Option Implied Stock Return Distributions

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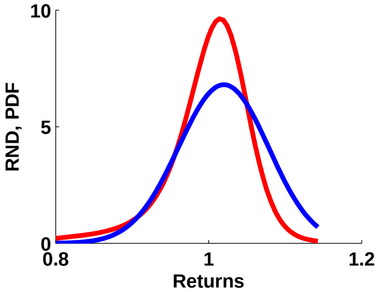

Figure 1: Physical density (red), risk neutral density (blue) DAX30 Index on 20060517

Differences between the two densities documented in: Barone-Adesi et al (2013), Campbell and Cochrane (1999), Christofferson et al (2012)

Why "yes!" to physical densities?

- decision making with respect to monetary policies
- assessment of the impact of announced or implemented changes in monetary policies
- construction of optimal portfolios

The link between physical and risk neutral densities

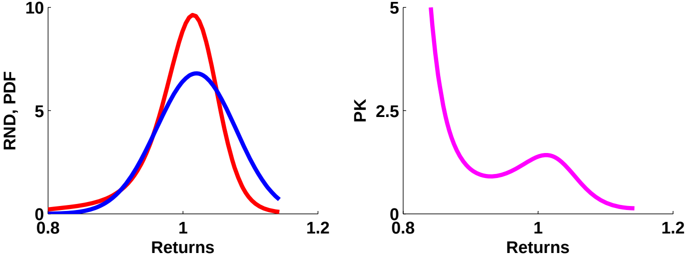

Figure 2: Physical density (red), risk neutral density (blue) (left) and corresponding pricing kernel (right) DAX30 Index on 20060517

Option implied physical densities

$$\mathcal{K}_\theta = \frac{q}{p} \rightarrow p = \frac{q}{\mathcal{K}_\theta}$$

where p physical density, q risk neutral density, $\mathcal{K}$ pricing kernel, θ unknown parameter vector

How to model the pricing kernel (PK)?

- in preference asset pricing models, the PK is proportional to the marginal utility function: $\mathcal{K} \sim U^{'}$
- representative agent (RA) characterized by an increasing, concave, continuos, twice differentiable utility function U: risk averse RA
- PK is decreasing

Bliss and Panigirtzoglou (2004): $U^{'}(S_t) = S_t^{-\gamma}$ (power) or $U^{'}(S_t) = e^{-\gamma S_t}$ (exponential)

The Empirical Pricing Kernel (EPK) Paradox

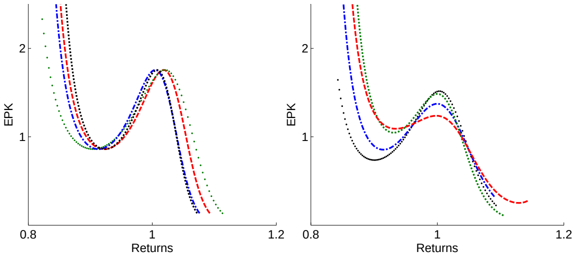

Figure 3: Intertemporal pricing kernel on European Option Market: for various maturities on 20060602 (left) and for fixed maturity one month and different estimation dates 20060215, 20060322, 20060419, 20060517 (right): Grith et al. (2012)

More evidence on the EPK Paradox

Figure 4: DAX 30 EPK's, 20010101-20011231, Giacomini and Härdle (2008)

Theoretical Explanations for the EPK puzzle

- **state dependence**: Benzoni, Collin-Dufresne & Goldstein (2011), Chabi-Yo, Garcia & Renault (2008), Christoffersen, Heston & Jacobs (2012)
- **heterogeneity in beliefs**: Ziegler (2007), Bakshi & Madan (2008), Bakshi, Madan & Panayotov (2010), Hens & Reichlin (2012)
- **misestimations/distortions**: Polkovnichenko & Zhao (2012), Hens & Reichlin (2012)
- **investors' sentiment**: Barone-Adesi, Mancini & Shefrin (2013)
- **ambiguity aversion**: Gollier (2011)
- **incomplete markets**: Hens & Reichlin (2012)

Research questions

- Does the forecasting performance of p-density improve when using a flexible pricing kernel which allows for non-monotonicity?
- Is the EPK paradox confirmed in this setting?

Outline

1. Motivation ✓
2. Methodology
3. Simulation study
4. Empirical study
5. Conclusions and further research

The model

$$p = \frac{q}{K_\theta}$$

- q is not observed, but can be estimated from option data
- The pricing kernel K is known up to some parametric specfication

Pricing kernel I

Grith, Härdle, and Krätschmer (2012)

- financial investors with reference dependent preferences
$$u_i^0(y) = \frac{y^{(1-\gamma)}}{1-\gamma} \text{ and } u_i^1(y) = b\frac{y^{(1-\gamma)}}{1-\gamma},$$
for some positive constant $b > 0$ and $\gamma > 0$ coefficient of relative risk aversion, depending on a reference point x_i in index return space; $i = 1, \ldots, m$
- cdf of reference points

$$F(r_T) = \frac{1}{m}\sum_{i=1}^{m} \mathbf{I}\left\{r_T \in (0, x_i]\right\}$$

where $R_T = \frac{S_T}{S_0}$ in a one-period model and r_T is a realization of R_T

Pricing kernel II

$$\mathcal{K}_{b,F}(r_{T,t}) = \left[\frac{r_{T,t}}{1 + F(r_{T,t})\,(b-1)}\right]^{-\gamma}$$

for every realization $r_{T,t}$ of $R_{T,t}$, the stock gross return at maturity.

F can be approximated by a mixture of known distributions. For example, let $\Phi_k(x) = \Phi\left(\frac{x-\mu_k}{\sigma_k}\right)$, where Φ is the standard normal cdf $k = 1, \ldots, L$.

$$F(x) = \int_0^x \sum_{k=1}^{L} \beta_k \phi_k(u)du = \sum_{k=1}^{L} \beta_k \int_0^x \phi_k(u)du = \sum_{k=1}^{L} \beta_k \Phi_k(x)$$

for ϕ_k densities of Φ_k

Risk neutral distribution (RND)

European call price - arbitrage free market

$$
\begin{aligned}
& C(X,\tau, rf_{t,\tau}, \delta_{t,\tau}, S_t) \qquad (1)\\
= & e^{-r_{t,\tau},\tau} \int_0^{\infty} max(S_T - X, 0) q(S_T \mid \tau, rf_{t,\tau}, \delta_{t,\tau}, S_t) dS_T
\end{aligned}
$$

S_t- underlying asset price at t, X - strike price, τ - time to maturity, $T = t + \tau$ - expiration date, $rf_{t,\tau}$ - risk free rate, $\delta_{t,\tau}$ - dividend

Breeden and Litzenberger (1978)

$$
q(S_T) = e^{r\tau} \frac{\partial^2 C}{\partial X} \bigg|_{X=S_T} \qquad (2)
$$

Estimation of RND

Rookley method: for fixed one month maturity estimate a smooth call price function with respect to the moneyness X/S_t

- implied volatility σ_{IV} substitute the call price
- $\hat{\sigma}_{IV}$, $\hat{\sigma}'_{IV}$, $\hat{\sigma}''_{IV}$ improve efficiency
- local polynomial smoothing of degree 3
- quartic kernel
- little sensitivity to the bandwidth choice

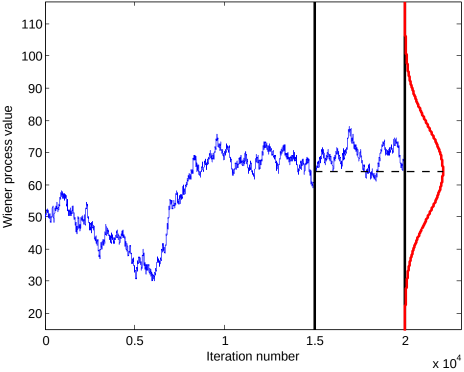

Figure 5: Probability density function of a Wiener process at a certain point in time

Estimation: two alternatives

- maximization of the p-value of Berkowitz test (evaluates the forecasting performance of the estimated densities): Bliss and Panigirtzoglou (2004), Kang and Kim (2006), Alonso et al (2009)
- maximum likelihood estimation: Liu et al (2007)

The Berkowitz test I

Required transformations

Let $\{S_t\}_{t=1}^n$ an independently and identically distributed (i.i.d.) process, with true densities $\{p_t(S_t)\}_{t=1}^n$.

- First transformation: $y_t = \int_{-\infty}^{S_t} \hat{p}_t(u)du$

$$y_t \sim i.i.d. \, \mathrm{U}(0,1)$$

- Second transformation: $z_t = \Phi^{-1}(y_t)$

Under $H_0 : \hat{p}_t(\cdot) = p_t(\cdot)$, we have

$$z_t \sim i.i.d. \, \mathrm{N}(0,1)$$

The Berkowitz test II

$$H_0 : \begin{aligned} \mu &= 0 \\ \sigma^2 &= 1 \\ \rho &= 0 \end{aligned}$$

$$H_1 : \mu \neq 0, \sigma^2 \neq 1, \rho \neq 0$$

Fit $AR(1)$ model for z_t:

$$z_t - \mu = \rho(z_{t-1} - \mu) + \varepsilon_t$$

where μ is the mean of z_t, σ^2 is the variance of ε_t and ρ is the correlation coefficient in the AR model.

The Berkowitz test III

Define likelihood ratio test:

$$LR = -2\left\{L(0,1,0) - L(\hat{\mu},\hat{\sigma}^2,\hat{\rho})\right\}$$

where L is the log-likelihood function of a Gaussian $AR(1)$ model.

▸ LogLikelihood

Under H_0 the test statistic follows a $\chi^2(3)$ distribution.

Maximum Likelihood Estimation

Define the log likelihood:

$$\ell(\theta, S_{T,1}, \ldots, S_{T,n}) = \sum_{t=1}^{n} \log \hat{p}_t(\theta, S_{T,t})$$

$$\max_{\theta} \ell(\theta, s_{T,1}, \ldots, s_{T,n})$$

where $S_{T,i}$ represents the value of the stock at the maturity of the option evaluated at time i and $s_{T,i}$ is the realization of $S_{T,i}$

Estimation of p-densities under the Black-Scholes Model

Consider a stock index which follows the process:

$$d \log S_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

with μ mean, σ volatility, W_t Wiener process Risk neutral density q is log-normal, $\tau = T - t$

$$q_t(S_T) = \frac{1}{S_T\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{1}{2}\left\{\frac{\log(S_T/S_t) - \left(rf - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right\}^2\right]$$

PK is a decreasing function in S_T for fixed S_t

$$\mathcal{K}(S_t, S_T) = \left(\frac{S_T}{S_t}\right)^{-\frac{\mu - rf}{\sigma^2}} \exp\left\{\frac{(\mu - rf)\left(\mu + rf - \sigma^2\right)\tau}{2\sigma^2}\right\}$$
$$= \beta\left(\frac{S_T}{S_t}\right)^{-\delta}$$

$\beta = \exp\left\{\frac{(\mu - rf)\left(\mu + rf - \sigma^2\right)\tau}{2\sigma^2}\right\}$ and $\delta = \frac{\mu - rf}{\sigma^2} \geq 0$ constant relative risk aversion (CRRA) coefficient

Parameter selection

$$\tau = 1/12$$
$$\sigma = 0.18$$
$$rf = 0.01$$
$$\mu = 0.03$$

Then $\beta = 1.0002$, $\delta = 0.6173$

Simulation setting: parameter δ is unknown - estimation via minimization of Berkowitz test and maximum likelihood estimation

Number of repetitions: 1000

Number of realizations considered inside each repetition: 50/ 100/ 150/ 200

$\beta = 1.0002$; $\delta =?$

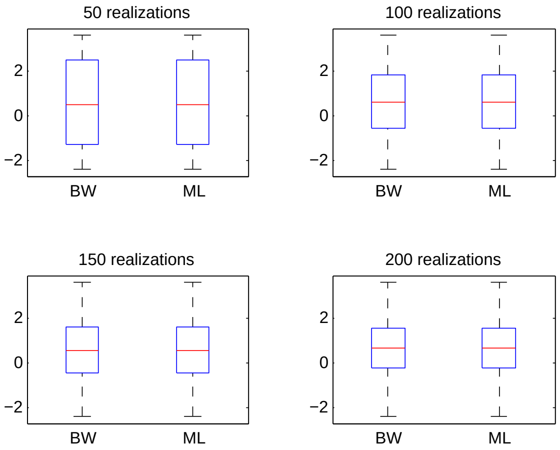

Figure 6: Boxplots of estimated parameter δ: Berkowitz test method (BW), maximum likelihood (ML) for 50, 100, 150 and respectively 200 realizations

Data

- **Source**: Reseach Data Center (RDC)
  http://sfb649.wiwi.hu-berlin.de
- Reuters DAX 30 Index opening price
- EUREX European Option Data: call/put settlement prices
- daily observations, time window length: 2002 - 2011

Data

Figure 7: Daily Risk Neutral Densities 2011; traded maturities (blue) and τ=28 days (red)

Extracting the samples...

Sample	Days	Start date	Sample	Days	Start date
1	122	20020102	11	129	20020116
2	124	20020103	12	128	20020117
3	116	20020104	13	121	20020118
4	119	20020107	14	124	20020121
5	128	20020108	15	128	20020122
6	123	20020109	16	126	20020123
7	123	20020110	17	126	20020124
8	123	20020111	18	120	20020125
9	124	20020114	19	116	20020128
10	128	20020115	20	123	20020129

Adjust the data with the non-monotonic PK

$$\mathcal{K}(r_{T,t},\theta) = \left[\frac{r_{T,t}}{1+\Phi\left(\frac{r_{T,t}-\mu}{\sigma}\right)(b-1)}\right]^{-1}$$

with fixed $\gamma = 1$, $\Phi(\mu,\sigma)$ the cdf of the normal distribution characterized by mean μ and standard deviation σ and $\theta = (b,\mu,\sigma)$ represents the vector of parameters to be estimated.

If $\widehat{\sigma} = 0$, then the model reduces to:

$$\mathcal{K}(r_{T,t},b,\mu) = \begin{cases} \frac{1}{r_{T,t}} & \text{if } r_{T,t} < \mu \\ \frac{b}{r_{T,t}} & \text{if } r_{T,t} \geq \mu \end{cases}$$

where μ corresponds to the switching point in this case.

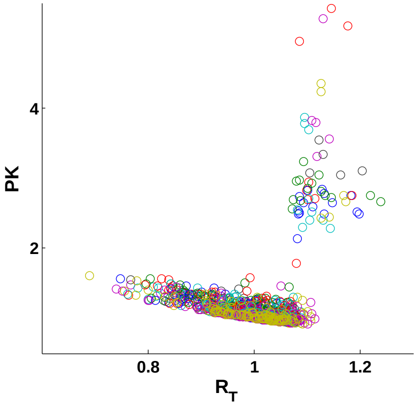

Figure 8: Realized PK for 13 out of the 20 samples

Figure 9: Risk Neutral Densities (blue) and estimated p densities (red)

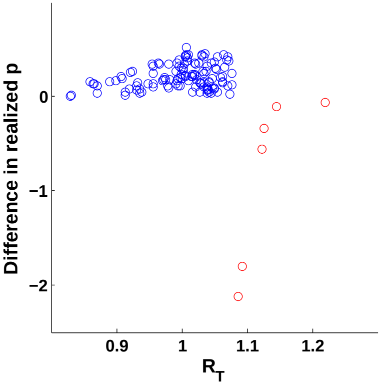

Figure 10: Differences in realized p for sample 5: realized p with non-monotonic PK and monotonic power $\mathcal{K}$. Blue circles: the non-monotonic PK outperforms monotonic power $\mathcal{K}$. Red circles: vice versa.

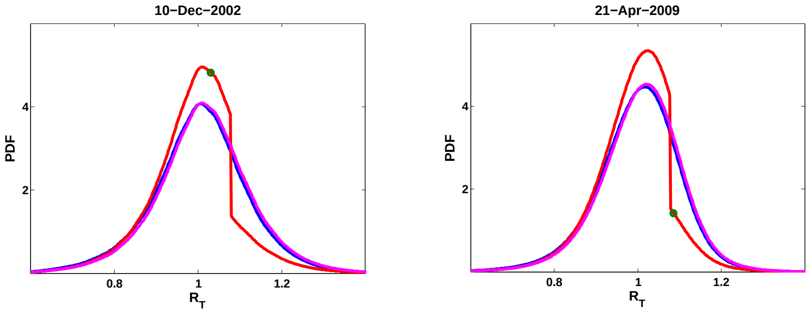

Figure 11: RND(blue), p density with non-monotonic PK (red) and p density with monotonic power PK (magenta) on 20021210 and 20090421

In sample performance: BIC values

Sample	NonM	Power	Exp	Sample	NonM	Power	Exp
1	167.47	168.39	168.35	11	195.25	195.37	195.34
2	167.77	167.78	167.79	12	193.54	191.80	191.71
3	162.82	162.03	162.00	13	188.90	185.76	185.72
4	170.59	169.71	169.66	14	182.03	181.40	181.38
5	188.29	187.05	186.99	15	187.75	187.10	187.08
6	182.48	181.95	181.90	16	181.16	180.69	180.65
7	185.03	183.99	183.97	17	183.91	183.66	183.56
8	176.80	176.42	176.40	18	167.02	167.31	167.27
9	182.44	181.98	181.91	19	158.80	159.17	159.15
10	191.99	190.67	190.63	20	167.19	166.98	166.89

Table 1: BIC for NonM, Power, Exp models. NonM, Power, Exp represent p density models with non-monotonic, power and exponential PK respectively.

Conclusions and Further Research

- the shape of the unconditional *PK* is generally decreasing, with an increasing part in the high returns domain
- the *p* density obtained from RND corrected with a flexible non-monotonic PK outperforms the *p* density from RND corrected with a monotonic PK
- behavior of non-monotonic PK model with level of $\gamma > 1$ should be further investigated
- modeling *p* density conditional on volatility would definitely be a step further

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Journal of Financial Economics In press, 2012

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Why Does Implied Risk Aversion Smile?
Review of Financial Studies **20**:3 859–904, 2007

Log likelihood function for a Gaussian $AR(1)$ process

▸ Berkowitz test Hamilton (1994)

$$
\begin{aligned}
L = & -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left\{\sigma^2 / \left(1-\rho^2\right)\right\} \\
& - \frac{[y_1 - \{\mu/(1+\rho)\}]}{2\sigma^2/(1-\rho^2)} \\
& - \{(T-1)/2\}\log(2\pi) - \{(T-1)/2\}\log\left(\sigma^2\right) \\
& - \sum_{t=2}^{T}\left\{\frac{y_t - \mu(1-\rho) - \rho y_{t-1}}{2\sigma^2}\right\}
\end{aligned}
$$