in \mathbb{R}^{n\times p}$, $\beta \in \mathbb{R}^p$, $\{\varepsilon_i\}_{i=1}^n$ i.i.d., independent of $\{X_i; i = 1, ..., n\}$

The optimization problem for the lasso estimator:

$$\begin{aligned} \hat{\beta}^{\text{lasso}} = & \quad \arg\min_{\beta\in\mathbb{R}^p} f(\beta) \\ & \quad \text{subject to} \quad g(\beta) \geq 0 \end{aligned} \tag{2}$$

where

$$\begin{aligned} f(\beta) &= \frac{1}{2}(y - X\beta)^\top (y - X\beta) \\ g(\beta) &= t - \|\beta\|_1 \end{aligned}$$

where t is the size constraint on $\|\beta\|_1$ Back to "Tail Events"

Lasso Duality

If (1) is convex programming problem, then the Lagrangian is

$$L(\beta, \lambda) = f(\beta) - \lambda g(\beta).$$

and the primal-dual relationship is

$$\underbrace{\operatorname*{minimize}_{\beta} \sup_{\lambda \geq 0} L(\beta, \lambda)}_{\text{primal}} \geq \underbrace{\operatorname*{maximize}_{\lambda \geq 0} \inf_{\beta} L(\beta, \lambda)}_{\text{dual}}$$

Then the dual function $L^*(\lambda) = \inf_{\beta} L(\beta, \lambda)$ is

$$L^*(\lambda) = \frac{1}{2} y^\top y - \frac{1}{2} \hat{\beta}^\top X^\top X \hat{\beta} - t \frac{(y - X\hat{\beta})^\top X \hat{\beta}}{\|\hat{\beta}\|_1}$$

with $(y - X\hat{\beta})^\top X\hat{\beta} / \|\hat{\beta}\|_1 = \lambda$ Back to "Tail Events"

Paths of Lasso Coefficients

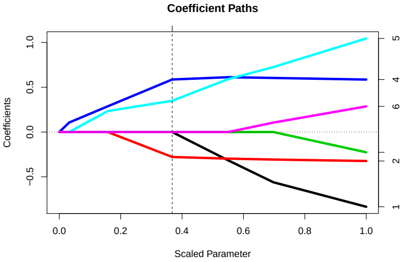


Figure 20: Lasso shrinkage of coefficients in the hedge funds dataset example (6 covariates were chosen for illustration); each curve represents a coefficient as a function of the scaled parameter $\hat{s} = t/\|\beta\|_1$; the dashed line represents the model selected by the BIC information criterion ($\hat{s} = 3.7$)

Example of Lasso Geometry

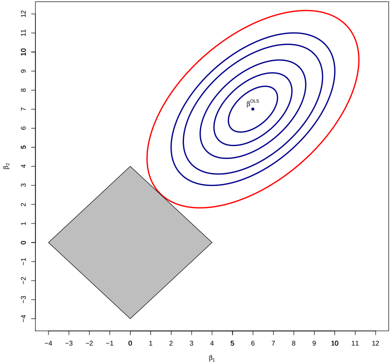


Figure 21: Contour plot of the residual sum of squares objective function centered at the OLS estimate $\widehat{\beta}^{ols} = (6, 7)$ and the constraint region $\sum |\beta_j| \leq t$ MVAlassocontour

Quantile Regression

The loss $\rho_\tau(u) = u\{\tau - \mathbf{I}(u < 0)\}$ gives the (conditional) quantiles $F^{-1}_{y|x}(\tau) \stackrel{\text{def}}{=} q_\tau(x)$.

Minimize

$$\hat{\beta}_\tau = \arg \min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} \rho_\tau(Y_i - X_i^\top \beta).$$

Re-write:

$$\underset{(\xi,\zeta)\in\mathbb{R}^{2n}_{+}}{\text{minimize}} \quad \left\{ \tau 1_n^\top \xi + (1-\tau) 1_n^\top \zeta \middle| X\beta + \xi - \zeta = Y \right\}$$

with ξ, ζ are vectors of "slack" variables Back to "Tail Events"

Non-Positive (NP) Lasso-Penalized QR

The lasso-penalized QR problem with an additional non-positivity constraint takes the following form:

$$
\begin{aligned}
\underset{(\xi,\zeta,\eta,\tilde{\beta})\in\mathbb{R}_{+}^{2n+p}\times\mathbb{R}^{p}}{\text{minimize}} \quad & \tau 1_n^\top \xi + (1-\tau) 1_n^\top \zeta + \lambda 1_n^\top \eta \\
\text{subject to} \quad & \xi - \zeta = Y + X\tilde{\beta}, \\
& \xi \geq 0, \\
& \zeta \geq 0, \\
& \eta \geq \tilde{\beta}, \\
& \eta \geq -\tilde{\beta}, \\
& \tilde{\beta} \geq 0, \qquad \tilde{\beta} \overset{\text{def}}{=} -\beta
\end{aligned}
\tag{3}
$$

Back to "Tail Events"

Solution

Transform into matrix (I_p is $p \times p$ identity matrix; $E_{p\times n} = \begin{pmatrix} I_n \\ 0 \end{pmatrix}$):

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax = b, \;\; Bx \leq 0 \end{aligned}$$

where $A = \begin{pmatrix} I_n & -I_n & 0 & X \end{pmatrix}$, $b = Y$, $x = \begin{pmatrix} \xi & \zeta & \eta & \beta \end{pmatrix}^\top$,

$$c = \begin{pmatrix} \tau 1_n \\ (1-\tau)1_n \\ \lambda 1_p \\ 0 1_p \end{pmatrix}, \quad B = \begin{pmatrix} -E_{p\times n} & 0 & 0 & 0 \\ 0 & -E_{p\times n} & 0 & 0 \\ 0 & 0 & -I_p & I_p \\ 0 & 0 & -I_p & -I_p \\ 0 & 0 & 0 & I_p \end{pmatrix}$$

Back to "Tail Events"

Solution - Continued

The previous problem may be reformulated into *standard form*

$$
\begin{array}{ll}
\text{minimize} & c^\top x \\
\text{subject to} & Cx = d, \\
 & x + s = u, \;\; x \geq 0, s \geq 0
\end{array}
$$

and the dual problem is:

$$
\begin{array}{ll}
\text{maximize} & d^\top y - u^\top w \\
\text{subject to} & C^\top y - w + z = c, \;\; z \geq 0, w \geq 0
\end{array}
$$

Back to "Tail Events"

Solution - Continued

The KKT conditions for this linear program are

$$F(x, y, z, s, w) = \left\{ \begin{array}{c} Cx - d \\ x + s - u \\ C^\top y - w + z - c \\ x \circ z \\ s \circ w \end{array} \right\} = 0,$$

with $y \geq 0$, $z \geq 0$ dual slacks, $s \geq 0$ primal slacks, $w \geq 0$ dual variables.

This can be solved by a primal-dual path following algorithm based on the *Newton method*

Back to "Tail Events"

Adaptive Lasso Procedure

Lasso estimates $\hat{\beta}$ can be inconsistent (Zou, 2006) in some scenarios.

Lasso soft-threshold function gives biased results

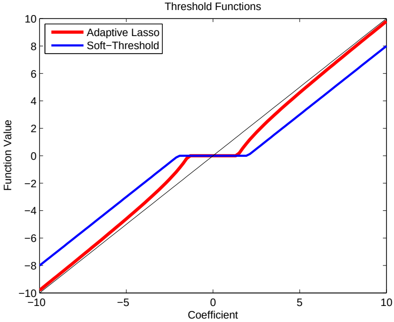


Figure 22: Threshold functions for simple and adaptive Lasso

Adaptive Lasso Procedure

The adaptive Lasso (Zou, 2006) yields a sparser solution and is less biased.

L_1 - penalty replaced by a re-weighted version; $\hat{\omega}_j = 1/|\hat{\beta}_j^{\text{init}}|^{\gamma}$, $\gamma = 1$, $\hat{\beta}^{\text{init}}$ is from (2)

The adaptive lasso estimates are given by:

$$\hat{\beta}_{\lambda}^{\text{adapt}} = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} (Y_i - X_i^{\top}\beta)^2 + \lambda \|\hat{\omega}^{\top}\beta\|_1$$

(Bühlmann, van de Geer, 2011): $\hat{\beta}_j^{\text{init}} = 0$, then $\hat{\beta}_j^{\text{adapt}} = 0$

Back to "Tail Events"

Simple and Adaptive Lasso Penalized QR

Simple lasso-penalized QR optimization problem is:

$$\hat{\beta}_{\tau,\lambda} = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} \rho_\tau(Y_i - X_i^\top \beta) + \lambda \|\beta\|_1 \tag{4}$$

Adaptive lasso-penalized QR model uses the re-weighted penalty:

$$\hat{\beta}_{\tau,\lambda}^{\text{adapt}} = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} \rho_\tau(Y_i - X_i^\top \beta) + \lambda \|\hat{\omega}^\top \beta\|_1 \tag{5}$$

Adaptive lasso-penalized QR procedure can ensure oracle properties for the estimator Details

Back to "Tail Events"

Algorithm for Adaptive Lasso Penalized QR

The optimization for the adaptive lasso quantile regression can be re-formulated as a lasso problem:

- the covariates are rescaled: $\tilde{X} = (X_1 \circ \hat{\beta}_1^{\text{init}}, \ldots, X_p \circ \hat{\beta}_p^{\text{init}})$;
- the lasso problem (4) is solved:

$$\hat{\hat{\beta}}_{\tau,\lambda} = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} \rho_\tau(Y_i - \tilde{X}_i^\top \beta) + \lambda \|\beta\|_1$$

- the coefficients are re-weighted as $\hat{\beta}_{\tau,\lambda}^{\text{adapt}} = \hat{\hat{\beta}}_{\tau,\lambda} \circ \hat{\beta}^{\text{init}}$

▸ Back to "Tail Events"

Oracle Properties of an Estimator

An estimator has oracle properties if (Zheng et al., 2013):

- it selects the correct model with probability converging to 1;
- the model estimates are consistent with an appropriate convergence rate;
- estimates are asymptotically normal with the same asymptotic variance as that knowing the true model

Back to "Simple and Adaptive Lasso Penalized QR"

Oracle Properties for Adaptive Lasso QR

In the linear model, let $Y = X\beta + \varepsilon = X^1\beta^1 + X^2\beta^2 + \varepsilon$, where $X = (X^1, X^2)$, $X^1 \in \mathbb{R}^{n\times q}$, $X^2 \in \mathbb{R}^{n\times(p-q)}$; β^1_q are true nonzero coefficients, $\beta^2_{p-q} = 0$ are noise coefficients; $q = \|\beta\|_0$.

Also assume that $\lambda q/\sqrt{n} \to 0$ and $\lambda/\{\sqrt{q}\log(n \vee p)\} \to \infty$ and certain regularity conditions are satisfied ▸ Details

Oracle Properties for Adaptive Lasso QR

Then the adaptive L_1 QR estimator has the oracle properties (Zheng et al., 2013):

1. Variable selection consistency:

$$\mathsf{P}(\beta^2 = 0) \geq 1 - 6\exp\left\{-\frac{\log(n \vee p)}{4}\right\}.$$

2. Estimation consistency: $\|\beta - \hat{\beta}\| = \mathcal{O}_p(\sqrt{q/n})$

3. Asymptotic normality: $u_q^2 \stackrel{\text{def}}{=} \alpha^\top \Sigma_{11} \alpha$, $\forall \alpha \in \mathbb{R}^q$, $\|\alpha\| < \infty$,

$$n^{1/2} u_q^{-1} \alpha^\top (\beta^1 - \hat{\beta}^1) \xrightarrow{\mathcal{L}} \mathsf{N}\left\{0, \frac{(1-\tau)\tau}{f^2(\gamma^*)}\right\}$$

   where γ^* is the τth quantile and f is the pdf of ε

Risk-Return Asset Allocation

Log returns $X_t \in \mathbb{R}^p$:

$$\begin{aligned} \min_{w_t \in \mathbb{R}^p} \quad & \sigma^2_{P,t}(w_t) \stackrel{\text{def}}{=} w_t^\top \Sigma_t w_t \\ \text{s.t.} \quad & \mu_{P,t}(w_t) = r_T, \\ & w_t^\top 1_p = 1, \\ & w_{i,t} \geq 0 \end{aligned} \tag{6}$$

where r_T "target" return, $\Sigma_t \stackrel{\text{def}}{=} E_{t-1}\{(X_t - \mu)(X_t - \mu)^\top\}$, Σ_t is modeled with a GARCH model Details Back to "Benchmark Strategies"
Return to "TEDAS Gestalten"

The Orthogonal GARCH Model

- $X_t \in \mathbb{R}^{n\times p}$, $\Gamma_t = B_t \in \mathbb{R}^{p\times p}$ matrix of standardized eigenvectors of $n^{-1}X_t^\top X_t$ ordered according to decreasing magnitude of eigenvalues
- $F_t = P_t \stackrel{\text{def}}{=} X_t\Gamma_t$ PCs of X_t
- factors f, introduce noise u_i, i.e. $y_j = b_{j1}f_1 + b_{j2}f_2 + \ldots + b_{jk}f_k + u_i$ or $Y_t = F_t B_t^\top + U_t$
- then $\Sigma_t = \mathrm{Var}(X_t) = \mathrm{Var}(F_t B_t^\top) + \mathrm{Var}(U_t) = B_t\Delta_t B_t^\top + \Omega_t$, $\Delta_t = \mathrm{Var}(F_t)$ diagonal matrix of PC variances at t

Return to "Risk-Return Asset Allocation"

Dynamic Conditional Correlations Model

Assume: $r_t | \mathcal{F}_{t-1} \sim \mathrm{N}(0, D_t R_t D_t)$, $\varepsilon_t = D_t^{-1} r_t$,

$$D_t^2 = \mathrm{diag}(\omega_i) + \mathrm{diag}(\alpha_i) \odot r_{t-1} r_{t-1}^\top + \mathrm{diag}(\beta_i) \odot D_{t-1}^2,$$

$$Q_t = S \odot (1 1^\top - A - B) + A \odot \{P_{t-1} \varepsilon_{t-1} \varepsilon_{t-1}^\top P_{t-1}\} + B \odot Q_{t-1},$$

$$R_t = \{\mathrm{diag}(Q_t)\}^{-1} Q_t \{\mathrm{diag}(Q_t)\}^{-1}$$

where $r_t \in \mathbb{R}^p$, $D_t = diag(\sigma_{it}) \in \mathbb{R}^{p \times p}$, $\varepsilon_t \in \mathbb{R}^p$ standardized returns with $\varepsilon_{it} \stackrel{\mathrm{def}}{=} r_{it} \sigma_{it}^{-1}$, 1 vector of ones; $P_{t-1} \stackrel{\mathrm{def}}{=} \{\mathrm{diag}(Q_t)\}^{1/2}$, ω_i, α_i, β_i, A, B coefficients, $\odot$ Hadamard (elementwise) product

▸ Return to "TEDAS Gestalten"

The DCC Model - Continued

- correlation targeting: $S = (1/T)\sum_{t=1}^{T} \varepsilon_t \varepsilon_t^\top$
- $Q_0 = \varepsilon_0 \varepsilon_0^\top$ positive definite, each subsequent Q_t also positive definite
- consistent but inefficient estimates: the log-likelihood function

$$L(\theta, \phi) = -\frac{1}{2} \sum_{t=1}^{T} \left\{ n \log(2\pi) + 2 \log |D_t| + \log |R_t| + \varepsilon_t^\top R_t^{-1} \varepsilon_t \right\},$$

where θ parameters in D and ϕ additional correlation parameters in R Back

The DCC Model - Continued

Re-write:

$$L(\theta, \phi) = L_V(\theta) + L_C(\theta, \phi),$$

with volatility part $L_V(\theta)$ and correlation part $L_C(\theta, \phi)$,

$$\begin{aligned} L_V(\theta) &= -\frac{1}{2} \sum_{t=1}^{T} \left\{ n \log(2\pi) + \log |D_t|^2 + r_t^\top D_t^{-2} r_t \right\} \\ &= -\frac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{d} \left\{ \log(2\pi) + \log(\sigma_{it}^2) + \frac{r_{it}^2}{\sigma_{it}^2} \right\}, \end{aligned}$$

$$L_C(\theta, \phi) = -\frac{1}{2} \sum_{t=1}^{T} \left\{ \log |R_t| + \varepsilon_t^\top R_t^{-1} \varepsilon_t - \varepsilon_t^\top \varepsilon_t \right\}.$$

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Cornish-Fisher VaR Optimization

Log returns $X_t \in \mathbb{R}^p$:

$$\begin{aligned} \underset{w \in \mathbb{R}^d}{\text{minimize}} \quad & W_t\{-q_\alpha(w_t) \cdot \sigma_p(w_t)\} \\ \text{subject to} \quad & w_t^\top \mu = \mu_p, \;\; w_t^\top 1 = 1, \;\; w_{t,i} \geq 0 \end{aligned}$$

here $W_t \stackrel{\text{def}}{=} W_0 \cdot \prod_{j=1}^{t-1} w_{t-j}^\top (1 + X_{t-j})$, $\tilde{w}$, W_0 initial wealth,
$\sigma_p^2(w) \stackrel{\text{def}}{=} w_t^\top \Sigma_t w_t$,

$$q_\alpha(w) \stackrel{\text{def}}{=} z_\alpha + (z_\alpha^2 - 1)\frac{S_p(w)}{6} + (z_\alpha^3 - 3z_\alpha)\frac{K_p(w)}{24} - (2z_\alpha^3 - 5z_\alpha)\frac{S_p(w)^2}{36},$$

here $S_p(w)$ skewness, $K_p(w)$ kurtosis, z_α is N$(0, 1)$ α-quantile If $S_p(w)$, $K_p(w)$ zero, then obtain Markowitz allocation

▸ Return to "TEDAS Gestalten"

Risk Parity (Equal risk contribution)

Let $\sigma(w) = \sqrt{w^\top \Sigma w}$. Euler decomposition:

$$\sigma(w) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \sigma_i(w) = \sum_{i=1}^{n} w_i \frac{\sigma(w)}{\partial w_i}$$

where $\frac{\sigma(w)}{\partial w_i}$ is the marginal risk contribution and $\sigma_i(w) = w_i \frac{\sigma(w)}{\partial w_i}$ the risk contribution of i-th asset. The idea of ERC strategy is to find risk balanced porfolio, such that:

$$\sigma_i(w) = \sigma_j(w)$$

i.e. the risk contribution is the same for all assets of the portfolio

Return to "Benchmark Strategies"

60/40 allocation strategy

60/40 portfolio allocation strategy implies the investing of 60% of the portfolio value in stocks (often via a broad index such as S&P500) and 40% in government or other high-quality bonds, with regular rebalancing to keep proportions steady.

Return to "Benchmark Strategies"

Regularity Conditions for Adaptive Lasso QR

A1 Sampling and smoothness: $\forall x$ in the support of X_i, $\forall y \in \mathbb{R}$, $f_{Y_i|X_i}(y|x)$, $f \in \mathcal{C}^k(\mathbb{R})$, $|f_{Y_i|X_i}(y|x)| < \overline{f}$, $|f'_{Y_i|X_i}(y|x)| < \overline{f'}$; $\exists \underline{f}$, such that $f_{Y_i|X_i}(x^\top \beta_\tau|x) > \underline{f} > 0$

A2 Restricted identifiability and nonlinearity: let $\delta \in \mathbb{R}^p$, $T \subset \{0, 1, ..., p\}$, δ_T such that $\delta_{Tj} = \delta_j$ if $j \in T$, $\delta_{Tj} = 0$ if $j \notin T$; $T = \{0, 1, ..., s\}$, $\overline{T}(\delta, m) \subset \{0, 1, ..., p\} \backslash T$, then $\exists m \geq 0$, $c \geq 0$ such that

$$\inf_{\boldsymbol{\delta} \in A, \boldsymbol{\delta} \neq 0} \frac{\delta^\mathsf{T} \,\mathsf{E}(X_i X_i^\top)\delta}{\|\delta_{T \cup \overline{T}(\delta,m)}\|^2} > 0, \qquad \frac{3\underline{f}^{3/2}}{8\overline{f}'} \inf_{\delta \in A, \delta \neq 0} \frac{\mathsf{E}[|X_i^\mathsf{T}\delta|^2]^{3/2}}{\mathsf{E}[|X_i^\mathsf{T}\delta|^3]} > 0,$$

where $A \stackrel{\text{def}}{=} \{\delta \in \mathbb{R}^p : \|\delta_{T^c}\|_1 \leq c\|\delta_T\|_1, \|\delta_{T^c}\|_0 \leq n\}$

Back

Regularity Conditions - Continued

A3 Growth rate of covariates:

$$\frac{q^3\{\log(n \vee p)\}^{2+\eta}}{n} \to 0, \eta > 0$$

A4 Moments of covariates: Cramér condition

$$\mathsf{E}[|x_{ij}|^k] \leq 0.5 C_m M^{k-2} k!$$

for some constants C_m, M, $\forall k \geq 2$, $j = 1, ..., p$

A5 Well-separated regression coefficients: $\exists b_0 > 0$, such that $\forall j \leq q$, $|\hat{\beta}_j| > b_0$

Notation

- $\hat{q}_\tau \stackrel{\text{def}}{=} \widehat{F}_n^{-1}(\tau)$, with

$$\widehat{F}_n(Y_t) \stackrel{\text{def}}{=} \int_{-\infty}^{Y_t} \hat{f}_n(u)\,du = \frac{1}{n}\sum_{i=1}^{n} H\left(\frac{Y_t - Y_i}{h}\right), \qquad (7)$$

  where $\hat{f}_n(Y_t) \stackrel{\text{def}}{=} (1/nh)\sum_{i=1}^{n} K\{(Y_t - Y_i)/h\}$,
  $H(x) = \int_{-\infty}^{x} K(u)du$, $K(\cdot) = \varphi(\cdot)$;
  Silverman (1986) rule-of-thumb:

$$h = 1.06 s n^{-1/5}\ , \ s \text{ sample standard deviation of } Y$$

- $\hat{\beta}_{\tau,\lambda_n}$ are the estimated non-zero ALQR coefficients

▸ Back to Motivation

References

W. K. Härdle, S. Nasekin, D. K. C. Lee , K. F. Phoon

TEDAS - Tail Event Driven Asset Allocation

manuscript ID, 15-239, submitted to Journal of Empirical Finance. 23.07.2015

W. K. Härdle, S. Nasekin, D. K. C. Lee , X. Ni and A. Petukhina

Tail Event Driven ASset allocation: evidence from equity and mutual funds markets

manuscript ID, 15-239, submitted to Journal of Asset management. 09.10.2015

References

H. Markowitz,

Portfolio Selection

Journal of Finance, Vol. 7, No. 1 (Mar., 1952), pp. 77-91

R. A. Schüssler

Dynamic Optimization of Asset Allocation Strategies under Downside Risk Control: An Application to Futures Markets

http://ssrn.com/abstract=2502383

References

Qi Zheng, Colin Gallagher, K.B. Kulasekera

Adaptive Penalized Quantile Regression for High-Dimensional Data

Journal of Statistical Planning and Inference, 143 (2013) 1029-1038

Hui Zou

The Adaptive Lasso and Its Oracle Properties

Journal of the American Statistical Association, Dec., 2006, Vol. 101, No. 476