Copulae Based Factor Model for Credit Risk Analysis

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Motivation

Figure 1: Credit Risk depends the state of economy

Motivation

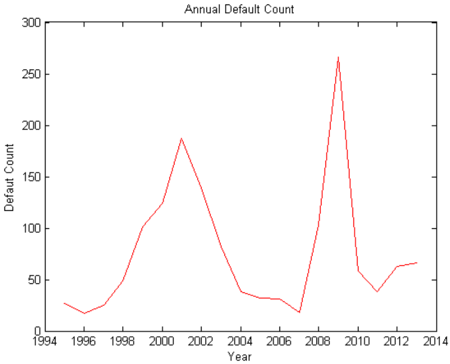


Figure 2: Annual Default Counts from 1995-2013

Motivation

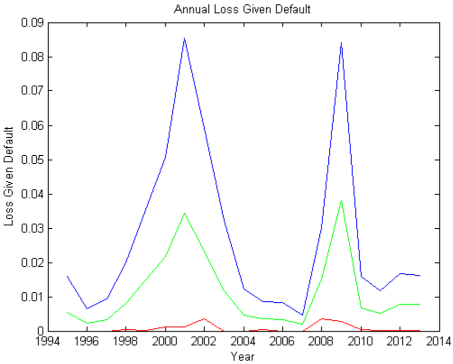
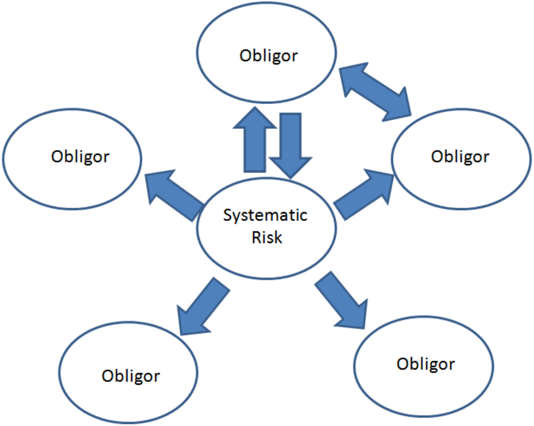


Figure 3: Annual average LGD: IG , SG and All , from 1995-2013

Motivation



Objectives

(i) Credit Risk Modeling

- Factor loading conditional on hectic and quiet state
- State-dependent recovery rate

(ii) Model Comparison

- Four models

Implication to Basel III

- Highlight systematic risk after 2008-2009 crisis
- Credit risk versus Business cycle
- How credit risk moves over the business cycle
- Contribution of systematic risk on credit risk is state-dependent

Standard Technology

- Default Event Modeling
  - Latent variable, U_i, is a linear combination of systematic and idiosyncratic vector
  - Copula enables flexible and realistic default dependent structure

Outline

1. Motivation ✓
2. Factor Copulae & Stochastic Recoveries
3. Methodology
4. Empirical Results
5. Conclusions

Factor Copulae & Stochastic Recoveries

- Factor copula model is a flexible measurement of portfolio credit risk: Krupskii and Joe (2013)
- Correlation breakdown structure: Ang and Bekaert (2002), Anderson et al. (2005)
- Recovery rate varies with the market conditions: Amraoui et al. (2012)

Candidate Models

- FC model - One-factor Gaussian copula model with constant correlation structure and constant recoveries
- RFL model - Conditional factor loading and constant recoveries
- RR model - One-factor Gaussian copula and stochastic recoveries
- RRFL model - Conditional factor loading and stochastic recoveries

Default Modeling

- One-factor non-Standardized Gaussian Copula model

$$U_i = \alpha_i Z + \sqrt{1-\alpha_i^2}\varepsilon_i \qquad i = 1, \ldots, N$$

- Z: systematic factor, ε_i: idiosyncratic factors
- Z and ε_i are independent, and ε_i is uncorrelated with each other
- The correlation coefficient between U_i and U_j is

$$\rho_{ij} = \frac{\alpha_i \alpha_j \sigma^2}{\sqrt{\alpha_i^2(\sigma^2-1)+1}\sqrt{\alpha_j^2(\sigma^2-1)+1}}$$

Default Modeling

- The default indicator

$$\mathbf{I}\{\tau_i \leq t\} = \mathbf{I}\left[U_i \leq F^{-1}\{P_i(t)\}\right]$$

- U_i: the proxies for firm asset and liquidation value
- $P_i(t)$: hazard rate and marginal probability that obligor i defaults before t.
  - From Moody's report
  - Extract from Credit spreads
  - Extract from Credit default swaps spreads

Default Modeling

- Portfolio Loss for each obligor

$$L = \sum_{i=1}^{N} G_i \mathbf{I}\{\tau_i \leq t\} = \sum_{i=1}^{N} G_i \mathbf{I}\left[U_i \leq F^{-1}\{P_i(t)\}\right]$$

- G_i is the loss given default (LGD) (i-th obligor's exposure = 1).
- $F^{-1}(\cdot)$ donates the inverse cdf of any distribution.

Copulae

- For N dimensions distribution F with marginal distribution $F_{X_1}, \cdots, F_{X_N}$, Copula function:

$$F(x_1, \cdots, x_N) = C\left\{F_{X_1}(x_1), \cdots, F_{X_N}(x_N)\right\}$$

- Gaussian Copula $C = C(\Sigma)$

Hoeffding on BBI: 

Conditional Default Model

- Conditional factor copulae model

$$U_i|_{\mathrm{S=H}} = \alpha_i^H Z + \sqrt{1-(\alpha_i^H)^2}\varepsilon_i$$

$$U_i|_{\mathrm{S=Q}} = \alpha_i^Q Z + \sqrt{1-(\alpha_i^Q)^2}\varepsilon_i$$

- Conditional default probability

$$P(\tau_i < t|\mathrm{S}) = F\left[\frac{F^{-1}\{P_i(t)\} - \alpha_i^S Z}{\sqrt{1-(\alpha_i^S)^2}}\right] = P_i(Z|\mathrm{S}) \quad \mathrm{S} \in \{\mathrm{H,Q}\}$$

- α^H, α^Q are conditional factor loading. link
- $\mathrm{P(S{=}H){=}}\omega$, $\mathrm{P(S{=}Q){=}}1-\omega$

State-dependent Recovery Rate ▸ link

- The LGD on name i, $G_i(Z)$ is related to common factor Z and the marginal default probability P_i
- Given fixed expected loss, $(1 - R_i)P_i = (1 - \bar{R}_i)\bar{P}_i$

$$G_i(Z|\mathrm{S{=}H}) = (1 - \bar{R}_i)\frac{F\left[\left\{F^{-1}\left(\overline{P}_i\right) - \alpha_i^H Z\right\}/\sqrt{1-(\alpha_i^H)^2}\right]}{F\left[\left\{F^{-1}\left(P_i\right) - \alpha_i^H Z\right\}/\sqrt{1-(\alpha_i^H)^2}\right]}$$

$$G_i(Z|\mathrm{S{=}Q}) = (1 - \bar{R}_i)\frac{F\left[\left\{F^{-1}\left(\overline{P}_i\right) - \alpha_i^Q Z\right\}/\sqrt{1-(\alpha_i^Q)^2}\right]}{F\left[\left\{F^{-1}\left(P_i\right) - \alpha_i^Q Z\right\}/\sqrt{1-(\alpha_i^Q)^2}\right]}$$

- We set $\bar{R}_i = 0$ in the simplest case.

Conditional Expected Loss

- With these two specifications, the conditional default probability $P_i(Z|\mathrm{S{=}H,Q})$ and conditional LGD, $G_i(Z|\mathrm{S{=}H,Q})$, conditional expected loss,

$$\mathrm{E}(L_i|Z) = \omega G_i(Z|\mathrm{S{=}H})P_i(Z|\mathrm{S{=}H}) + (1-\omega)G_i(Z|\mathrm{S{=}Q})P_i(Z|\mathrm{S{=}Q})$$

Monte Carlo Simulation and MSE

- One-factor non-standardized Gaussian Copula
  - $Z \sim N(-0.08, 1.02), \varepsilon_i \sim N(0, 1)$
  - Z and ε_i are generated 252 observations.
- Conditional probability that date t was belonging to the hectic is $\pi(Z = z)$

$$\mathrm{P}(S = H|Z = z) = \pi(Z = z) = \frac{\omega\varphi(z|\theta^H)}{(1-\omega)\varphi(z|\theta^Q) + \omega\varphi(z|\theta^H)}$$

- α_i^H, α_i^Q are derived from the daily stock returns of S&P 500 and of collected default companies during the crisis (2008-2009) period.
  - Five-year period prior to the crisis period is the estimation period.

Project to Default Time

- Using the definition of survival rate (Hull, 2006)

$$\tau_i|\mathrm{S} = -\frac{\log\{1 - F(U_i|\mathrm{S})\}}{P_i}$$

- P_i is the hazard rate and marginal probability that obligor i will default during the first year conditional on no earlier default obtained from Moody's report.
- $\tau_i|\mathrm{S}$ is corresponding to
  $\mathrm{E}[\mathrm{I}(\tau_i|\mathrm{S} < 1)] = \mathrm{P}(\tau_i|\mathrm{S} < 1) = P_i(Z|\mathrm{S})$.

State-dependent recovery rate simulation ▸ link

- $\overline{P}_i$ is a adjusted default probability calibrated by plugging hazard rate P_i
- $(1 - R_i)P_i = (1 - \bar{R}_i)\bar{P}_i$
- $\bar{R}_i$ is a lower bound for state-dependent recovery rates [0,1].
- We set $\bar{R}_i = 0$ in the simplest case.
- Given α_i^S and simulated Z, we generate $G_i(Z|S)$

Expected loss function

- With these two specifications, we study the expected loss function under the given scenarios

$$
\begin{aligned}
\mathrm{E}(L_i|Z) &= \pi(Z=z)G_i(Z|\mathrm{S{=}H})P_i(Z|\mathrm{S{=}H}) \\
&+ (1-\pi)(Z=z)G_i(Z|\mathrm{S{=}Q})P_i(Z|\mathrm{S{=}Q})
\end{aligned}
$$

- $\pi(Z=z)$ is a proxy of unconditional probability ω.

Estimation of the MSE

- Estimated Square Error (SE)

$$SE = (actual\ default\ loss\ - expected\ default\ loss)^2$$

- Actual default loss is from Moody's report.
- Calculate the mean of square errors referred to as Mean Square Error (MSE)
- Compare minimum MSE to evaluate FC, RFL, RR, and RRFL model

Data

- Forecast Period: 2008 and 2009
- Daily USD S&P 500 and stock return of the defaults
- Estimated period: 5 years before the default year
- Source: Datastream

Data

- Recovery rate: Realized recovery rate R_i (weighted by volume) before default year by Moody's

- Hazard rate: Average historical default probability from Moody's report

Empirical Results

Model	Probability	Mean	STD
Period	2003-2007		
Unconditional (one normal)	1	0.004	0.009
Conditional on quiet	0.591	0.001	0.005
Conditional on hectic	0.409	-0.001	0.012
Period	2004-2008		
Unconditional (one normal)	1	0.001	0.007
Conditional on quiet	0.325	0.001	0.002
Conditional on hectic	0.675	-0.001	0.009

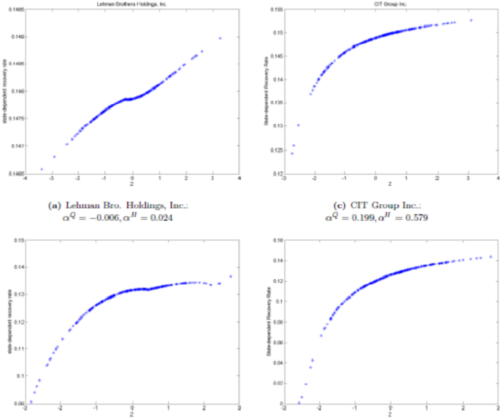
Table 1: Estimate Mixture of Normal Distribution by employing an EM algorithm

STD represent standard deviation

Conditional Factor Loading

Company	Uncond.	Quiet	Hectic
Abitibi-Consolidated Com. of Can.	0.2917	0.1651	0.2898
Abitibi-Consolidated Inc.	0.3267	0.1880	0.3190
FRANKLIN BANK	0.3886	0.2116	0.3119
GLITNIR BANKI	0.0444	0.0293	0.0674
LEHMAN BROS	0.0375	-0.0060	0.0235

Table 2: Correlation coefficients between S&P500 index returns and the return of default companies in 2008



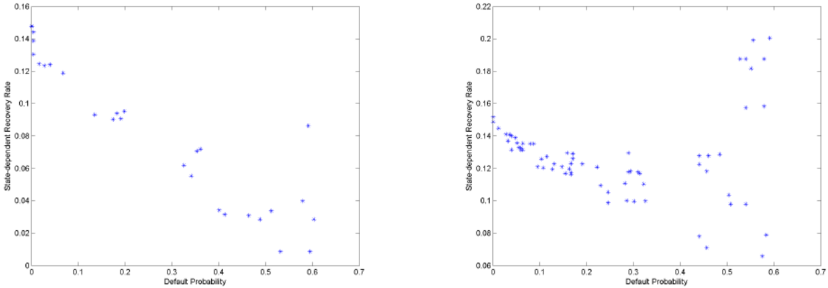
(a) Lehman Bro. Holdings, Inc.: $\alpha^Q = -0.006, \alpha^H = 0.024$

(c) CIT Group Inc.: $\alpha^Q = 0.199, \alpha^H = 0.579$

(b) Franklin Bank Corp.: $\alpha^Q = 0.2116, \alpha^H = 0.3119$

(d) Eddie Bauer Holdings Inc.: $\alpha^Q = 0.5993, \alpha^H = 0.601$

Figure 1: The relationship between recovery rate and S&P 500. In panel (a) and (b), illustrate the pattern of recoveries rate of 2 defaults in 2008. In panel (c) and (d), 2 defaults in 2009.



(a) 2008 (b) 2009

Figure 2: The relationship between recovery rate and default probabilities

Estimation of MSE

Company	FC	RFC	RR	RRFL
Abitibi-Consolidated Com. of Can.	0.0522	0.0526	0.0246	0.0240
Abitibi-Consolidated Inc.	0.1030	0.1041	0.0623	0.0608
Franklin Bank Corp.	0.9904	0.9881	0.9774	0.9765
Glitnir banki hf	0.9406	0.9404	0.9404	0.9399
Lehman Bro. Hold., Inc.	0.8223	0.8223	0.8223	0.8222

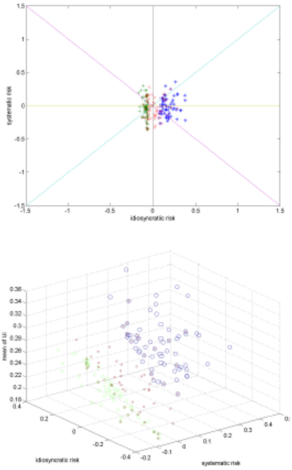
Table 3: The Estimation of MSE

The estimation of MSE of four different models for each default company in 2008.

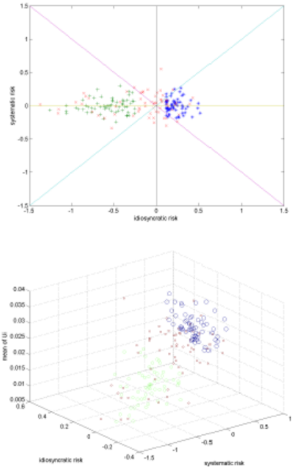
Empirical Results

Year	FC	RFL	RR	RRFL	Total
2008	1	1	10	19	31
	3.2%	3.2%	32.3%	61.3%	
2009	3	5	23	31	62
	4.8%	7.9%	37.1%	50.0%	

Table 4: Number and Percentage of defaults with minimum MSE



(a) 2008



(b) 2009

Figure 3: The 2D and 3D scatters plot of relative contribution

Conclusions

(i) Model the dependence in a more flexible and realistic way
  - Build the quiet and hectic regimes
  - Connect the recovery rate to the common factor

(ii) The conditional factor copulae together with state-dependent recoveries model could predict the default event during the crisis period

(iii) Coherent with the goals of Basel III

Further Work

(i) Alternative marginals: Generalized extreme value distribution or t-distribution.

(ii) Alternative copulae: T-copulae.

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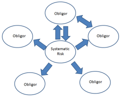

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Conditional factor loading

back

- $(Z, U_i) \sim$

$$\begin{cases} N\left(\begin{bmatrix} \mu_Z^Q \\ \mu_i^Q \end{bmatrix}, \begin{bmatrix} (\sigma_Z^Q)^2 & (\sigma_Z^Q)\alpha^Q(\sigma_i^Q) \\ (\sigma_Z^Q)\alpha^Q(\sigma_i^Q) & (\sigma_i^Q)^2 \end{bmatrix} \right) \\ N\left(\begin{bmatrix} \mu_Z^H \\ \mu_i^H \end{bmatrix}, \begin{bmatrix} (\sigma_Z^H)^2 & (\sigma_Z^H)\alpha^H(\sigma_i^H) \\ (\sigma_Z^H)\alpha^H(\sigma_i^H) & (\sigma_i^H)^2 \end{bmatrix} \right) \end{cases}$$

- where P(S=H)=ω, P(S=Q)=$1-\omega$
- Volatility in hectic periods is higher than in a quiet periods, $(\sigma_i^H)^2 > (\sigma_i^Q)^2$.
- α^Q and α^H are the correlation coefficient between each obligor and S&P 500 in quiet and hectic period