

An Axiomatic and Data Driven View on the EPK Paradox

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Motivation

□ Pricing kernel (PK)

- ▶ Consumption based models
 - marginal rate of consumption substitution
- ▶ Arbitrage free models
 - Radon-Nikodym derivative of the physical measure w.r.t. the risk neutral measure

▶ Risk Neutral Valuation

▶ PK - Black-Scholes

□ Empirical pricing kernel (EPK)

- ▶ Any estimate of the PK
- ▶ EPK paradox - locally increasing EPK



EPK Paradox

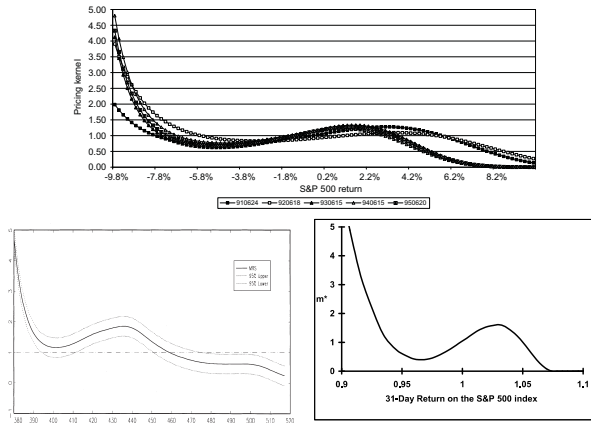


Figure 1: EPK's: Engle and Rosenberg (2002), Ait-Sahalia and Lo (2000), Brown and Jackwerth (2004)



EPK Paradox

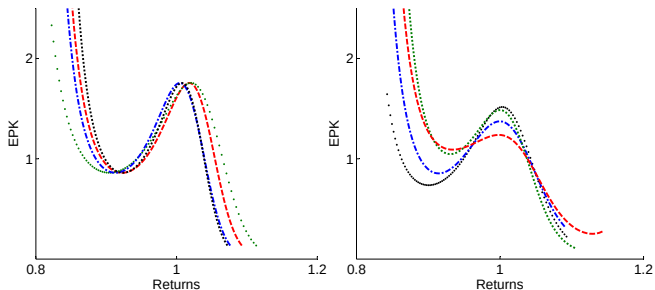


Figure 2: EPK's for various maturities (left) and different estimation dates for fixed maturity 1M (right), Grith et al. (2010)



EPK Paradox

Figure 3: EPK's across moneyiness κ and maturity τ for DAX from 20010101 – 20011231, Giacomini and Härdle (2008)



EPK Paradox

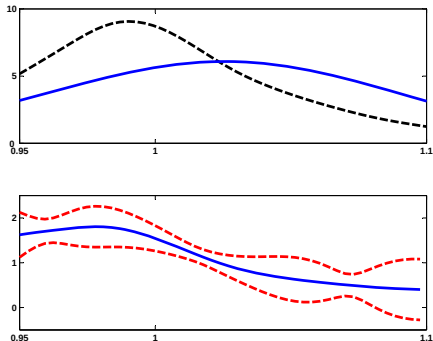


Figure 4: Upper panel: estimated risk neutral density $\hat{q}$ and historical density $\hat{p}$. Lower panel: EPK and 95% uniform confidence bands on 20080228, Härdle et al. (2010)



Objectives

- Pricing kernel derivation
 - ▶ Adjust individual and aggregate preferences
 - ▶ State-dependent (state variable: market return)
 - ▶ Simulation study

- Fitting EPK's
 - ▶ Identifiability of parameters
 - ▶ Empirical study



Research Questions

- How to modify standard expected utility theory to rationalize the EPK paradox?
- How well can 'observed' EPK's be fitted?
- How sensitive are results with respect to the preference parameters?
- How to estimate the time variation of estimated parameters/functions?



Outline

1. Motivation ✓
2. Microeconomic Framework
3. Pricing Kernel
4. Fitting EPK's
5. Empirical Study
6. Statistical Properties
7. Conclusions



Assumptions

□ Financial markets

- ▶ Finite investment time horizon $[0, T]$ and r risk free interest rate
- ▶ Risky asset with prices $\{S_t\}_{0 \leq t \leq T}$ and return $R_T = S_T/S_0$
- ▶ Arbitrage free market, at least one equivalent martingale measure with density π

□ m Consumers

- ▶ Endowment $e_i(R_T)$ and consumption $c_i(R_T)$, $i = 1, \dots, m$
- ▶ State-dependent utility function



State-Dependent Utility - Literature Review

□ Axiomatisation

- ▶ Dreze and Rustichini (2004)
- ▶ Evans and Viscusi (1991)
- ▶ Mas-Colell, Winston und Green (1995)

□ Empirical evidence

- ▶ Karni, Schmeidler and Vind (1983)



Individual Preferences

- Consumer i 's extended expected utility, Mas-Colell et al. (1995)

$$U^i \{c_i(R_T)\} = E [u^i \{R_T, c_i(R_T)\}] ,$$

with $u^i : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ - state dependent utility index

$$u^i \{R_T, c_i(R_T)\} = u_i^0 \{c_i(R_T)\} \mathbf{1} \{R_T \in [0, x_i]\} + u_i^1 \{c_i(R_T)\} \mathbf{1} \{R_T \in (x_i, \infty)\}$$

$x_i \in [0, \infty)$ - reference point of consumer i ; $x_1 \leq \dots \leq x_m$

$u_i^0, u_i^1 : \mathbb{R}_+ \rightarrow \mathbb{R}$ - utility indices

- strictly increasing, concave and twice cts differentiable



Individual Preferences

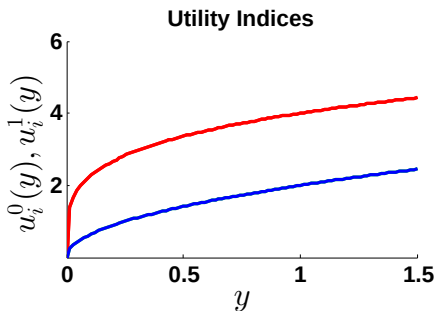


Figure 5: Utility indices $u_i^0(y) = y^{0.25}/0.25$ (bearish market) and $u_i^0(y) = y^{0.50}/0.50$ (bullish market)



Equilibrium

- Individual optimization

$$\bar{c}_i(R_T) = \arg \max_{c_i(R_T)} U^i \{c_i(R_T)\}$$

- Market clearing

$$\sum_{i=1}^m \bar{c}_i(R_T) = \sum_{i=1}^m e_i(R_T) \stackrel{\text{def}}{=} \bar{e}(R_T)$$

- ▶ Pareto optimal $\bar{c}_1(R_T), \dots, \bar{c}_m(R_T)$



Aggregated Preferences

- Aggregated extended expected preferences

$$U_{\alpha} \{ \bar{e}(R_T) \} = E [u_{\alpha} \{ R_T, \bar{e}(R_T) \}],$$

with $u_{\alpha} : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ - aggregated indirect utility

$$\begin{aligned} u_{\alpha} \{ R_T, \bar{e}(R_T) \} = & u_{\alpha,1} \{ \bar{e}(R_T) \} \mathbf{I} \{ R_T \in [0, x_1] \} + \\ & + \sum_{i=1}^{m-1} u_{\alpha,i+1} \{ \bar{e}(R_T) \} \mathbf{I} \{ R_T \in (x_i, x_{i+1}] \} + \\ & + u_{\alpha,m+1} \{ \bar{e}(R_T) \} \mathbf{I} \{ R_T \in (x_m, \infty) \} \end{aligned}$$

$$u_{\alpha,j} \{ \bar{e}(R_T) \} = \sum_{k=1}^m \alpha_k u_k^0 \{ \bar{c}_k(R_T) \} \mathbf{I} \{ k \geq j \} + \sum_{k=1}^m \alpha_k u_k^1 \{ \bar{c}_k(R_T) \} \mathbf{I} \{ k < j \}$$

for $j = 1, \dots, m+1$ and importance weights $\alpha = (\alpha_1, \dots, \alpha_m)^{\top}$



Pricing Kernel

Theorem

If $\bar{e}(r_T) = r_T$ then for every realization r_T of R_T

$$\begin{aligned}\tilde{\mathcal{K}}_{\pi}(r_T) = & \left. \frac{\partial u_{\alpha,1}\{y\}}{\partial y} \right|_{y=r_T} \mathbf{I}\{r_T \in [0, x_1]\} + \\ & + \sum_{i=1}^{m-1} \left. \frac{\partial u_{\alpha,i+1}\{y\}}{\partial y} \right|_{y=r_T} \mathbf{I}\{r_T \in (x_i, x_{i+1}]\} + \\ & + \left. \frac{\partial u_{\alpha,m+1}\{y\}}{\partial y} \right|_{y=r_T} \mathbf{I}\{r_T \in (x_m, \infty)\}.\end{aligned}$$

Note: $\tilde{\mathcal{K}}_{\pi}(r_T)$ is nonincreasing separately on the intervals $[0, x_1], (x_1, x_2], \dots, (x_m, \infty)$ but may be nonmonotone at x_i 's



Example 1

Example 1. Consider m investors with identical reference point x_1 that switch between constant relative risk aversion (CRRA) utilities $u_i^0(y) = y^{\gamma_i^0} / \gamma_i^0$ and $u_i^1(y) = y^{\gamma_i^1} / \gamma_i^1$, $0 < \gamma_i^0 < \gamma_i^1 < 1$.

$$\tilde{\mathcal{K}}_{\pi}(r_T) = r_T^{\gamma_{\alpha}^0 - 1} \mathbf{I}\{r_T \in [0, x_1]\} + r_T^{\gamma_{\alpha}^1 - 1} \mathbf{I}\{r_T \in (x_1, \infty)\},$$

$$1 - \gamma_{\alpha}^{\ell} = r_T / \sum_{i=1}^m \frac{\bar{c}_i(r_T)}{\gamma_i^{\ell} - 1}, \quad \ell = \{0, 1\} \text{ - implied CRRA coeff's}$$



Example 1

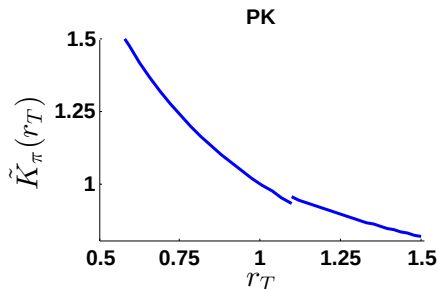
[▶ R code](#)

Figure 6: Pricing kernel $\tilde{K}_\pi(r_T)$ for $x_1 = 1.1$ and $\gamma_\alpha^0 = 0.25 < \gamma_\alpha^1 = 0.50$



Example 2

Example 2. Consider m investors with possibly different reference points x_i 's that switch between CRRA utilities $u^0(y) = b_0 \frac{y^\gamma}{\gamma}$ and $u^1(y) = b_1 \frac{y^\gamma}{\gamma}$. Let $F(r_T)$ be the cdf of the reference points

$$F(r_T) = m^{-1} \sum_{i=1}^m \mathbf{I}\{x_i \leq r_T\}$$

$$\mathcal{K}_{v,F}(r_T) = \tilde{\mathcal{K}}_\pi(r_T) = \left[\frac{r_T}{\{1 - F(r_T)\} b_0^{\frac{1}{1-\gamma}} + F(r_T) b_1^{\frac{1}{1-\gamma}}} \right]^{\gamma-1} \quad (1)$$

for parameters $v = (\gamma, b_0, b_1)^\top$, $0 < b_0 \leq b_1$



Example 2

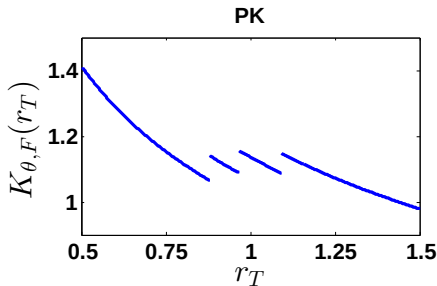
[▶ R code](#)

Figure 7: Pricing kernel $\mathcal{K}_{v,F}(r_T)$ with $\gamma = 0.5$, $b_0 = 1$, $b_1 = 1.2$ and $m = 3$ with uniformly generated reference points



Example 2

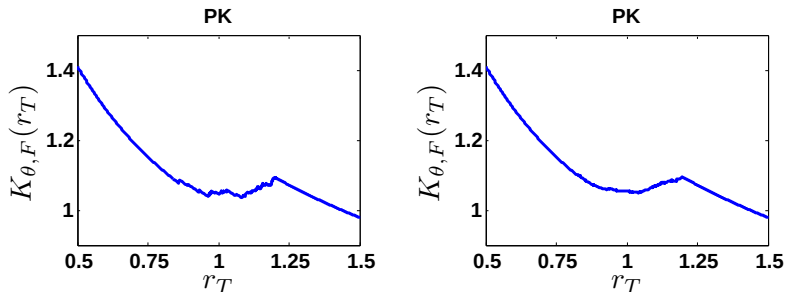
[► R code](#)

Figure 8: Pricing kernel $K_{\nu, F}(r_T)$ with $\gamma = 0.5$, $b_0 = 1$, $b_1 = 1.2$ and $m = 40$ (left) and $m = 400$ (right) with reference points generated from a triangular distribution



Example 2

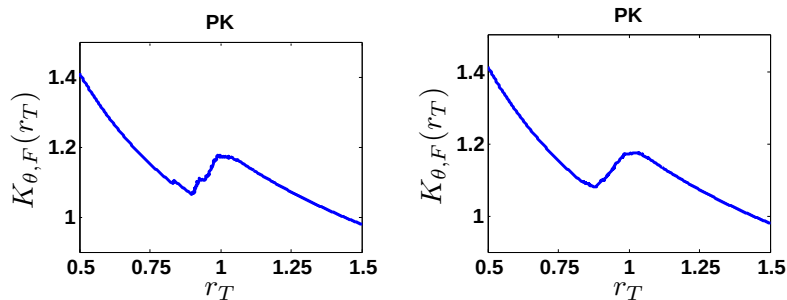
[► R code](#)

Figure 9: Pricing kernel $K_{v, F}(r_T)$ with $\gamma = 0.5$, $b_0 = 1$, $b_1 = 1.2$ and $m = 40$ (left) and $m = 400$ (right) with reference points generated from a normal distribution $N(0.95, 0.05)$



Example 2

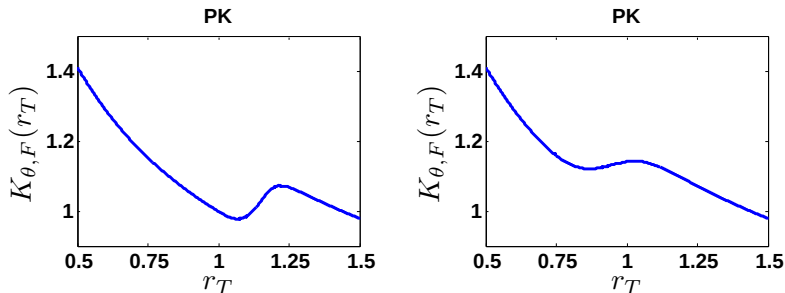


Figure 10: Pricing kernel $K_{v,F}(r_T)$ with $\gamma = 0.5$, $b_0 = 1$, $b_1 = 1.2$ and $m = 4000$ with reference points generated from a normal distribution $N(1.15, 0.05)$ (left) and $N(0.95, 0.10)$ (right)



Risk Aversion

- Arrow-Pratt coefficients of absolute risk aversion (ARA) at x_i 's from the left and from the right

$$\lim_{\delta \rightarrow 0_+} \frac{\tilde{\mathcal{K}}_{\pi}(x_i - \delta) - \tilde{\mathcal{K}}_{\pi}(x_i)}{\delta \tilde{\mathcal{K}}_{\pi}(x_i)}, \quad \lim_{\delta \rightarrow 0_+} \frac{\tilde{\mathcal{K}}_{\pi}(x_i + \delta) - \tilde{\mathcal{K}}_{\pi}(x_i)}{\delta \tilde{\mathcal{K}}_{\pi}(x_i)}$$



Risk Aversion

2 graphs that pertain to Example 1 (jump in ARA) and 2 (no jump in ARA)



Fitting EPK's

- Find v and F that minimize

$$\sum_{j=1}^n \left\{ \hat{\mathcal{K}}(s_j) - \mathcal{K}_{v,F}(s_j) \right\}^2 \quad (2)$$

for the estimate $\hat{\mathcal{K}}$ at points $\{s_j\}_{j=0}^n$

$$\mathcal{K}_{v,F}(x) = \left[\frac{x}{\{1 - F(x)\} b_0^{\frac{1}{1-\gamma}} + F(x) b_1^{\frac{1}{1-\gamma}}} \right]^{\gamma-1}$$

with $v = (\gamma, b_0, b_1)^\top$ and F cdf.



Parameters Identifiability

For $\delta = 1 - \gamma > 0$ and $b_0 \leq b_1$

$$x\mathcal{K}_{v,F}^{\frac{1}{\delta}}(x) = \{1 - F(x)\} b_0^{\frac{1}{\delta}} + F(x) b_1^{\frac{1}{\delta}} \quad (3)$$

is a monotonically increasing function bounded between $b_0^{\frac{1}{\delta}}$ and $b_1^{\frac{1}{\delta}}$.

- For discrete reference points v is identifiable ► Discrete RP
- For F continuous v is not identifiable



Data

Financial markets

- ▶ EUREX European option data on 20000920 and 20060621
- ▶ Daily DAX returns - past 500 observations until 20000920 and 20060621 respectively

Pricing kernels

- ▶ $\hat{\mathcal{K}}(r_T)$ - Grith et al. (2010)
- ▶ $\mathcal{K}_{v,F}(r_T) = \tilde{\mathcal{K}}_{\pi}(r_T)$ - semi-parametric PK (1)
- ▶ $\mathcal{K}_{\hat{v},\hat{F}}(r_T)$ - estimated $\mathcal{K}_{v,F}(r_T)$



Discrete RP

write the optimization problem in terms of c_I and x_I



Fitting Results: Discrete RP

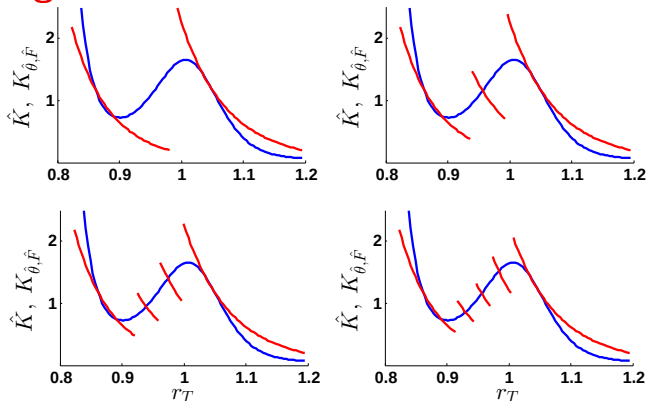


Figure 11: $\hat{K}(r_T)$ on 20000920 and $K_{\hat{v}, \hat{F}}(r_T)$ for $m = 1, 2, 3, 4$.
 $\hat{v} = (-12.61, 0.15, 2.27)^\top$



Fitting Results: Discrete RP

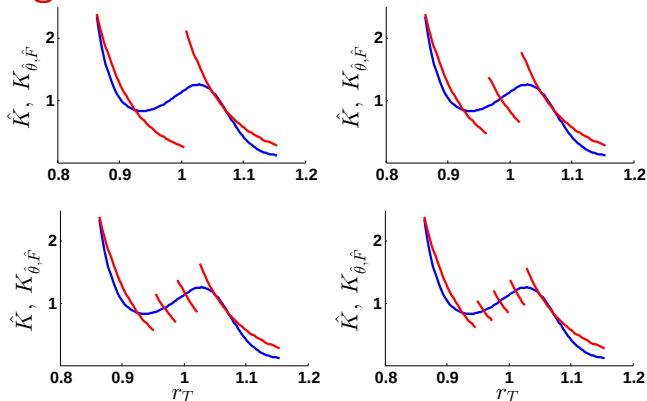


Figure 12: $\hat{K}(r_T)$ on 20060621 and $\hat{K}_{\hat{v}, \hat{F}}(r_T)$ for $m = 1, 2, 3, 4$.
 $\hat{v} = (-13.96, 0.27, 2.38)^\top$



Continuous F: Parametric Case

Assume

$$F(x) = \mathcal{F}(x) = \frac{T(x)^\phi}{\left[T(x)^\phi + \{1 - T(x)\}^\phi\right]^\phi}$$

$\phi > 0$ distortion parameters and T sigmoid distribution

$$T(x) = [1 + \exp\{-a(x - c)\}]^{-1}$$

$a > 0$ and $c \in \mathbb{R}$. Then find v and $\mathcal{F}$ that minimize

$$\sum_{j=1}^n \left\{ \hat{\mathcal{K}}(s_j) - \mathcal{K}_{v, \mathcal{F}}(s_j) \right\}^2 \quad (4)$$



Fitting Results: Continuous $\mathcal{F}$

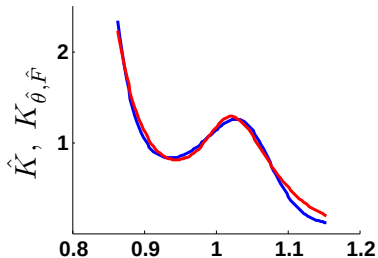


Figure 13: $\hat{K}(r_T)$ on 20060621 and $K_{\hat{\nu}, \hat{\mathcal{F}}}(r_T)$ for $\hat{\delta} = 21.10$ $\hat{b}_0 = 0.09$, $\hat{b}_1 = 3.99$, $\hat{a} = 65.01$, $\hat{c} = 0.97$, $\hat{\psi} = 0.58$



Fitting Results: F Continuous

For fixed δ find $\hat{b}_0, \hat{b}_1, \hat{a}, \hat{c}, \hat{\psi}$ minimize 2

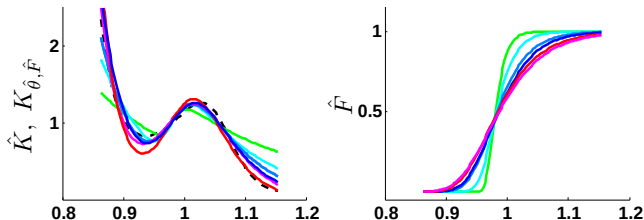


Figure 14: $\hat{K}(r_T)$ and $\hat{F}(r_T)$ on 20060621 for $\theta = 5$ (green), $\theta = 10$ (cyan), $\theta = 15$ (light blue), $\theta = 20$ (dark blue), $\theta = 25$ (magenta), $\theta = 30$ (red)



Continuous F: Semi-Parametric Case

Assume

$$F(x) = \int_0^x \sum_{k=1}^P \beta_k \psi_k(u) du = \sum_{k=1}^P \beta_k \int_0^x \psi_k(u) du = \sum_{k=1}^P \beta_k \Psi_k(x)$$

For fixed P and fixed δ find $\theta = (b_0, b_1, \beta_1, \dots, \beta_p)^\top$ that minimize

$$\sum_{j=1}^n \left\{ s_j \hat{\mathcal{K}}(s_j) - \sum_{k=1}^P \beta_k \Psi_k(s_j) (b_1^{\frac{1}{1-\gamma}} - b_0^{\frac{1}{1-\gamma}}) + b_0^{\frac{1}{1-\gamma}} \right\}^2 \quad (5)$$

under the restriction that F is a distribution.



- dynamic implementation + interpretation



Conclusions

Pricing kernel derivation

- ▣ Reference points determine jumps in the aggregate utility
- ▣ State-dependent preferences may explain the EPK paradox

Fitting EPK's

- ▣ Quality increases with the number of switching points
- ▣ Fully parametric PK specification successfully applied



Conclusions

Further Research

- ▣ Statistical estimation methodology for semi-parametric PK's
- ▣ Theoretical properties of $\hat{v}$ and $\hat{F}$
- ▣ Multidimensional reference points
- ▣ Dynamic implementation (PK's, reference points)



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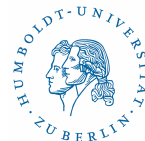
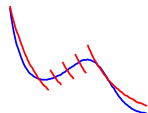
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Risk Neutral Valuation

► Motivation

- Present value of the payoffs $\psi(S_T)$

$$P_0 = E_Q \left[e^{-Tr} \psi(s_T) \right] = \int_0^\infty e^{-Tr} \psi(s_T) \mathcal{K}(s_T) p(s_T) ds_T$$

r risk free interest rate, $\{S_t\}_{t \in [0, T]}$ stock price process,
 p pdf of S_T , Q risk neutral measure, $\mathcal{K}(\cdot)$ pricing kernel



PK under the Black-Scholes Model ► Motivation

- Geometric Brownian motion for S_t

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

μ mean, σ volatility, W_t Wiener process

- Physical density p is log-normal, $\tau = T - t$

$$p_t(S_T) = \frac{1}{S_T \sqrt{2\pi\sigma^2\tau}} \exp \left[-\frac{1}{2} \left\{ \frac{\log(S_T/S_t) - \left(\mu - \frac{\sigma^2}{2} \right) \tau}{\sigma\sqrt{\tau}} \right\}^2 \right]$$

- Risk neutral density q is log-normal: replace μ by r



PK under the Black-Scholes Model ► Motivation

- PK is a decreasing function in S_T for fixed S_t

$$\begin{aligned}\mathcal{K}(S_t, S_T) &= \left(\frac{S_T}{S_t}\right)^{-\frac{\mu-r}{\sigma^2}} \exp\left\{\frac{(\mu-r)(\mu+r-\sigma^2)\tau}{2\sigma^2}\right\} \\ &= b \left(\frac{S_T}{S_t}\right)^{-\delta}\end{aligned}$$

$b = \exp\left\{\frac{(\mu-r)(\mu+r-\sigma^2)\tau}{2\sigma^2}\right\}$ and $\delta = \frac{\mu-r}{\sigma^2} \geq 0$ constant relative risk aversion (CRRA) coefficient



Example 1

► Example 1

```
# Step 1/3: Input parameters
R = t(matrix(seq(0.8, 1.2, by = 0.01), 1))
x0 <- 1.1
gamma0 <- 0.25
gamma1 <- 0.50

# Step 2/3: Define the PK
K = R[R <= x0, ] ^ (gamma0 - 1)
K2 = R[R >= x0, ] ^ (gamma1 - 1)

# Step 3/3: Plot the PK against simple gross market return
plot(R[R <= x0, ], K, type = 'l', lwd = 3, col = "blue",
     xlim = c(0.8, 1.2), ylim = c(0.8, 1.25), xlab = "r_T")
lines(R[R >= x0, ], K2, type = 'l', lwd = 3, col = "blue",
     xlim = c(0.8, 1.2), ylim = c(0.8, 1.25), xlab = "r_T")
```



Example 2

► Example 2

```
gamma = 0.5
```

```
b0 = 1
```

```
b1 = 1.2
```

```
# Step 1/3: Input parameters and F_n
```

```
n = 1000
```

```
s = seq(0.5, 1.5, 0.2/n)
```

```
m = 10 # number of switching points
```

```
x = runif(m, 0.8, 1.2)
```

```
F_n = ecdf(x)(s)
```

```
# Step 2/3: Define the PK
```

```
PK = (s/((1 - F_n)*b0^(1/(1-gamma)) + F_n*b1^(1/(1-gamma))))^(gamma-1)
```

```
# Step 3/3: Plot the PK against simple gross market return
```

```
plot( cbind(s, PK) )
```

EPK Paradox



Example 2

► Example 2

```
gamma = 0.5
```

```
b0 = 1
```

```
b1 = 1.2
```

```
# Step 1/3: Input parameters and F_n
```

```
n = 1000
```

```
s = seq(0.5, 1.5, 0.2/n)
```

```
m = 40 # number of switching points
```

```
x = 0.8 + 0.4*sqrt(runif(m))
```

```
F_n = ecdf(x)(s)
```

```
# Step 2/3: Define the PK
```

```
PK = (s/((1 - F_n)*b0^(1/(1-gamma)) + F_n*b1^(1/(1-gamma))))^(gamma-1)
```

```
# Step 3/3: Plot the PK against simple gross market return
```

```
plot( cbind(s, PK) )
```

EPK Paradox



Example 2

► Example 2

```
gamma = 0.5
b0 = 1
b1 = 1.2

# Step 1/3: Input parameters and F_n
n = 1000
s = seq(0.5, 1.5, 0.2/n)
m = 40 # number of switching points
F_n = pnorm( 20*(s-0.95) )

# Step 2/3: Define the PK
PK = (s/((1 - F_n)*b0^(1/(1-gamma)) + F_n*b1^(1/(1-gamma))))^(gamma-1)

# Step 3/3: Plot the PK against simple gross market return
plot( cbind(s, PK) )
```



Discrete RP

► Parameters Identifiability

$$F(r_T) = m^{-1} \sum_{i=1}^m \mathbf{1}\{x_i \leq r_T\}$$

For L distinct reference points $x_1 < x_2 < \dots < x_L$, on any arbitrary interval $(x_{l-1}, x_l]$ with $l = 1, \dots, L+1$

$$F(x) = F_L(x) = \text{const.} = c_l$$

Using (3)

$$x \mathcal{K}_{v,F}^{\frac{1}{\delta}}(x) = (1 - c_l) b_0^{\frac{1}{\delta}} + c_l b_1^{\frac{1}{\delta}} = \text{const.},$$

which identifies v .

