# Forecasting Limit Order Book Liquidity with Functional AutoRegressive Dynamics

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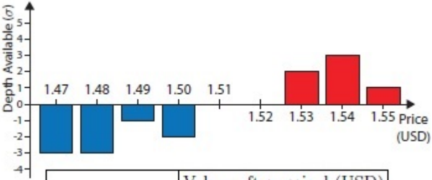
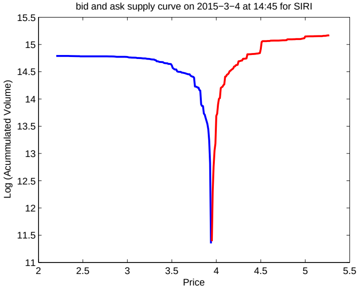
# Limit order book

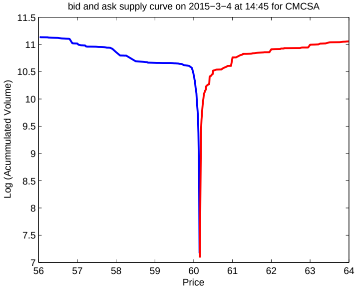



| Arriving order $x$ | Values after arrival (USD) | | | |
|---|---|---|---|---|
| | $b(t_x)$ | $a(t_x)$ | $m(t_x)$ | $s(t_x)$ |
| Initial Values | 1.50 | 1.53 | 1.515 | 0.03 |
| $(\$1.48, -3\sigma, t_x)$ | 1.50 | 1.53 | 1.515 | 0.03 |
| $(\$1.51, -3\sigma, t_x)$ | 1.51 | 1.53 | 1.52 | 0.02 |
| $(\$1.55, -3\sigma, t_x)$ | 1.50 | 1.54 | 1.52 | 0.04 |
| $(\$1.55, -5\sigma, t_x)$ | 1.50 | 1.55 | 1.525 | 0.05 |
| $(\$1.54, 4\sigma, t_x)$ | 1.50 | 1.53 | 1.515 | 0.03 |
| $(\$1.52, 4\sigma, t_x)$ | 1.50 | 1.52 | 1.51 | 0.02 |
| $(\$1.47, 4\sigma, t_x)$ | 1.48 | 1.53 | 1.505 | 0.05 |
| $(\$1.50, 4\sigma, t_x)$ | 1.49 | 1.50 | 1.495 | 0.01 |

# Liquidity demand and supply curve

SIRI and CMCSA bid $X_t^{(b)}$ and ask $X_t^{(a)}$ supply curve on March 4, 2015 at 14:45pm. Liquidity curves Data source




Bid ask curve

# Economic implications

- Liquidity demand and supply curves provide information on traders' expectations of the price
- Improve order execution strategy
- Smaller transaction cost
- Robust arbitrage pricing theory (Çetin, Jarrow and Protter, 2004)
- Forecasting with DSFM (Härdle, Hautsch and Mihoci, 2012)

# Liquidity measures

- Bid-ask spread (Benston and Hagerman, 1974; Stoll, 1978; Fleming and Remolona, 1999)
- Liquidity depth based on volumes at the best quotes or some particular price quotes, e.g. XLM (Cooper, Groth and Avera, 1985; Gomber, Schweickert and Theissen, 2015)

# Autoregressive models

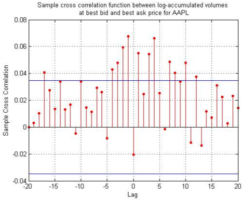
- Long memory autoregressive conditional Poisson model (Groß-Klußmann and Hautsch, 2013)
- Autoregressive model (Huberman and Halka, 2001)
- Vector autoregressive model (Chordia, Sarkar and Subrahmanyam, 2003)
- Local adaptive multiplicative error model (Härdle, Hautsch and Mihoci, 2015; Härdle, Mihoci and Ting, 2016)

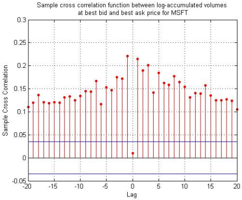
# Cross-correlations

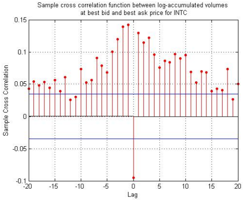
- Limit order demand and supply elasticities are cross-related (Dierker, Kim, Lee and Morck, 2014)
- Public or private information: switch sides
- Market-wide events: similar changes on both sides

# Cross-correlations

Sample cross correlation function between log-accumulated volumes at best bid and best ask price for AAPL, MSFT, and INTC.



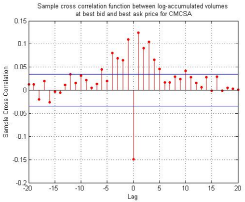



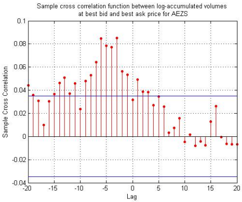



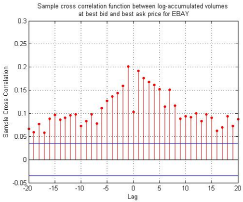

Cross Correlation

# Cross-correlations

Sample cross correlation function between log-accumulated volumes at best bid and best ask price for CMCSA, AEZS, and EBAY.




Cross Correlation

- Cross correlations suggest richer dynamics
- Richer dynamics of liquidity allows for more precise forecasts
- The bid/ask cross-dependency motivates to analyze two liquidity curves simultaneously

# Objectives

- Employ a Vector Functional AutoRegressive (VFAR) model:

$$\begin{bmatrix} X_t^{(a)} - \mu_a \\ X_t^{(b)} - \mu_b \end{bmatrix} = \begin{bmatrix} \rho^{aa} & \rho^{ab} \\ \rho^{ba} & \rho^{bb} \end{bmatrix} \begin{bmatrix} X_{t-1}^{(a)} - \mu_a \\ X_{t-1}^{(b)} - \mu_b \end{bmatrix} + \begin{bmatrix} \varepsilon_t^{(a)} \\ \varepsilon_t^{(b)} \end{bmatrix}$$

  where $X_t^{(a)}$ and $X_t^{(b)}$ are the curves on the (b)id and (a)sk side, while $\rho$ is a bounded linear operator.

- Asymptotic consistency of the VFAR estimator
- Finite sample performance in real data analysis

# Outline

1. Motivation ✓
2. Data
3. Vector Functional AutoRegression (VFAR)
4. Empirical Applications
5. Conclusion

# Data

LOB records of 12 stocks traded in NASDAQ stock market from 2 Jan 2015 to 6 Mar 2015 (44 trading days)

- Apple Inc. (AAPL)
- Microsoft Corporation (MSFT)
- Intel Corporation (INTC)
- Cisco Systems, Inc. (CSCO)
- Sirius XM Holdings Inc. (SIRI)
- Applied Materials, Inc. (AMAT)
- Comcast Corporation (CMCSA)
- AEterna Zentaris Inc. (AEZS)
- eBay Inc. (EBAY)
- Micron Technology, Inc. (MU)
- Whole Foods Market, Inc. (WFM)
- Starbucks Corporation (SBUX)

CRC649 LOBSTER

# Summary statistics

5-minutes snapshots of the LOB data.

| Ticker Symbol | Spread (USD) | | Bid vol | | Ask vol | |
|---|---|---|---|---|---|---|
| | min | max | min | max | min | max |
| AAPL | 0.01 | 0.07 | 52,267 | 710,020 | 61,305 | 1,298,696 |
| MSFT | 0.01 | 0.02 | 90,344 | 928,319 | 122,377 | 621,471 |
| INTC | 0.01 | 0.02 | 158,900 | 557,251 | 146,959 | 1,142,641 |
| CSCO | 0.01 | 0.02 | 134,790 | 1,316,058 | 266,455 | 4,458,672 |
| SIRI | 0.01 | 0.02 | 1,266,528 | 3,725,304 | 3,002,680 | 7,605,467 |
| AMAT | 0.01 | 0.03 | 78,944 | 334,794 | 180,749 | 787,983 |
| CMCSA | 0.01 | 0.06 | 23,668 | 128,916 | 40,638 | 146,724 |
| AEZS | 0.0001 | 0.05 | 145,635 | 767,785 | 472,689 | 1,158,740 |
| EBAY | 0.01 | 0.04 | 42,060 | 160,572 | 52,813 | 415,033 |
| MU | 0.01 | 0.04 | 95,907 | 497,910 | 102,357 | 595,200 |
| WFM | 0.01 | 0.12 | 34,538 | 114,386 | 41,019 | 159,488 |
| SBUX | 0.01 | 0.12 | 27,467 | 151,022 | 34,914 | 166,932 |

# Data pre-processing

- Sampling frequency: 5 minutes (Aït-Sahalia, Mykland and Zhang, 2005; Zhang and Aït-Sahalia, 2005)
- Discarded the first 15 minutes and the last 5 minutes (Härdle et al., 2012)
- Log-transformed the accumulated volumes (Potters and Bouchaud, 2003)
- Obtain 75 pairs of bid and ask liquidity curves, for each stock, at each trading day
- Total: 3300 pairs of bid and ask supply curves over the whole sample period of 44 trading days for each stock

# Vector Functional AutoRegression (VFAR)

$X_t^{(a)}(\tau),\ X_t^{(b)}(\tau) \in C_{(-\infty,\infty)}$. ▸ $X_t^{(a)}$

$$\begin{bmatrix} X_t^{(a)} - \mu_a \\ X_t^{(b)} - \mu_b \end{bmatrix} = \begin{bmatrix} \rho^{aa} & \rho^{ab} \\ \rho^{ba} & \rho^{bb} \end{bmatrix} \begin{bmatrix} X_{t-1}^{(a)} - \mu_a \\ X_{t-1}^{(b)} - \mu_b \end{bmatrix} + \begin{bmatrix} \varepsilon_t^{(a)} \\ \varepsilon_t^{(b)} \end{bmatrix} \tag{1}$$

where $(\mu_a, \mu_b)^\top$ are the mean functions. The operators $\rho$ is bounded linear operator from $\mathcal{H}$ to $\mathcal{H}$. The innovations $\{\varepsilon_t^{(a)}\}_{t=1}^n$ and $\{\varepsilon_t^{(b)}\}_{t=1}^n$ are strong $\mathcal{H}$-white noise, i.i.d. with zero mean and
$0 < \mathsf{E}\,\|\varepsilon_1^{(a)}\|^2 = \cdots = \mathsf{E}\,\|\varepsilon_n^{(a)}\|^2 < \infty$ and
$0 < \mathsf{E}\,\|\varepsilon_1^{(b)}\|^2 = \cdots = \mathsf{E}\,\|\varepsilon_n^{(b)}\|^2 < \infty$, $\varepsilon_t^{(a)}$ and $\varepsilon_t^{(b)}$ need not be cross-independent.

# Convolution kernel operator

The operators $\rho$ in (1):

$$
\begin{aligned}
X_t^{(a)}(\tau) - \mu_a(\tau) &= \int_0^1 \kappa_{ab}(\tau - s)\{X_{t-1}^{(b)}(s) - \mu_b(s)\}ds \\
&+ \int_0^1 \kappa_{aa}(\tau - s)\{X_{t-1}^{(a)}(s) - \mu_a(s)\}ds + \varepsilon_t^{(a)}(\tau) \\
X_t^{(b)}(\tau) - \mu_b(\tau) &= \int_0^1 \kappa_{bb}(\tau - s)\{X_{t-1}^{(b)}(s) - \mu_b(s)\}ds \\
&+ \int_0^1 \kappa_{ba}(\tau - s)\{X_{t-1}^{(a)}(s) - \mu_a(s)\}ds + \varepsilon_t^{(b)}(\tau) \qquad (2)
\end{aligned}
$$

where the kernel function $\kappa_{xy} \in L^2((-\infty, \infty))$ and $\|\kappa_{xy}\|_2 < 1$ for $xy = aa$, $ab$, $ba$, and $bb$.

# $B$-spline expansion

Expand the functions in (2) using $B$-spline basis function in $L^2((-\infty, \infty))$:

$$B_{j,m}(\tau) = \frac{\tau - w_j}{w_{j+m-1} - w_j} B_{j,m-1}(\tau) + \frac{w_{j+m} - \tau}{w_{j+m} - w_{j+1}} B_{j+1,m-1}(\tau), m \geq 2,$$

where $m$ is the order, and $w_1 \leq \cdots \leq w_{J+m}$ denotes the knot sequence. Note that

$$B_{j,1}(\tau) = \begin{cases} 1 & \text{if } w_j \leq \tau < w_{j+1}, \\ 0 & \text{otherwise.} \end{cases}$$

$B$-spline expansions

# Sieve

Introduce a sequence of subsets $\{\Theta_{J_n}\}$ - a sieve, where $\Theta_{J_n} \subseteq \Theta_{J_n+1}$ and the union of subsets $\bigcup \Theta_{J_n}$ is dense in the parameter space (Grenander, 1981).

The sieve is defined as follows:

$$\Theta_{J_n} = \left\{ \kappa_{xy} \in L^2 \mid \kappa_{xy}(\tau) = \sum_{l=1}^{J_n} c_l^{xy} B_{l,m}(\tau), \tau \in (-\infty, \infty), \sum_{l=1}^{J_n} l^2 (c_l^{xy})^2 \leq v J_n \right\} \tag{3}$$

VFAR & $B-$Splines

# Coefficients relationship

This provides the following relationship of the $B$-spline coefficients:

$$
\begin{aligned}
d^a_{t,h} &= p^a_h + \sum_{i=1}^{J_n} \left\{ \sum_{j=1}^{J_n} \left( \frac{w_{j+m} - w_{j+1}}{w_{j+m} - w_j} - \frac{w_{j+m+1} - w_{j+2}}{w_{j+m+1} - w_{j+1}} \right) c^{aa}_j - c^{aa}_h \right\} \frac{w_{i+m} - w_i}{m} d^a_{t-1,i} \\
&+ \sum_{i=1}^{J_n} \left\{ \sum_{j=1}^{J_n} \left( \frac{w_{j+m} - w_{j+1}}{w_{j+m} - w_j} - \frac{w_{j+m+1} - w_{j+2}}{w_{j+m+1} - w_{j+1}} \right) c^{ab}_j - c^{ab}_h \right\} \frac{w_{i+m} - w_i}{m} d^b_{t-1,i} + d^a_h(\varepsilon^{(a)}_t) \\
d^b_{t,h} &= p^b_h + \sum_{i=1}^{J_n} \left\{ \sum_{j=1}^{J_n} \left( \frac{w_{j+m} - w_{j+1}}{w_{j+m} - w_j} - \frac{w_{j+m+1} - w_{j+2}}{w_{j+m+1} - w_{j+1}} \right) c^{bb}_j - c^{bb}_h \right\} \frac{w_{i+m} - w_i}{m} d^b_{t-1,i} \\
&+ \sum_{i=1}^{J_n} \left\{ \sum_{j=1}^{J_n} \left( \frac{w_{j+m} - w_{j+1}}{w_{j+m} - w_j} - \frac{w_{j+m+1} - w_{j+2}}{w_{j+m+1} - w_{j+1}} \right) c^{ba}_j - c^{ba}_h \right\} \frac{w_{i+m} - w_i}{m} d^a_{t-1,i} + d^b_h(\varepsilon^{(b)}_t)
\end{aligned}
\tag{4}
$$

# Vector AutoRegressive form

Rewriting (4) as a matrix yields a form of Vector AutoRegressive (VAR) of order 1:

$$
\begin{bmatrix} d^a_{t,1} \\ \vdots \\ d^a_{t,J_n} \\ d^b_{t,1} \\ \vdots \\ d^b_{t,J_n} \end{bmatrix}
=
\begin{bmatrix}
p^a_1 & r^{aa}_{1,1} & \cdots & r^{aa}_{1,J_n} & r^{ab}_{1,1} & \cdots & r^{ab}_{1,J_n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
p^a_{J_n} & r^{aa}_{J_n,1} & \cdots & r^{aa}_{J_n,J_n} & r^{ab}_{J_n,1} & \cdots & r^{ab}_{J_n,J_n} \\
p^b_1 & r^{ba}_{1,1} & \cdots & r^{ba}_{1,J_n} & r^{bb}_{1,1} & \cdots & r^{bb}_{1,J_n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
p^b_{J_n} & r^{ba}_{J_n,1} & \cdots & r^{ba}_{J_n,J_n} & r^{bb}_{J_n,1} & \cdots & r^{bb}_{J_n,J_n}
\end{bmatrix}
\begin{bmatrix} 1 \\ d^a_{t-1,1} \\ \vdots \\ d^a_{t-1,J_n} \\ d^b_{t-1,1} \\ \vdots \\ d^b_{t-1,J_n} \end{bmatrix}
+
\begin{bmatrix} d^a_1(\varepsilon^{(a)}_t) \\ \vdots \\ d^a_{J_n}(\varepsilon^{(a)}_t) \\ d^b_1(\varepsilon^{(b)}_t) \\ \vdots \\ d^b_{J_n}(\varepsilon^{(b)}_t) \end{bmatrix}
\tag{5}
$$

where $r^{xy}_{h,i}$ denotes $\left\{ \sum_{j=1}^{J_n} \left( \frac{w_{j+m}-w_{j+1}}{w_{j+m}-w_j} - \frac{w_{j+m+1}-w_{j+2}}{w_{j+m+1}-w_{j+1}} \right) c^{xy}_j - c^{xy}_h \right\} \frac{w_{i+m}-w_i}{m}$, for $xy = aa$, $ab$, $ba$, and $bb$.

# Vector AutoRegressive form

Write compactly as the following:

$$Y = BZ + U \tag{6}$$

Assuming

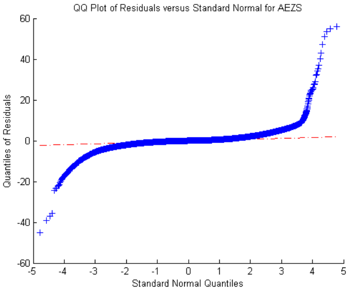
$$\mathbf{u} = \mathrm{vec}(U) = \begin{bmatrix} u_1 \\ \vdots \\ u_T \end{bmatrix} \sim \mathrm{N}(0, I_T \otimes \Sigma_u),$$

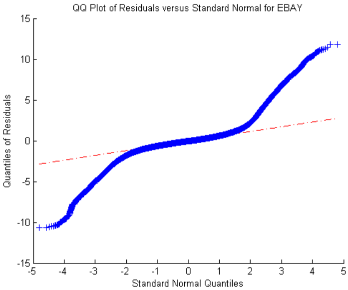
with pdf of $\mathbf{u}$:

$$f_{\mathbf{u}}(\mathbf{u}) = \frac{1}{(2\pi)^{KT/2}} \left| I_T \otimes \Sigma_u \right|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \mathbf{u}^\top (I_T \otimes \Sigma_u^{-1}) \mathbf{u} \right\}.$$

# Residual analysis

## QQ plots of residuals for AEZS, EBAY




QQ plot

# Maximum likelihood estimation

Using $\mathbf{u} = \mathbf{y} - (Z^\top \otimes I_K)\beta$, the likelihood function is:

$$g\left(X_t^{(a)}, X_t^{(b)}, X_{t-1}^{(a)}, X_{t-1}^{(b)}, \rho^{aa}, \rho^{ab}, \rho^{ba}, \rho^{bb}\right) = \left|\frac{\partial \mathbf{u}}{\partial \mathbf{y}^\top}\right| f_{\mathbf{u}}(\mathbf{u})$$

$$= \frac{1}{(2\pi)^{KT/2}} \left| I_T \otimes \Sigma_u \right|^{-\frac{1}{2}} \exp\left[ -\frac{1}{2}\left(\mathbf{y} - (Z^\top \otimes I_K)\beta\right)^\top (I_T \otimes \Sigma_u^{-1}) \left\{\mathbf{y} - (Z^\top \otimes I_K)\beta\right\}\right].$$

Log-likelihood

# Maximum likelihood estimation

$$
\begin{aligned}
\widehat{B} &= YZ^\top (ZZ^\top)^{-1} \\
\widehat{\Sigma_u} &= \frac{1}{T}(Y - BZ)(Y - BZ)^\top
\end{aligned}
\tag{7}
$$

The first column of $YZ^\top(ZZ^\top)^{-1}$ in (7) is the estimator for $v = \left(p_1^a, \cdots, p_{J_n}^a, p_1^b, \cdots, p_{J_n}^b\right)^\top$.

# Maximum likelihood estimation

To show the estimator for $c_j^{xy}$ for $xy = aa, ab, ba, bb$ as in (2), first define the following notations:

$$
\begin{aligned}
W &= \mathrm{diag}\Big(\frac{m}{w_{1+m} - w_1}, \cdots, \frac{m}{w_{J_n+m} - w_{J_n}}, \\
&\qquad\qquad \frac{m}{w_{1+m} - w_1}, \cdots, \frac{m}{w_{J_n+m} - w_{J_n}}\Big), \\
q_j &= \frac{w_{j+m} - w_{j+1}}{w_{j+m} - w_j} - \frac{w_{j+m+1} - w_{j+2}}{w_{j+m+1} - w_{j+1}}, \\
\theta_1 &= (c_1^{aa}, \cdots, c_{J_n}^{aa}, c_1^{ba}, \cdots, c_{J_n}^{ba})^\top, \\
\theta_2 &= (c_1^{ab}, \cdots, c_{J_n}^{ab}, c_1^{bb}, \cdots, c_{J_n}^{bb})^\top, \\
\boldsymbol{\theta} &= (\theta_1, \cdots, \theta_1, \theta_2, \cdots, \theta_2),
\end{aligned}
$$

# Maximum likelihood estimation

$$
Q = \begin{bmatrix}
\begin{matrix} q_1 - 1 & q_2 & \cdots & q_{J_n} \\ q_1 & q_2 - 1 & \cdots & q_{J_n} \\ \vdots & \vdots & \ddots & \vdots \\ q_1 & q_2 & \cdots & q_{J_n} - 1 \end{matrix} & \mathbf{0} \\
\mathbf{0} & \begin{matrix} q_1 - 1 & q_2 & \cdots & q_{J_n} \\ q_1 & q_2 - 1 & \cdots & q_{J_n} \\ \vdots & \vdots & \ddots & \vdots \\ q_1 & q_2 & \cdots & q_{J_n} - 1 \end{matrix}
\end{bmatrix},
$$

where $\boldsymbol{\theta}$ contains $J_n$ columns of $\theta_1$ and $J_n$ columns of $\theta_2$.

Therefore the estimator for $c_j^{xy}$ for $xy = aa, ab, ba, bb$ is:

$$
\widehat{\boldsymbol{\theta}} = Q^{-1} Y Z^\top (Z Z^\top)^{-1} (\mathbf{0}_{2J_n \times 1}, \mathbf{I}_{2J_n \times 2J_n})^\top W
$$

# Consistency result

## Theorem (1)

*Assume* $\{\Theta_{J_n}\}$ *is chosen such that conditions C1 and C2 are in force. Suppose that for each* $\delta > 0$, *we can find subsets* $\Gamma_1, \Gamma_2, \cdots, \Gamma_{l_{J_n}}$ *of* $\Theta_{J_n}$, $J_n = 1, 2, \cdots$ *such that*

(i) $D_{J_n} \subseteq \bigcup_{k=1}^{l_{J_n}} \Gamma_k$, *where* $D_{J_n} = \{\boldsymbol{\rho} \in \Theta_{J_n} | H(\boldsymbol{\rho}_{0|\Theta_{J_n}}, \boldsymbol{\rho}) \leq H(\boldsymbol{\rho}_{0|\Theta_{J_n}}, \boldsymbol{\rho}_{J_n}) - \delta\}$ *for every* $\delta > 0$ *and every* $J_n$.

(ii) $\sum_{n=1}^{+\infty} l_{J_n}(\varphi_{J_n})^n < +\infty$, *where given* $l$ *sets* $\Gamma_1, \cdots, \Gamma_l$ *in* $\Theta_{J_n}$, *where*

$$\varphi_{J_n} = \sup_k \inf_{t \geq 0} E_{\boldsymbol{\rho}_{0|\Theta_{J_n}}} \exp\left\{ t \log \frac{g(X_t^{(a)}, X_t^{(b)}, X_{t-1}^{(a)}, X_{t-1}^{(b)}, \Gamma_k)}{g(X_t^{(a)}, X_t^{(b)}, X_{t-1}^{(a)}, X_{t-1}^{(b)}, \boldsymbol{\rho}_{J_n})} \right\}.$$

*Then we have* $\sup_{\hat{\rho}_n \in M_J^n} \|\hat{\boldsymbol{\rho}}_n - \boldsymbol{\rho}_{0|\Theta_{J_n}}\|_{HS} \to 0$ *a.s.*

C1 & C2

# Consistency result

### Theorem (2)

*If $J_n = \mathcal{O}(n^{1/3-\eta})$ for $\eta > 0$, then $\|\hat{\boldsymbol{\kappa}}_{J_n} - \boldsymbol{\kappa}_{0|\Theta_{J_n}}\|_2 \to 0$ a.s. when $n \to +\infty$ and $\|\cdot\|_2$ is the $L^2$ norm in $C_{[0,1]}$.*

*$\hat{\boldsymbol{\kappa}}_{J_n} = (\hat{\kappa}_{aa,J_n}, \hat{\kappa}_{ab,J_n}, \hat{\kappa}_{ba,J_n}, \hat{\kappa}_{bb,J_n})$ is the set of sieve estimators on $\Theta_{J_n}$ and $\boldsymbol{\kappa}_{0|\Theta_{J_n}} = (\kappa_{aa,0|\Theta_{J_n}}, \kappa_{ab,0|\Theta_{J_n}}, \kappa_{ba,0|\Theta_{J_n}}, \kappa_{bb,0|\Theta_{J_n}})$ is the projection of the set of true kernel functions $\boldsymbol{\kappa}_0$ on $\Theta_{J_n}$.*

# Real data analysis

- Liquidity demand and supply curves over 44 trading days from date 2 Jan 2015 to 6 Mar 2015
- Cubic $B$-spline expansions with equally-spaced price percentile as nodes and $J_n = 20$ in the sieve
- 20 coefficients for the bid and another 20 for the ask liquidity curve

$R^2$ with different $J_n$

# Evaluation

The root mean squared estimation error (RMSE), mean absolute percentage error (MAPE), and $R^2$ are computed:

$$
\begin{aligned}
RMSE &= \sqrt{\frac{\sum_{xy=a,b}\sum_{t=1}^{T}\sum_{\tau}\left\{X_t^{(xy)}(\tau)-\hat{X}_t^{(xy)}(\tau)\right\}^2}{N}} \\
MAPE &= \frac{\sum_{xy=a,b}\sum_{t=1}^{T}\sum_{\tau}\frac{\left|X_t^{(xy)}(\tau)-\hat{X}_t^{(xy)}(\tau)\right|}{X_t^{(xy)}(\tau)}}{N} \\
R^2 &= 1-\frac{\sum_{xy=a,b}\sum_{t=1}^{T}\sum_{\tau}\left\{X_t^{(xy)}(\tau)-\hat{X}_t^{(xy)}(\tau)\right\}^2}{\sum_{xy=a,b}\sum_{t=1}^{T}\sum_{\tau}\left\{X_t^{(xy)}(\tau)-\bar{X}\right\}^2}
\end{aligned}
\tag{8}
$$

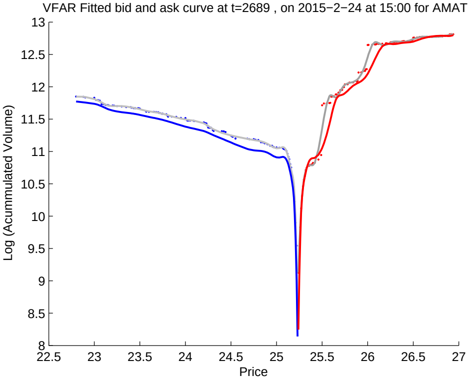
# VFAR fits

## $R^2$, RMSE, and MAPE for in-sample estimation of the nine stocks

| Ticker Symbol | $R^2$ | | RMSE | | MAPE | |
|---|---|---|---|---|---|---|
| | VFAR | Naive | VFAR | Naive | VFAR | Naive |
| AAPL | **92.03%** | 88.87% | **0.3419** | 0.4041 | **3.61%** | 3.80% |
| MSFT | **95.19%** | 93.53% | **0.1831** | 0.2124 | **0.95%** | 1.02% |
| INTC | **94.79%** | 93.13% | **0.1868** | 0.2144 | **0.92%** | 0.98% |
| CSCO | **96.16%** | 95.08% | **0.1926** | 0.2180 | **0.86%** | 0.91% |
| SIRI | **98.29%** | 97.96% | **0.0852** | 0.0931 | **0.29%** | **0.29%** |
| AMAT | **95.83%** | 94.46% | **0.1768** | 0.2040 | **0.89%** | 0.97% |
| CMCSA | **93.39%** | 90.79% | **0.1851** | 0.2185 | **1.20%** | 1.35% |
| AEZS | **98.48%** | 96.82% | **0.4247** | 0.6148 | **2.18%** | 2.28% |
| EBAY | **94.88%** | 93.20% | **0.2250** | 0.2591 | **1.55%** | 1.64% |
| MU | **95.14%** | 93.45% | **0.2157** | 0.2504 | **1.17%** | 1.26% |
| WFM | **95.52%** | 94.00% | **0.2023** | 0.2339 | **1.57%** | 1.59% |
| SBUX | **94.77%** | 92.81% | **0.2241** | 0.2627 | **2.51%** | 2.63% |

# VFAR fits

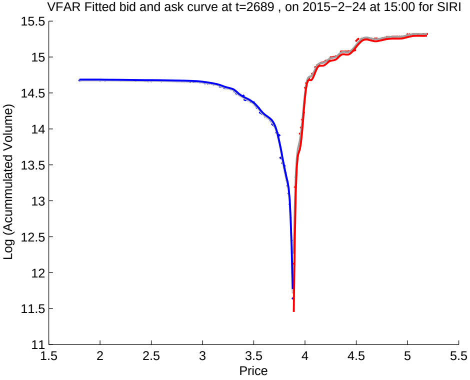
AMAT estimated bid (and ask) supply curves vs. the actually observed ones on 24 February 2015 at 3p.m. ▸ VFAR fits




Estimated VFAR curve

# VFAR fits

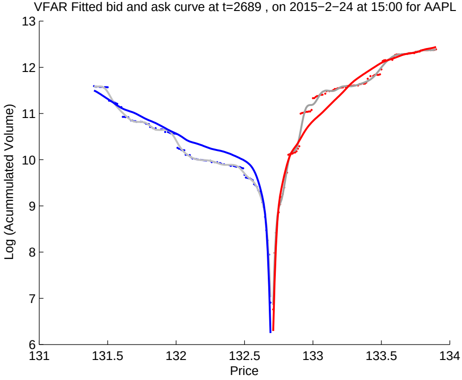
SIRI estimated bid (and ask) supply curves vs. the actually observed ones on 24 February 2015 at 3p.m.




Estimated VFAR curve

# VFAR fits

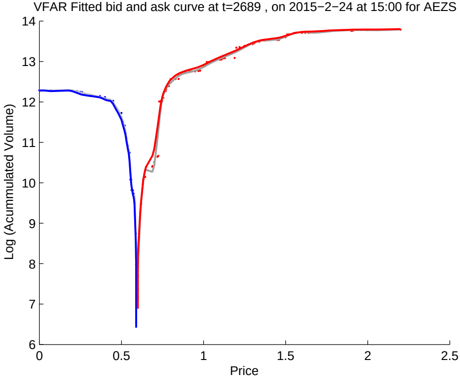
AAPL estimated bid (and ask) supply curves vs. the actually observed ones on 24 February 2015 at 3p.m.




Estimated VFAR curve

# VFAR fits

AEZS estimated bid (and ask) supply curves vs. the actually observed ones on 24 February 2015 at 3p.m.




Estimated VFAR curve

# Forecast

- Make an out-of-sample forecast for the liquidity curves starting from the 31st trading day onwards and predict $1-, 5-$ and $10-$step ahead forecasts that correspond to $5-, 25-$ and $50-$minute ahead liquidity curves respectively
- The first pair of forecasted curves is for time $t = 2251$, based on the past 30 trading days of $30 \times 75 = 2250$ functional objects
- Move forward one period, i.e. 5 minutes at a time, and re-estimate and forecast until the last time point at $t = 3300$

# Comparison with Naive Forecast

## RMSE for multistep ahead VFAR and naive forecast of the nine stocks

| Ticker Symbol | RMSE | | | | | |
|---|---|---|---|---|---|---|
| | 1-step | | 5-steps | | 10-steps | |
| | VFAR | Naive | VFAR | Naive | VFAR | Naive |
| AAPL | **0.3742** | 0.4318 | **0.4757** | 0.5678 | **0.5063** | 0.6193 |
| MSFT | **0.1809** | 0.1986 | **0.2444** | 0.2744 | **0.2707** | 0.3082 |
| INTC | **0.1927** | 0.2156 | **0.2559** | 0.2974 | **0.2845** | 0.3377 |
| CSCO | **0.2124** | 0.2317 | **0.3063** | 0.3447 | **0.3614** | 0.4111 |
| SIRI | **0.0894** | 0.0932 | **0.1271** | 0.1404 | **0.1416** | 0.1543 |
| AMAT | **0.1928** | 0.2141 | **0.2533** | 0.2940 | **0.2863** | 0.3374 |
| CMCSA | **0.1959** | 0.2240 | **0.2392** | 0.2952 | **0.2512** | 0.3208 |
| AEZS | **0.4814** | 0.6525 | **0.5483** | 0.7699 | **0.5812** | 0.8075 |
| EBAY | **0.2335** | 0.2620 | **0.2931** | 0.3477 | **0.3228** | 0.3961 |
| MU | **0.2180** | 0.2507 | **0.2797** | 0.3355 | **0.3094** | 0.3787 |
| WFM | **0.2035** | 0.2303 | **0.2575** | 0.3167 | **0.2746** | 0.3529 |
| SBUX | **0.2285** | 0.2623 | **0.2782** | 0.3366 | **0.2975** | 0.3687 |

# Comparison with Naive Forecast

## MAPE for multistep ahead VFAR and naive forecast of the nine stocks

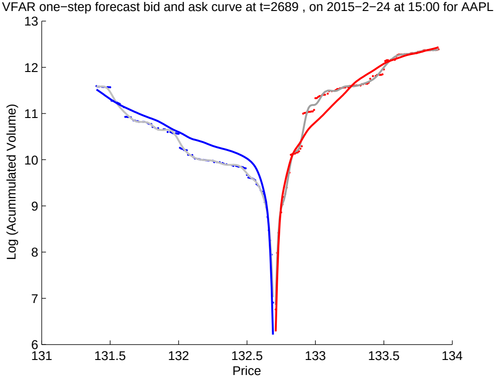
| Ticker Symbol | MAPE | | | | | |
|---|---|---|---|---|---|---|
| | 1-step | | 5-steps | | 10-steps | |
| | VFAR | Naive | VFAR | Naive | VFAR | Naive |
| AAPL | **3.61%** | 3.73% | **4.21%** | 4.68% | **4.49%** | 5.09% |
| MSFT | 0.93% | **0.90%** | 1.42% | **1.39%** | **1.63%** | 1.67% |
| INTC | **0.95%** | 0.97% | **1.46%** | 1.53% | **1.74%** | 1.86% |
| CSCO | **0.88%** | 0.89% | **1.44%** | 1.46% | **1.76%** | 1.85% |
| SIRI | 0.30% | **0.27%** | 0.52% | **0.47%** | 0.62% | **0.56%** |
| AMAT | **1.00%** | 1.01% | **1.47%** | 1.55% | **1.72%** | 1.87% |
| CMCSA | **1.15%** | 1.22% | **1.56%** | 1.76% | **1.69%** | 2.01% |
| AEZS | **2.22%** | 2.32% | **2.85%** | 3.20% | **3.14%** | 3.51% |
| EBAY | **1.31%** | 1.38% | **1.79%** | 1.99% | **2.03%** | 2.36% |
| MU | **1.18%** | 1.27% | **1.70%** | 1.90% | **1.99%** | 2.29% |
| WFM | **1.25%** | 1.31% | **1.76%** | 1.97% | **1.94%** | 2.32% |
| SBUX | **1.81%** | 1.87% | **2.27%** | 2.52% | **2.48%** | 2.86% |

# Findings

- Like the VFAR forecasts, the naive forecasts also have RMSE and MAPE increasing as one forecasts more steps into the future
- All VFAR forecasts outperform the naive forecasts in terms of RMSE
- Only in 5 (out of 36) instances, the naive forecasts perform better than VFAR forecasts in terms of MAPE

# VFAR forecasts

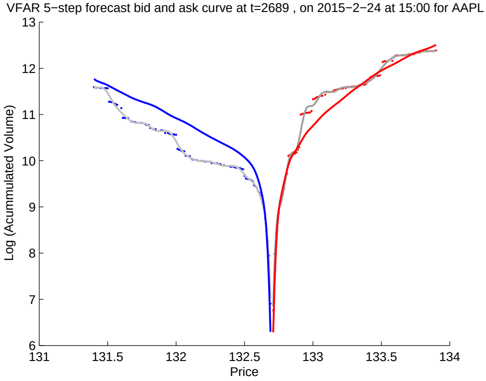
## AAPL 5−minute ahead forecasted bid (and ask) supply curves vs. the actually observed ones for 24 February 2015 at 3p.m. ▸ 1−step




Forecasted VFAR curve

# VFAR forecasts

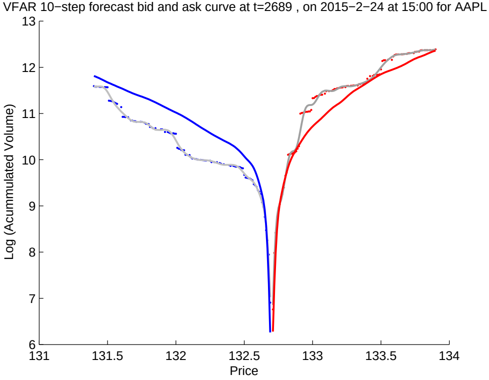
AAPL 25−minute ahead forecasted bid (and ask) supply curves vs. the actually observed ones for 24 February 2015 at 3p.m. ▸ 5−steps




Forecasted VFAR curve

# VFAR forecasts

AAPL 50−minute ahead forecasted bid (and ask) supply curves vs. the actually observed ones for 24 February 2015 at 3p.m. ▸ 10−steps




Forecasted VFAR curve

# Conclusion

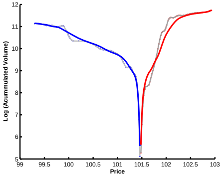
- Proposed the VFAR(1) modeling.
- Developed consistent ML estimators for VFAR(1) model with closed forms.
- In real data analysis, VFAR approach is more successful as compared to the naive model.
- Demand and supply curves are modelled and forecasted successfully.

# Forecasting Limit Order Book Liquidity with Functional AutoRegressive Dynamics

Ying Chen

Wee Song Chua

Wolfgang Karl Härdle

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# $B$-spline expansion

$$X_t^{(a)}(\tau) = \sum_{j=1}^{\infty} d_{t,j}^{a} B_{j,m}(\tau), \qquad X_t^{(b)}(\tau) = \sum_{j=1}^{\infty} d_{t,j}^{b} B_{j,m}(\tau),$$

$$\epsilon_t^{(a)}(\tau) = \sum_{j=1}^{\infty} d_{j}^{a}(\epsilon_t^{(a)}) B_{j,m}(\tau), \qquad \epsilon_t^{(b)}(\tau) = \sum_{j=1}^{\infty} d_{j}^{b}(\epsilon_t^{(b)}) B_{j,m}(\tau),$$

$$\kappa_{aa}(\tau) = \sum_{j=1}^{\infty} c_{j}^{aa} B_{j,m}(\tau), \qquad \kappa_{bb}(\tau) = \sum_{j=1}^{\infty} c_{j}^{bb} B_{j,m}(\tau),$$

$$\kappa_{ab}(\tau) = \sum_{j=1}^{\infty} c_{j}^{ab} B_{j,m}(\tau). \qquad \kappa_{ba}(\tau) = \sum_{j=1}^{\infty} c_{j}^{ba} B_{j,m}(\tau).$$

back

# VFAR & $B-$Splines

$$
\begin{aligned}
X_t^{(a)}(\tau) &= \sum_{j=1}^{J_n} d_{t,j}^a B_{j,m}(\tau) \\
&= \int_0^1 \left\{ \sum_{j=1}^{J_n} \sum_{i=1}^{J_n} c_j^{aa} d_{t-1,i}^a B_{j,m}(\tau - s) B_{i,m}(s) \right\} ds \\
&+ \int_0^1 \left\{ \sum_{j=1}^{J_n} \sum_{i=1}^{J_n} c_j^{ab} d_{t-1,i}^b B_{j,m}(\tau - s) B_{i,m}(s) \right\} ds + \sum_{j=1}^{J_n} d_j^a (\varepsilon_t^{(a)}) B_{j,m}(\tau) \\
&= \sum_{h=1}^{J_n} \sum_{i=1}^{J_n} \left\{ \sum_{j=1}^{J_n} \left( \frac{w_{j+m} - w_{j+1}}{w_{j+m} - w_j} - \frac{w_{j+m+1} - w_{j+2}}{w_{j+m+1} - w_{j+1}} \right) c_j^{aa} - c_h^{aa} \right\} \frac{w_{i+m} - w_i}{m} d_{t-1,i}^a B_{h,m}(\tau) \\
&+ \sum_{h=1}^{J_n} \sum_{i=1}^{J_n} \left\{ \sum_{j=1}^{J_n} \left( \frac{w_{j+m} - w_{j+1}}{w_{j+m} - w_j} - \frac{w_{j+m+1} - w_{j+2}}{w_{j+m+1} - w_{j+1}} \right) c_j^{ab} - c_h^{ab} \right\} \frac{w_{i+m} - w_i}{m} d_{t-1,i}^b B_{h,m}(\tau) \\
&\quad + \sum_{j=1}^{J_n} d_j^a (\varepsilon_t^{(a)}) B_{j,m}(\tau)
\end{aligned} \tag{9}
$$

# Maximum likelihood estimation

Log-likelihood function:

$$
\begin{aligned}
&\ell\left(X_1^{(a)},\cdots,X_T^{(a)},X_1^{(b)},\cdots,X_T^{(b)};\rho^{aa},\rho^{ab},\rho^{ba},\rho^{bb}\right)\\
&=-\frac{KT}{2}\log 2\pi-\frac{T}{2}\log\left|\Sigma_u\right|-\frac{1}{2}\left(\mathbf{y}-(Z^\top\otimes I_K)\beta\right)^\top(I_T\otimes\Sigma_u^{-1})\left\{\mathbf{y}-(Z^\top\otimes I_K)\beta\right\}\\
&=-\frac{KT}{2}\log 2\pi-\frac{T}{2}\log\left|\Sigma_u\right|-\frac{1}{2}\sum_{t=1}^{T}\left(y_t-v-Cy_{t-1}\right)^\top\Sigma_u^{-1}\left(y_t-v-Cy_{t-1}\right)\\
&=-\frac{KT}{2}\log 2\pi-\frac{T}{2}\log\left|\Sigma_u\right|-\frac{1}{2}\sum_{t=1}^{T}\left(y_t-Cy_{t-1}\right)^\top\Sigma_u^{-1}\left(y_t-Cy_{t-1}\right)\\
&\qquad\qquad+v^\top\Sigma_u^{-1}\sum_{t=1}^{T}\left(y_t-Cy_{t-1}\right)-\frac{T}{2}v^\top\Sigma_u^{-1}v\\
&=-\frac{KT}{2}\log 2\pi-\frac{T}{2}\log\left|\Sigma_u\right|-\frac{1}{2}\operatorname{Tr}\left\{(Y-BZ)^\top\Sigma_u^{-1}(Y-BZ)\right\}
\end{aligned}
$$

back

# Consistency result

Let $H(\boldsymbol{\rho}, \boldsymbol{\psi})$ denote the conditional entropy between a set of operators $\boldsymbol{\rho} = (\rho^{aa}, \rho^{ab}, \rho^{ba}, \rho^{bb})$ and a given set of operators $\boldsymbol{\psi}$:

$$H(\boldsymbol{\rho}, \boldsymbol{\psi}) = \mathrm{E}_{\boldsymbol{\rho}} \left[ \log g(X_t^{(a)}, X_t^{(b)}, X_{t-1}^{(a)}, X_{t-1}^{(b)}, \boldsymbol{\psi}) \right].$$

We consider the following conditions:

C1: If there exists a sequence $\{\boldsymbol{\rho}_{J_n}\}$ such that $\boldsymbol{\rho}_{J_n} \in \Theta_{J_n} \forall n$ and $H(\boldsymbol{\rho}_{0|\Theta_{J_n}}, \boldsymbol{\rho}_{J_n}) \to H(\boldsymbol{\rho}_{0|\Theta_{J_n}}, \boldsymbol{\rho}_{0|\Theta_{J_n}})$, then $\left\| \boldsymbol{\rho}_{J_n} - \boldsymbol{\rho}_{0|\Theta_{J_n}} \right\|_{HS} \to 0$. Here $\boldsymbol{\rho}_{0|\Theta_{J_n}}$ denotes the projection of the set of true operators $\boldsymbol{\rho}_0$ on the sieve $\Theta_{J_n}$.

C2: There exists a sequence $\{\boldsymbol{\rho}_{J_n}\}$ described in **C1** such that $H(\boldsymbol{\rho}_{0|\Theta_{J_n}}, \boldsymbol{\rho}_{J_n}) \to H(\boldsymbol{\rho}_{0|\Theta_{J_n}}, \boldsymbol{\rho}_{0|\Theta_{J_n}})$.

back

# $R^2$ with different values of $J_n$

| $J_n$ | Ticker Symbol | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | AAPL | MSFT | INTC | CSCO | SIRI | AMAT | CMCSA | AEZS | EBAY | MU | WFM | SBUX |
| 6 | 89.55% | 91.14% | 89.74% | 93.08% | 95.60% | 89.87% | 83.96% | 98.29% | 87.97% | 90.46% | 88.78% | 88.17% |
| 7 | 90.62% | 92.74% | 91.70% | 94.14% | 96.61% | 92.55% | 88.93% | 98.46% | 91.09% | 92.26% | 91.47% | 91.12% |
| 8 | 91.21% | 93.60% | 92.83% | 94.79% | 97.15% | 93.98% | 90.93% | 98.57% | 92.69% | 93.33% | 93.03% | 92.75% |
| 9 | 91.48% | 94.05% | 93.39% | 95.15% | 97.45% | 94.59% | 91.67% | 98.61% | 93.50% | 93.92% | 93.93% | 93.50% |
| 10 | 91.68% | 94.47% | 93.90% | 95.49% | 97.69% | 95.06% | 92.27% | **98.64%** | 94.12% | 94.41% | 94.62% | 94.07% |
| 11 | 91.78% | 94.72% | 94.21% | 95.72% | 97.86% | 95.34% | 92.66% | 98.64% | 94.46% | 94.67% | 94.99% | 94.35% |
| 12 | 91.84% | 94.86% | 94.38% | 95.84% | 97.95% | 95.47% | 92.85% | 98.64% | 94.59% | 94.79% | 95.13% | 94.45% |
| 13 | 91.88% | 94.96% | 94.51% | 95.93% | 98.03% | 95.58% | 92.99% | 98.61% | 94.68% | 94.89% | 95.24% | 94.53% |
| 14 | 91.91% | 95.05% | 94.60% | 96.00% | 98.10% | 95.67% | 93.12% | 98.58% | 94.74% | 94.96% | 95.32% | 94.58% |
| 15 | 91.94% | 95.10% | 94.67% | 96.06% | 98.16% | 95.74% | 93.23% | 98.53% | 94.78% | 95.02% | 95.37% | 94.64% |
| 16 | 91.95% | 95.12% | 94.69% | 96.08% | 98.18% | 95.74% | 93.24% | 98.45% | 94.79% | 95.04% | 95.40% | 94.67% |
| 17 | 91.99% | 95.16% | 94.73% | 96.12% | 98.22% | 95.78% | 93.31% | 98.43% | 94.82% | 95.08% | 95.45% | 94.70% |
| 18 | 92.00% | 95.16% | 94.75% | 96.13% | 98.24% | 95.80% | 93.33% | 98.40% | 94.84% | 95.10% | 95.46% | 94.72% |
| 19 | 92.02% | 95.19% | 94.77% | 96.15% | 98.27% | **95.84%** | 93.38% | 98.40% | 94.86% | 95.13% | 95.50% | 94.75% |
| 20 | **92.03%** | **95.19%** | **94.79%** | **96.16%** | **98.29%** | 95.83% | **93.39%** | 98.48% | **94.88%** | **95.14%** | **95.52%** | **94.77%** |

back