Uniform Confidence for Pricing Kernels

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Motivation

- Arbitrage free market; riskless bond with constant interest rate r; risky derivatives,
- Underlying price process S_t, $t \in [0, T]$,
- How to price the derivative ?

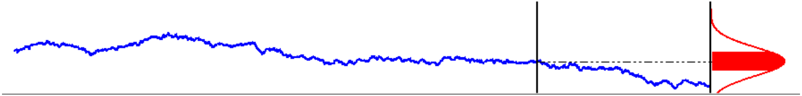


Figure 1: conditional measure at time of maturity T built opon a path of the stochastic process for underlying asset with information up to time t .

Risk Neutral Price z_t at time t from a payoff $\psi(S_T)$

$$
\begin{aligned}
z_t &= \int_0^\infty exp(-r\tau)\psi(S_T)dQ(S_T)|S_t = s_t \\
&= \int_0^\infty exp(-r\tau)\psi(S_T)\frac{q_t(S_T)}{p_t(S_T)}dP(S_T)|S_t = s_t
\end{aligned}
$$

where

- $p_t(S_T) \stackrel{\text{def}}{=} p(S_T)|S_t = s_t$ subjective density function,
- $q_t(S_T) \stackrel{\text{def}}{=} q(S_T)|S_t = s_t$ risk neutral density function.

Empirical Pricing Kernel (EPK)

Pricing Kernel (PK) a stochastic discount factor., i.e.

$$\pi_{t,\tau} = exp(-r\tau)\frac{q_t(S_T)}{p_t(S_T)}$$

EPK is therefore an estimation of PK:

$$\widehat{\pi_{t,\tau}} = exp(-r\tau)\frac{\widehat{q_t(S_T)}}{\widehat{p_t(S_T)}}$$

Empirical Pricing Kernel

Estimation approach:

- Black-Scholes Model, Black-Scholes(1973)
- GARCH Model, Engle(1982)
- Nonparametric diffusion model, Ait-Sahalia(1997)

All lead to the same paradoxa.

How true is the EPK Paradoxa?

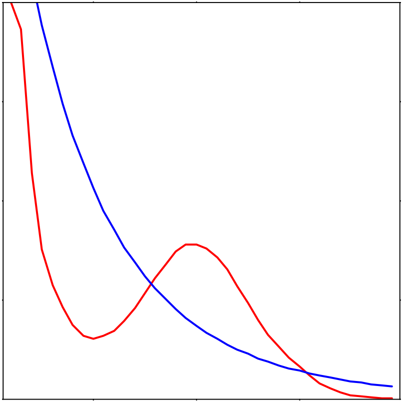


Figure 2: Empirical and theoretical pricing kernel, DAX 19990205, $\tau = 10$ days.

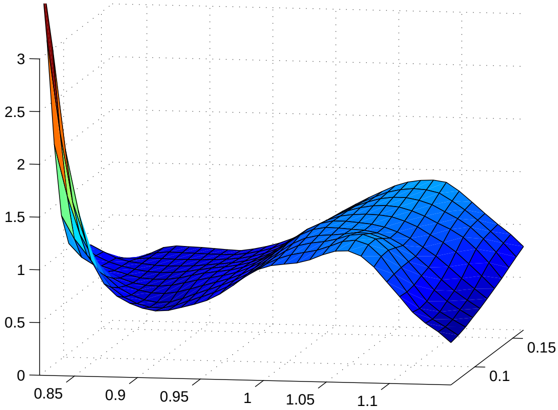
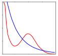


Figure 3: Estimated PK across moneyness and maturity, DAX on 20010710



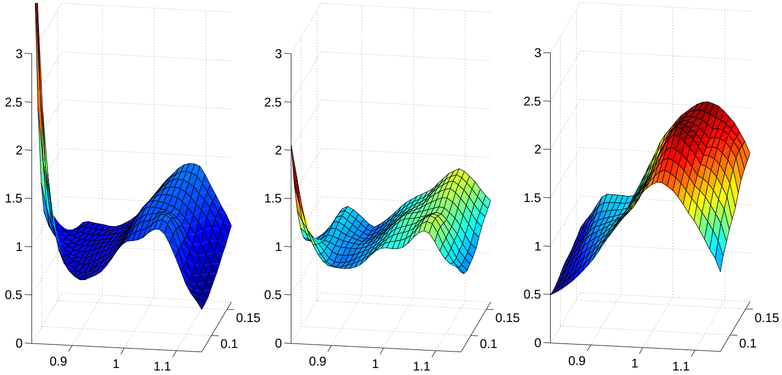


Figure 4: Empirical PK across moneyness and marurity , estimated form DAX on 20010710, 20010904 and 20011130

Our aims

- Nonparametric estimation for EPK
- Use uniform confidence band to do simple visual inspection
- Bootstrap to solve the slow convergence problem

Outline

1. Motivation ✓
2. Uniform Confidence Band
3. Bootstrap
4. Empirical Results

Risk Neutral Density (RND) Estimation

In Breeden and Litzenberger (1978), RND could be estimated from the option prices:

$$q_t(S_T) = exp(r\tau)\frac{\partial^2 C_t(k,\tau)}{\partial k^2}|_{k=S_T}$$

with option price function $C_t(k,\tau)$; strike price k.

Ait-Sahalia and Lo (1997) purposes to estimate $C_t(k,\tau)$ nonparametrically and then differentiate it twice w.r.t. k.

For RND, fix τ, we have:

$$Y_i = C(k_i) + \varepsilon_i, i = 1, \ldots, n_q \tag{1}$$

With call option price function $C(k_i)$ and option price Y_i. ε_is are i.i.d.

Under some regularity conditions, the estimate for RND is the local polynomial estimate of the second derivative of $C(k_i)$.

For p, the kernel density estimate is based on historical underlying assets prices,

$$\widehat{p_t(S_T)} = n_p^{-1} \sum_{j=1}^{n_p} K_{h_{n_p}}(S - S_j)$$

Convergence Rate

In Stone (1980), under some regularity conditions,

$$\sup_{S_T} |\widehat{p_t(S_T)} - p_t(S_T)| = \mathcal{O}_p\{(n_p / \log n_p)^{-1/(2p+1)}\} = \mathcal{O}_p(b_p),$$

assuming Lipschitz continuous with parameter p.
In Claeskens and Keilegom (2003), under some regularity conditions,

$$\sup_{S_T} |\widehat{q_t(S_T)} - q_t(S_T)| = \mathcal{O}((n_q \log n_q)^{-2/11}) = \mathcal{O}_p(b_q)$$

As in our setting, $\frac{b_p}{b_q} \ll 1$, so we have

$$|\frac{\widehat{q_t(S_T)}}{\widehat{p_t(S_T)}} - \frac{q_t(S_T)}{p_t(S_T)}| \approx |\frac{\hat{q}p - \hat{p}q}{p^2}|$$

Uniform Confidence Band

Thus, we have our $(1-\alpha)100\%$ confidence band for pricing kernel π_τ as

$$[\psi : \sup_{S_T}\{|2!\widehat{\pi_\tau(S_T)} - \psi_\tau(S_T)|\,\widehat{Var(\hat{q})}^{-0.5}\} \leq L_\alpha]$$

where

$$L_\alpha = 2!\hat{p}(n_q h_{n_q}^5)^{-0.5}\Big[(-2\log h_{n_q})^{0.5} + (-2\log h_{n_q})^{-0.5}\{x_\alpha + \log(C/2\pi)\}\Big]$$

and

$$x_\alpha = -\log\{-0.5\log(1-\alpha)\}$$

Challenge 1: Extend to τ direction(or higher dimension.)

- Make the ratio of the EPKs bounded by a function of τ, reparametrize?
- Single index model?

Challenge 2: Convexification of $\widehat{q_t(S_T)}$

- Numerical approach to clean data?
- Restricted optimization?

Bootstrap Confidence Band

The convergence rate to the limiting distribution is quite slow. Bootstrap method remedies this problem:

- Bootstrap Estimation of the leading term
- Bootstrap Method: Smooth Bootstrap or Wild Bootstrap