Copula-Based Factor Model for Credit Risk Analysis

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Systematic Risk

Figure 1: Credit Risk depends the state of economy.

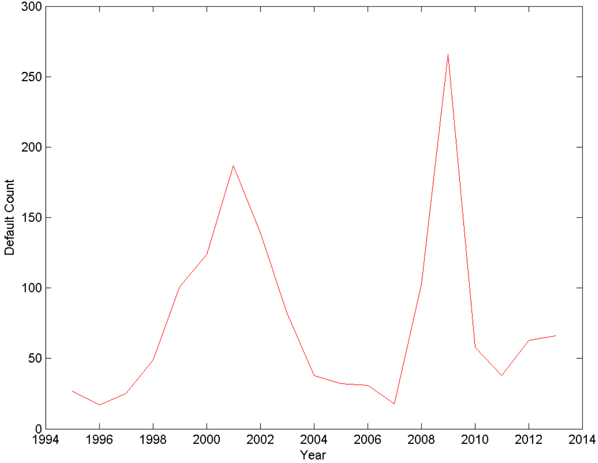


Figure 2: Annual Default Counts from 1995-2013.

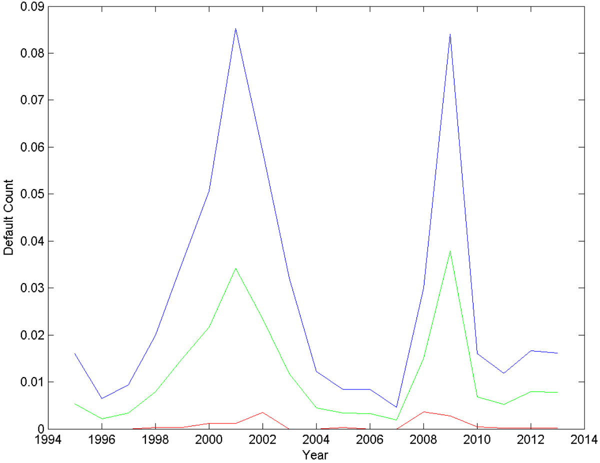
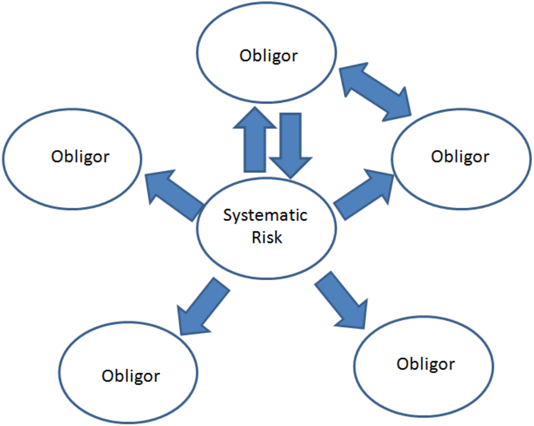


Figure 3: Annual average Loss Given Default rate: IG , SG and All of 1995-2013.


Obligor
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Systematic Risk
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Objectives

(i) Credit Risk Modeling

- Factor loading conditional on hectic and quiet state.
- State-dependent recovery rate.

(ii) Model Comparison

- Four models

Implication to Basel III

- Highlight systemic risk after 2008-2009 crisis.
- Credit risk versus business cycle.
- How credit risk moves over the business cycle.
- Contribution of systematic risk on credit risk is state-dependent.

Standard Technology

- Default event modeling
  - Latent variable is a linear combination of systematic and idiosyncratic shocks.
  - Copula enables flexible and realistic default dependence structure.

Outline

1. Motivation ✓
2. Factor Copulae & Stochastic Recoveries
3. Methodology
4. Empirical Results
5. Conclusions

Factor Copulae & Stochastic Recoveries

- Factor copula model is a flexible measurement of portfolio credit risk: Krupskii and Joe (2013)
- Correlation breakdown structure: Ang and Bekaert (2002), Anderson et al. (2004)
- Recovery rate varies with the market conditions: Amraoui et al. (2012)

Candidate Models

- FC model - One-factor Gaussian copula model with constant correlation structure and constant recoveries.
- RFL model - Conditional factor loading and constant recoveries.
- RR model - One-factor Gaussian copula and stochastic recoveries.
- RRFL model - Conditional factor loading and stochastic recoveries.

Default Modeling

- One-factor non-standardized Gaussian copula model

$$U_i = \alpha_i Z + \sqrt{1-\alpha_i^2}\varepsilon_i \qquad i = 1, \dots, N.$$

- Z: systematic factor, ε_i: idiosyncratic factors.
- Z and ε_i are independent, and ε_i are uncorrelated among each other, i=1,...,N.
- U_i: the proxies for firm asset and liquidation value.
- Correlation coefficient between U_i and U_j is

$$\rho_{ij} = \frac{\alpha_i \alpha_j \sigma^2}{\sqrt{\alpha_i^2(\sigma^2-1)+1}\sqrt{\alpha_j^2(\sigma^2-1)+1}}.$$

- The default indicator

$$\mathbf{I}\{\tau_i \leq t\} = \mathbf{I}\left[U_i \leq F^{-1}\{P_i(t)\}\right].$$

- τ_i indicates the default time of each obligor.
- $F^{-1}(\cdot)$ donates the inverse cdf of any distribution.
- $P_i(t)$: hazard rate and marginal probability that obligor i defaults before t.
  - From Moody's report.
  - Extract from Credit spreads.
  - Extract from Credit default swap spreads.

- Portfolio loss for each obligor

$$L = \sum_{i=1}^{N} G_i \mathbf{I}\{\tau_i \leq t\} = \sum_{i=1}^{N} G_i \mathbf{I}\left[U_i \leq F^{-1}\{P_i(t)\}\right].$$

- G_i is the loss given default (LGD) (i-th obligor's exposure = 1).

Copulae

- For n dimensions distribution F with marginal distribution $F_{X_1}, \cdots, F_{X_n}$, Copula function:

$$F(x_1, \cdots, x_n) = C\left\{F_{X_1}(x_1), \cdots, F_{X_n}(x_n)\right\}$$

Hoeffding on BBI:

Conditional Default Model

- Conditional factor copulae model

$$U_i|_{\mathrm{S=H}} = \alpha_i^H Z + \sqrt{1-(\alpha_i^H)^2}\varepsilon_i$$

$$U_i|_{\mathrm{S=Q}} = \alpha_i^Q Z + \sqrt{1-(\alpha_i^Q)^2}\varepsilon_i$$

- α^H, α^Q are conditional factor loading. link
- Conditional default probability

$$P(\tau_i < t|\mathrm{S}) = F\left[\frac{F^{-1}\{P_i(t)\} - \alpha_i^S Z}{\sqrt{1-(\alpha_i^S)^2}}\right] = P_i(Z|\mathrm{S}) \quad \mathrm{S} \in \{\mathrm{H,Q}\}$$

- with P(S=H)=ω, and P(S=Q)=$1-\omega$

State-Dependent Recovery Rate

- The LGD on name i, $G_i(Z)$ is related to common factor Z and the marginal default probability P_i link
- Given fixed expected loss, $(1-R_i)P_i = (1-\bar{R}_i)\bar{P}_i$

$$G_i(Z|\mathrm{S{=}H}) = (1-\bar{R}_i)\frac{F\left[\left\{F^{-1}\left(\overline{P}_i\right)-\alpha_i^H Z\right\}/\sqrt{1-(\alpha_i^H)^2}\right]}{F\left[\left\{F^{-1}\left(P_i\right)-\alpha_i^H Z\right\}/\sqrt{1-(\alpha_i^H)^2}\right]}.$$

$$G_i(Z|\mathrm{S{=}Q}) = (1-\bar{R}_i)\frac{F\left[\left\{F^{-1}\left(\overline{P}_i\right)-\alpha_i^Q Z\right\}/\sqrt{1-(\alpha_i^Q)^2}\right]}{F\left[\left\{F^{-1}\left(P_i\right)-\alpha_i^Q Z\right\}/\sqrt{1-(\alpha_i^Q)^2}\right]}.$$

- We set $\bar{R}_i = 0$ in the simplest case.

Conditional Expected Loss

- Conditional default probability $P_i(Z|\text{S}=\text{H,Q})$ and conditional LGD, $G_i(Z|\text{S}=\text{H,Q})$, conditional expected loss,

$$\text{E}(L_i|Z) = \omega G_i(Z|\text{S}=\text{H})P_i(Z|\text{S}=\text{H}) + (1-\omega)G_i(Z|\text{S}=\text{Q})P_i(Z|\text{S}=\text{Q}).$$

Monte Carlo Simulation and MSE

- One-factor non-standardized Gaussian Copula
  - $Z \sim \mathrm{N}(-0.03, 3.05), \varepsilon_i \sim N(0,1)$.
  - Z and ε_i are generated 1000 observations.
- Conditional probability that date t was belonging to the hectic is $\pi(Z = z)$.

$$\mathrm{P}(S = H | Z = z) = \pi(Z = z) = \frac{\omega \varphi(z|\theta^H)}{(1-\omega)\varphi(z|\theta^Q) + \omega\varphi(z|\theta^H)}.$$

- α_i^H, α_i^Q are derived from the daily stock returns of S&P 500 and of collected default companies during the crisis period.
  - Five-year period prior to the crisis period is the estimation period.

Project to Default Time

- Using the definition of survival rate (Hull, 2006)

$$\tau_i|\mathsf{S} = -\frac{\log\{1 - F(U_i|\mathsf{S})\}}{P_i}.$$

- P_i is the hazard rate and marginal probability that obligor i will default.
- $\tau_i|\mathsf{S}$ is corresponding to

$$\mathsf{E}[\mathsf{I}(\tau_i|\mathsf{S} < 1)] = \mathsf{P}(\tau_i|\mathsf{S} < 1) = P_i(Z|\mathsf{S}).$$

State-Dependent Recovery Rate Simulation

- $(1 - R_i)P_i = (1 - \bar{R}_i)\bar{P}_i$.
- $\overline{P}_i$ is a adjusted default probability calibrated by plugging hazard rate P_i. link
- $\bar{R}_i$ is a lower bound for state-dependent recovery rates [0,1].
- We set $\bar{R}_i = 0$ in the simplest case.
- Given α_i^S and simulated Z, we generate $G_i(Z|S)$.

Expected Loss Function

- With these two specifications, we study the expected loss function under the given scenarios

$$\begin{aligned} \mathrm{E}(L_i|Z) &= \pi(Z=z)G_i(Z|\mathrm{S{=}H})P_i(Z|\mathrm{S{=}H}) \\ &+ (1-\pi(Z=z))G_i(Z|\mathrm{S{=}Q})P_i(Z|\mathrm{S{=}Q}) \end{aligned}$$

- $\pi(Z=z)$ is better than unconditional probability ω.

Estimation of the AE

- Absolute Error (AE)

$$AE = (\text{actual portfolio loss - expected portfolio loss}).$$

- Actual portfolio loss is from Moody's report.
- Exposure of each obligor is 100 million.
- Compare minimum AE, MAE to evaluate FC, RFL, RR, and RRFL model.

Data

- Forecast Period: 31 and 62 firms in 2008 and 2009
- Daily USD S&P 500 and stock return of the defaults
- Estimated period: 5 years before the default year
- Source: Datastream

Data

- Recovery rate: Realized recovery rate R_i (weighted by volume) before default year by Moody's

- Hazard rate: Average historical default probability from Moody's report

Empirical Results

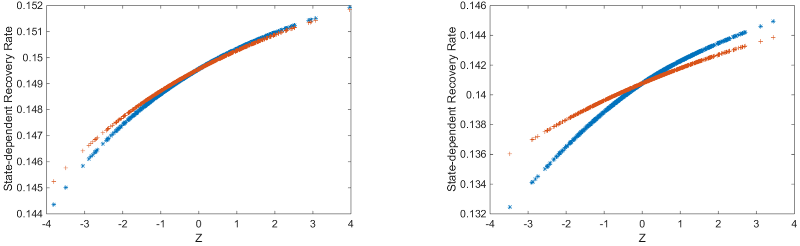
Model	Probability	Mean	STD
Period	2003-2007		
Unconditional (one normal)	100.00%	0.03%	0.77%
Conditional on quiet	58.68%	0.10%	0.43%
Conditional on hectic	41.32%	-0.08%	1.07%
Period	2004-2008		
Unconditional (one normal)	100.00%	0.03%	0.83%
Conditional on quiet	56.77%	0.10%	0.38%
Conditional on hectic	43.23%	-0.06%	1.17%

Table 1: Estimate Mixture of Normal Distribution by employing an EM algorithm SD means standard deviation

Conditional Factor Loading

Company	Uncond.	Quiet	Hectic
Abitibi-Consolidated Com. of Can.	0.29	0.17	0.29
Abitibi-Consolidated Inc.	0.33	0.19	0.32
FRANKLIN BANK	0.39	0.21	0.31
GLITNIR BANKI	0.04	0.03	0.07
LEHMAN BROS	0.04	-0.01	0.02

Table 2: Correlation coefficients between S&P500 index returns and the return of default companies in 2008.



(a) 2008 Lehman Bro. (b) 2009 E*TRADE

Figure 4: The relationship between state-dependent recovery rates and S&P 500, Z. '*' in blue illustrates the pattern of state-dependent recovery rate, and '+' in red plots the recoveries proposed by Amraoui et al.(2012)

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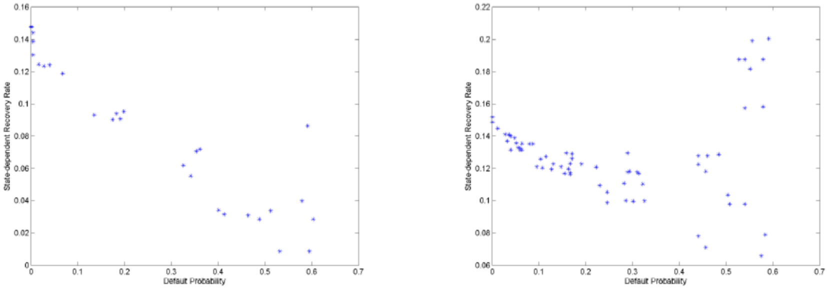


Figure 5: The relationship between recovery rate and default probabilities, left panel 2008 and right panel 2009

Estimation of MAE

	FC	RFL	RR	RRFL
2008				
APL	2035.02	2035.02	2035.02	2035.02
EPL	509.60	527.06	687.01	690.86
AE	1525.42	1507.96	1348.01	1344.16
MAE	47.12	47.67	42.13	42.01
EPL/APL	25.04%	25.90%	33.76%	33.95%

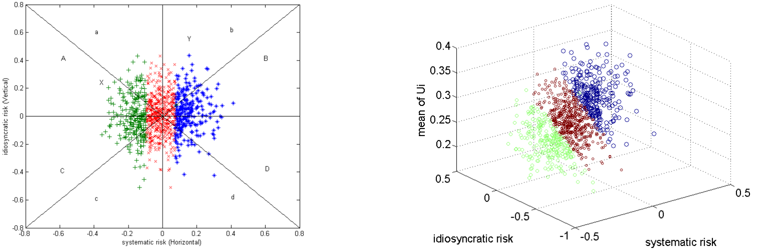
Table 3: The mean of actual portfolio loss (APL), expected portfolio loss (EPL) and AE, MAE (in million)

Robustness test

	FC	RFL	RR	RRFL
2008				
APL	1401.31	1401.31	1401.31	1401.31
EPL	560.50	533.82	589.54	591.40
AE	840.81	867.49	811.77	809.91
MAE	35.03	36.15	33.82	33.75
EPL/APL	40.00%	38.09%	42.07%	42.20%

Table 4: The actual portfolio loss (APL), expected portfolio loss (EPL), AE, and MAE (in million) for robustness

Basel III: Relative Contribution



(a) 2008 (b) 2008

Figure 6: The 2D and 3D scatters plot of relative contribution
The first group (marked as '+' in green) indicates that they are generated in distress. The second group (marked as '*' in blue) indicates that they are generated in a bullish atmosphere. The third group (marked as 'x' in red) collects the rest.

Conclusions

(i) Model the dependence in a more flexible and realistic way.

- Build the quiet and hectic regimes.
- Connect the recovery rate to the common factor.

(ii) The conditional factor copulae together with state-dependent recoveries model could predict the default event during the crisis period.

(iii) Coherent with the goals of Basel III.

Further Work

(i) Alternative marginals: Generalized extreme value distribution or t-distribution.

(ii) Alternative copula: t-copula.

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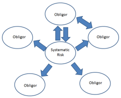

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Conditional Factor Loading ▸ back

- $(Z, U_i) \sim$

$$
\begin{cases}
N\left(\begin{bmatrix} \mu_Z^Q \\ \mu_i^Q \end{bmatrix}, \begin{bmatrix} (\sigma_Z^Q)^2 & (\sigma_Z^Q)\alpha^Q(\sigma_i^Q) \\ (\sigma_Z^Q)\alpha^Q(\sigma_i^Q) & (\sigma_i^Q)^2 \end{bmatrix} \right) \\
N\left(\begin{bmatrix} \mu_Z^H \\ \mu_i^H \end{bmatrix}, \begin{bmatrix} (\sigma_Z^H)^2 & (\sigma_Z^H)\alpha^H(\sigma_i^H) \\ (\sigma_Z^H)\alpha^H(\sigma_i^H) & (\sigma_i^H)^2 \end{bmatrix} \right)
\end{cases}
$$

- where P(S=H)=ω, P(S=Q)=$1-\omega$
- Volatility in hectic periods is higher than in a quiet periods, $\sigma_i^H > \sigma_i^Q$.
- α^Q and α^H are the correlation coefficient between each obligor and S&P 500 in quiet and hectic period