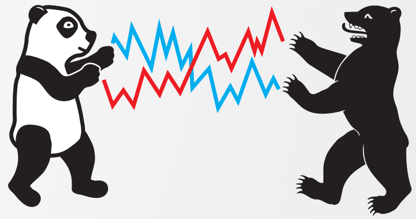
Pricing green financial products

Wolfgang Karl Härdle
Brenda López Cabrera
Awdesch Melzer

Ladislaus von Bortkiewicz Chair of Statistics
Humboldt-Universität zu Berlin
lvb.wiwi.hu-berlin.de



Hedging weather risk

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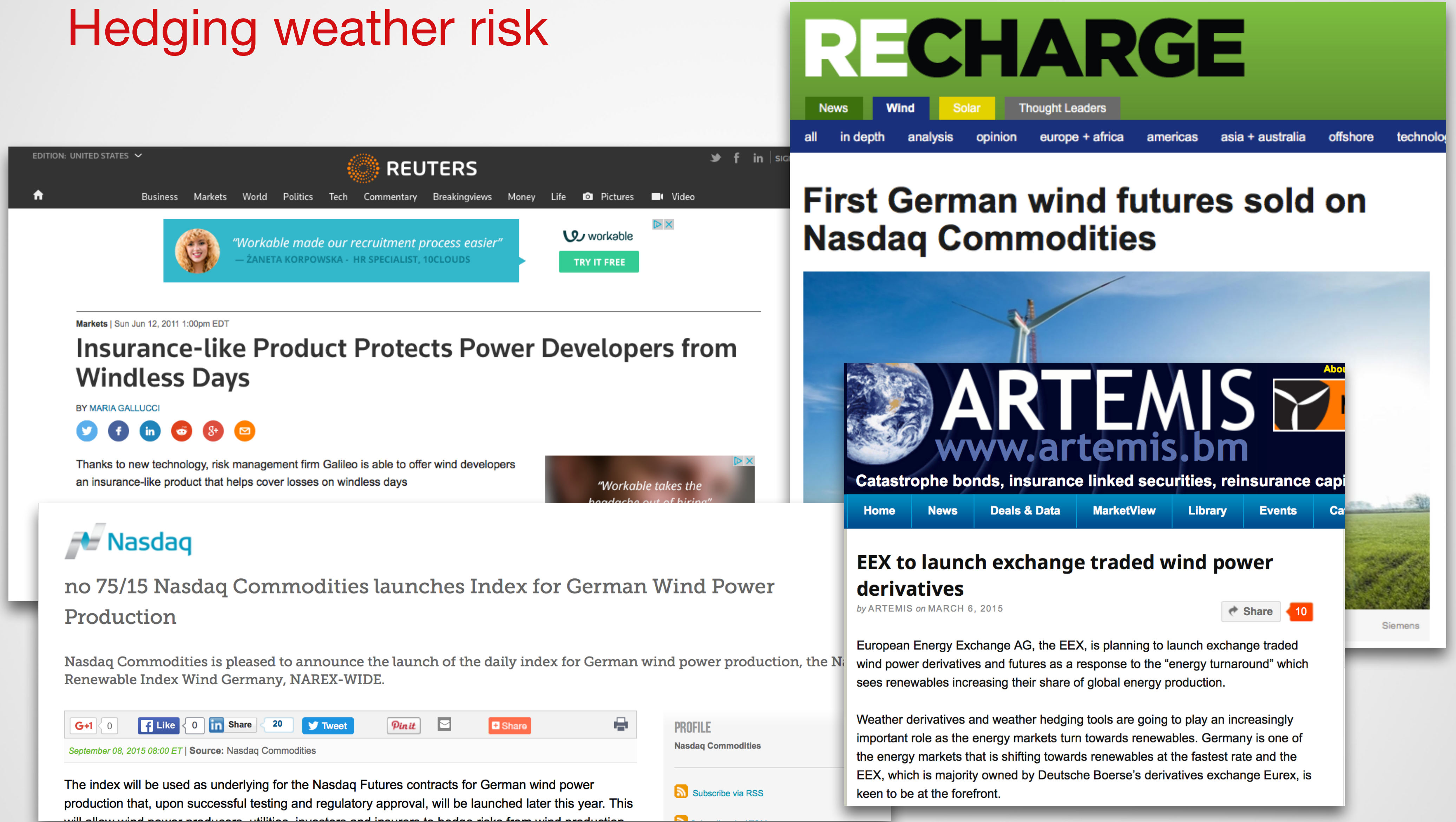


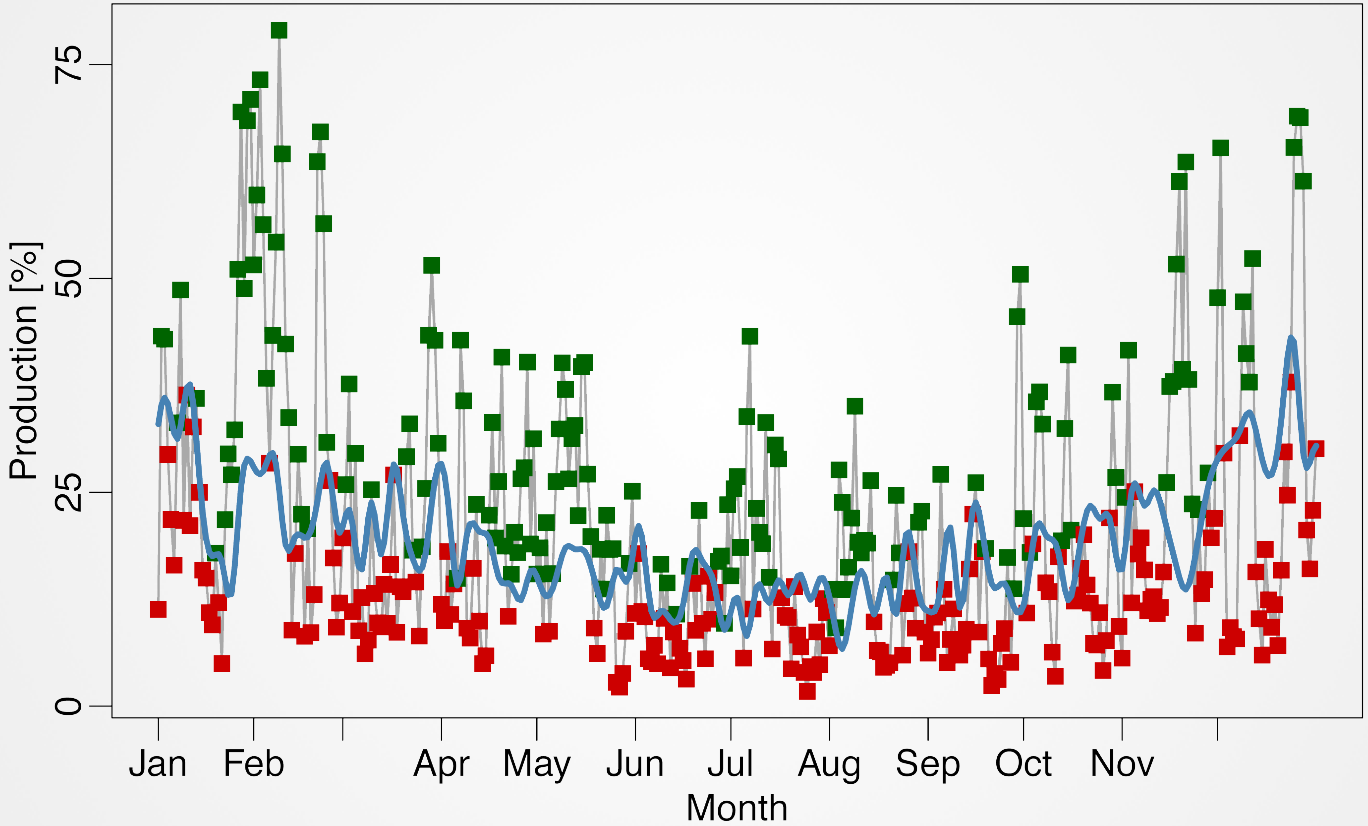
Risk in wind power production

- Project completion risk
- Operating risk
- Revenue risk

  subsidies, e.g. feed-in tariff, floating market price premium are per unit electricity

  - Price risk
  - Volume risk

Volume risk



Relative to expected production (stylised index curve),
investors benefit from holding WPF,
wind power generators benefit from selling WPF.

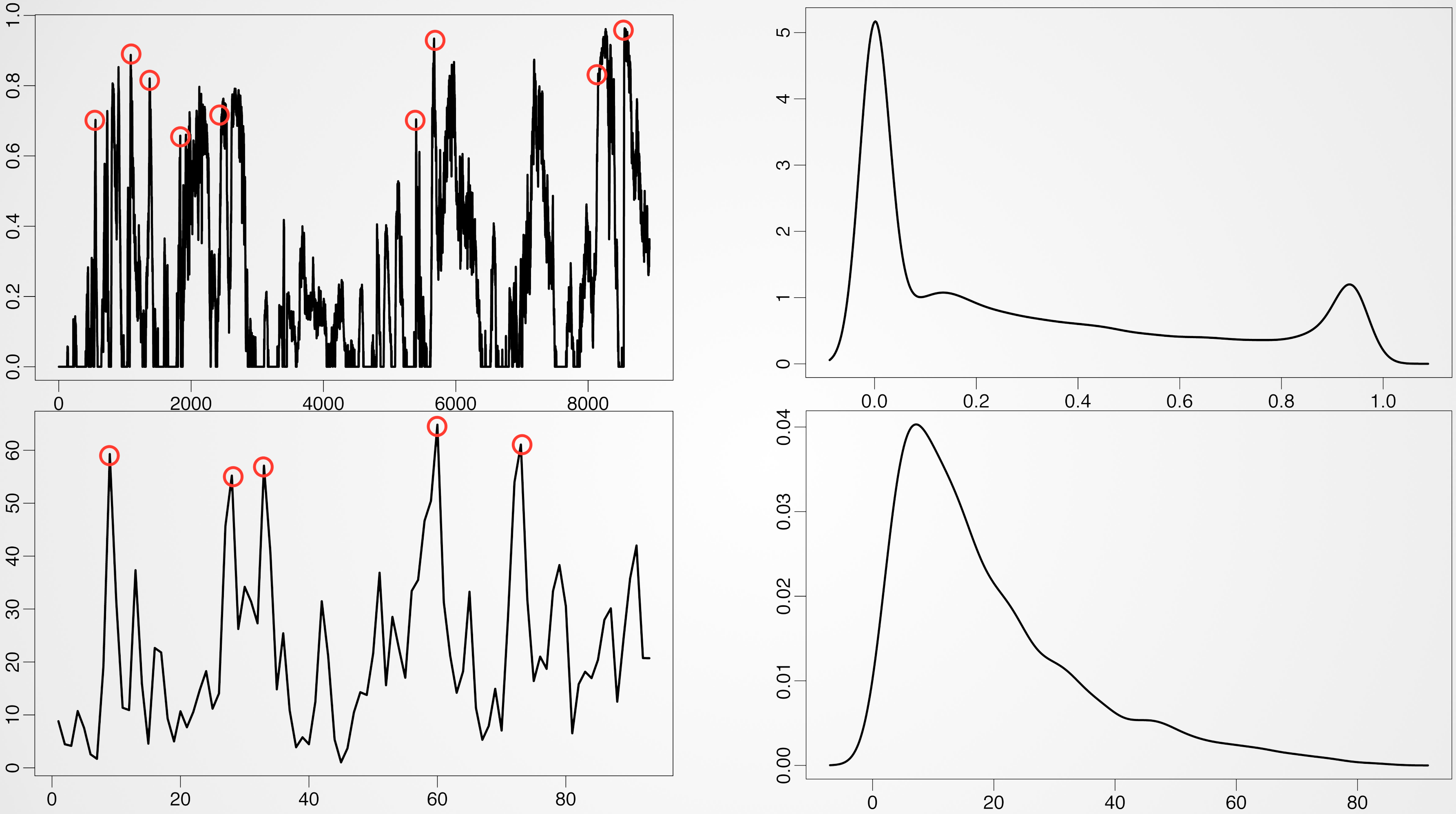
Wind power futures at Nasdaq

- **Contract:** A contract settling against the expected power production of future delivery periods $[\tau_1, \tau_2]$
- **Underlying:** NAREX-WIDE: average utilisation relative to the available capacity

$$NAREX(\tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} U(s) ds$$

with $U(s) = \frac{WPL(s)}{C(s)}$, where *WPL(s)* is the long term wind power load and *C(s)* the capacity at time *s*.

Wind power utilisation: non-standard pdf



Top: 5 min production data of 1 wind farm
Bottom: daily aggregates on country level

Research approaches for volume risk

- Equilibrium models
  - Bessembinder & Lemmon (1998)
  - Gersema & Wozabal (2017)
- No-arbitrage pricing models
  - Benth & Saltyte Benth (2009)
  - Alexandridis & Zapranis (2013)

Gaussian vs non-Gaussian models

Gaussian stochastic pricing CAR(p)

- Wind Futures: Benth & Saltyte Benth (2009), Alexandridis & Zapranis (2013)
- Weather Derivatives: Benth et al. (2011), Benth et al. (2007), Härdle & López Cabrera (2012), Härdle et al (2016), …

Non-Gaussian pricing CARMA(p,q)-Lévy

- Electricity Futures: Benth et al. (2014)
- Electricity pricing: Veraart (2016)
- Theory: Barndorff-Nielson & Shephard (2001), …

FEB Four Algorithm

FEB Four Algorithm

Econometrics

utilisation

$\downarrow$

deseasonalisation

$\downarrow$

ARMA(p, q)

$\downarrow$

seasonal variance

normalising increments

Fin. Mathematics

$CARMA(p, q)$

$\downarrow$

Futures pricing

$\downarrow$

Market price of risk

Benth et al. (2007), Härdle et al. (2016)

Research questions

- Is there a simple model mimicking weather dynamics with spikes and heavy-tailed distributions
- How to achieve Gaussian increments?
  - Logit-transform
  - Smooth Inter-Quartile (Expectile)-Range
- In-sample fit: Gaussian vs. non-Gaussian?
- Out-of-sample performance:
  - Calibration of market price of risk
  - Futures price forecasting

Outline

1. Motivation
2. FEB4 algorithm
3. Data
4. Model comparison
5. Conclusion

FEB Four Algorithm

Econometrics

utilisation Model Transform

$\downarrow$

deseasonalisation

$\downarrow$

ARMA(p, q)

$\downarrow$

seasonal variance

normalising increments

Fin. Mathematics

$CARMA(p, q)$ Lévy Gauss

$\downarrow$

Futures pricing

$\downarrow$

Market price of risk

NAREX-WIDE index

Data from transmission system operators (TSO)

Approximation of NAREX-WIDE index by using

1. Consistent time series of WP generation; training (2010-2015), test (2016) at 15 min resolution
2. Aggregate production TSO control areas (Tennet, Amprion, 50Hertz, Transnet BW)
3. Scale production by capacity (time series)
4. Calculate daily average utilisation factor

NAREX-WIDE is provided by MeteoGroup

Nasdaq WPF: quarterly, monthly and weekly: 01/2016-now

Gaussian and Lévy fit

Lévy				Gaussian		
b_1	a_1	a_2	a_3	a_1	a_2	a_3
-0.687	1.469	0.164	0.758	2.276	1.767	0.430

Left: CARMA(3,1) coefficient estimates of the CARMA-Lévy process; Right: CARMA(3,0) of the Gaussian process. AIC for the transformed Gaussian model: 6209.124

GH fit for Lévy increments

Distribution	λ	$\bar{\alpha}$	χ	ψ
GHYP	$\lambda \in \mathbb{R}$	$\bar{\alpha} > 0$	$\chi > 0$	$\psi > 0$
NIG	$\lambda = -\frac{1}{2}$	$\bar{\alpha} > 0$	$\chi > 0$	$\psi > 0$
Student-t (ν df)	$\lambda = -\frac{\nu}{2} < 1$	$\bar{\alpha} = 0$	$\chi > 0$	$\psi = 0$
HYP	$\lambda = -\frac{d+1}{2}$	$\bar{\alpha} > 0$	$\chi > 0$	$\psi > 0$
VG	$\lambda > 0$	$\bar{\alpha} = 0$	$\chi = 0$	$\psi > 0$

Model	Sym	λ	$\bar{\alpha}$	μ	σ	γ	AIC
VG	F	1.426	0.000	−5.386	10.410	5.390	16530.366
GHYP	F	1.418	0.052	−5.365	10.416	5.367	16532.365
HYP	F	1.000	0.008	−4.580	10.886	4.587	16547.773
NIG	F	−0.500	1.229	−5.663	10.308	5.664	16551.142

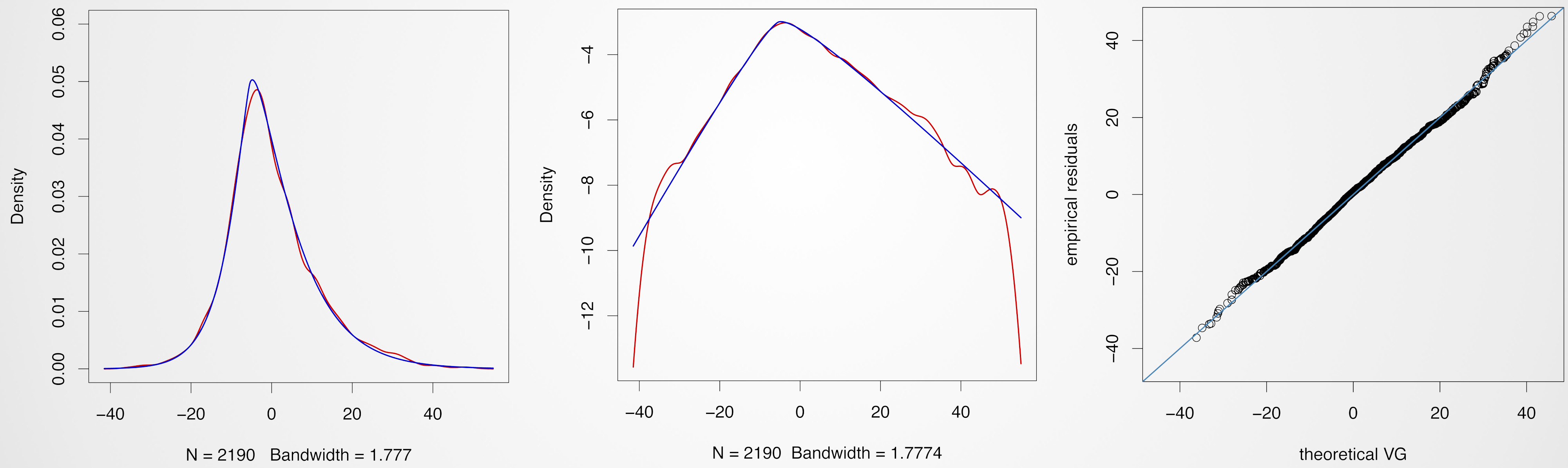
Estimation results: Fit of CARMA-Lévy increments to a selection of generalised hyperbolic distributions

Gaussian case: AIC estimate for transformed data

Following Akaike (1978) on AIC for transformed data gives

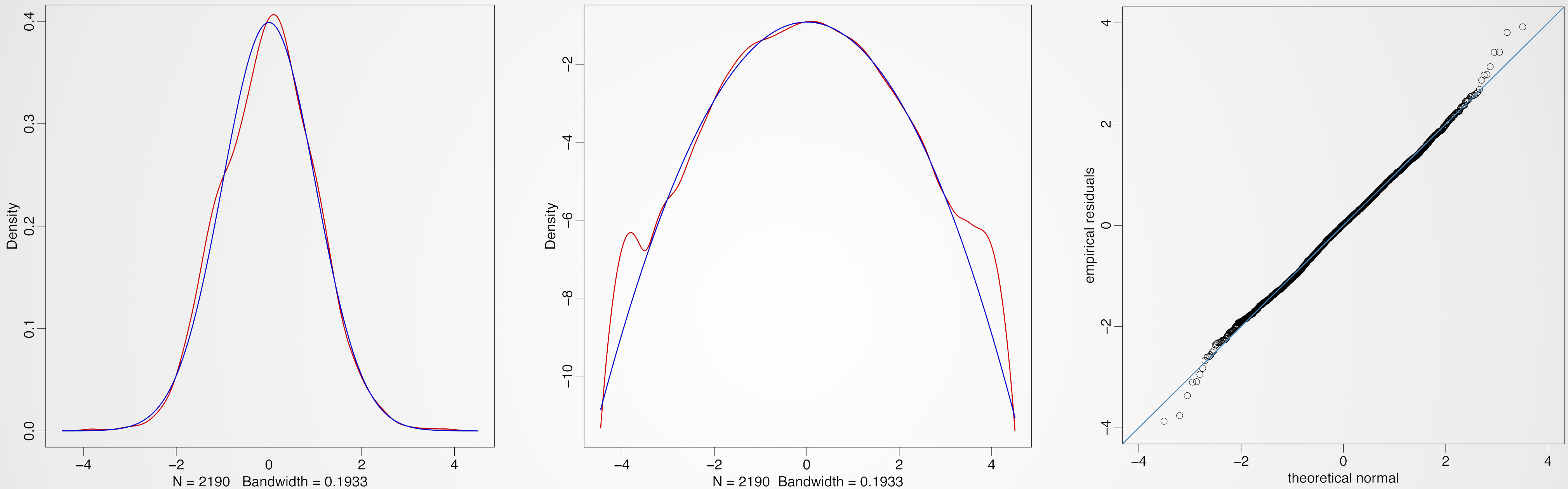
$$
\begin{aligned}
\text{AIC(transformed-model)} + \text{Adjustment} &= \text{AIC} + 2\sum_{t=1}^{T} \log\left(\frac{\partial \frac{U_t}{1-U_t}}{\partial U_t}\right) \\
&= \text{AIC} + 2\sum_{t=1}^{T} \log\left(\frac{1}{U_t - U_t^2}\right) \\
&= 6209.124 + 2 \cdot (-4874.352) \\
&= 15957.83
\end{aligned}
$$

Variance Gamma increments



Left to right: empirical vs theoretical pdf, log pdf, q-q-plot

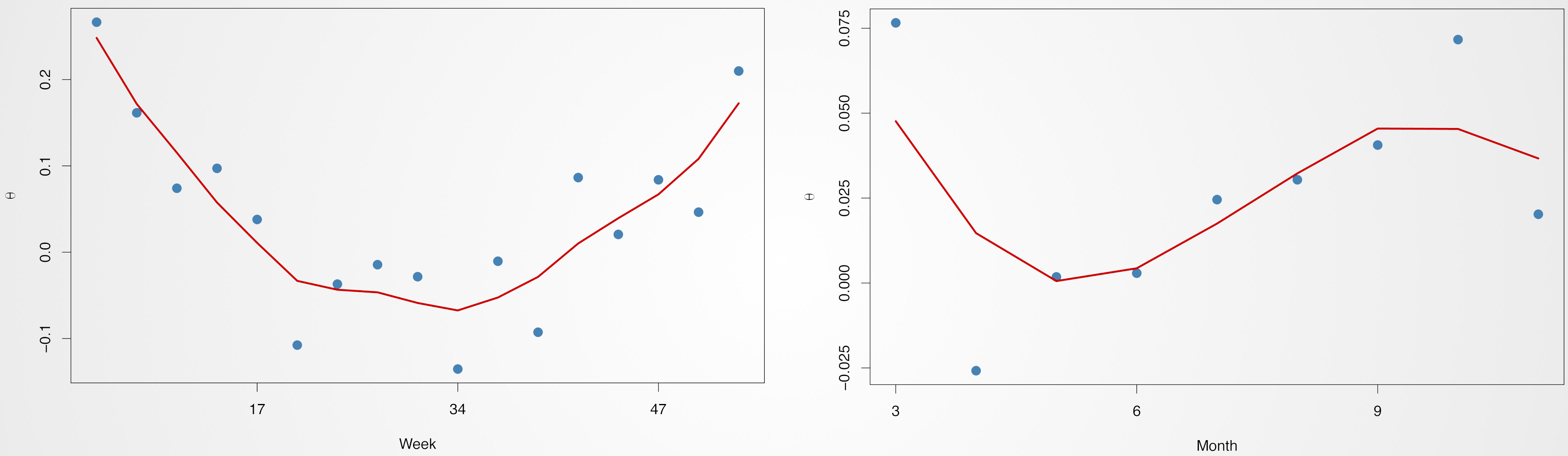
Gaussian increments



Left to right: empirical vs theoretical pdf, log pdf, q-q-plot

PGFP

Seasonal MPR - contract wise



Seasonal structure of MPR. Left: MPR for weekly contracts. Right: MPR for mostly contracts. Samples consist of 17 weekly and 9 monthly contracts

PGFP

MPR is strongly affected by seasonal variance!

RMSE	θ	$\widehat{\theta}^{OLS}$	$\widehat{\theta}_t^{smooth}$	$\widehat{\theta}_{CV}^{OLS}$	$\widehat{\theta}_{t,CV}^{smooth}$
mean(θ)	0	0.059	0.056	0.059	0.056
$\Delta(U_t, F_t)$	5.438	5.438	5.438	5.438	5.438
$\Delta(U_t, \widehat{F}_t)$	12.860	11.092	6.025	11.092	6.021
$\Delta(F, \widehat{F}_t)$	10.328	9.027	3.489	9.027	3.399

MAPE	θ	$\widehat{\theta}^{OLS}$	$\widehat{\theta}_t^{smooth}$	$\widehat{\theta}_{CV}^{OLS}$	$\widehat{\theta}_{t,CV}^{smooth}$
mean(θ)	0	0.059	0.056	0.059	0.056
$\Delta(U_t, F_t)$	21.271	21.271	21.271	21.271	21.271
$\Delta(U_t, \widehat{F}_t)$	43.014	44.164	22.772	44.164	22.638
$\Delta(F, \widehat{F}_t)$	35.302	38.404	18.560	38.404	18.107

Out-of-sample backtesting - weekly contracts. Top: RMSE. Bottom: MAPE. Tuning parameter for smoothing splines: $\zeta \approx 0.0002$. Cross-validation with $\zeta \approx 0.0003$

RMSE	θ	$\widehat{\theta}^{OLS}$	$\widehat{\theta}_t^{smooth}$	$\widehat{\theta}_{CV}^{OLS}$	$\widehat{\theta}_{t,CV}^{smooth}$
mean(θ)	0	0.036	0.036	0.036	0.036
$\Delta(U_t, F_t)$	1.833	1.833	1.833	1.833	1.833
$\Delta(U_t, \widehat{F}_t)$	3.801	2.667	2.214	2.667	2.117
$\Delta(F, \widehat{F}_t)$	4.065	2.647	1.622	2.647	1.504

MAPE	θ	$\widehat{\theta}^{OLS}$	$\widehat{\theta}_t^{smooth}$	$\widehat{\theta}_{CV}^{OLS}$	$\widehat{\theta}_{t,CV}^{smooth}$
mean(θ)	0	0.036	0.036	0.036	0.036
$\Delta(U_t, F_t)$	10.033	10.033	10.033	10.033	10.033
$\Delta(U_t, \widehat{F}_t)$	17.525	13.550	11.759	13.550	11.376
$\Delta(F, \widehat{F}_t)$	18.149	11.353	6.187	11.353	5.729

Out-of-sample backtesting - monthly contracts. Top: RMSE. Bottom: MAPE. Tuning parameter for smoothing splines: $\zeta \approx 0.0002$. Cross-validation with $\zeta \approx 0.0003$

Conclusion

- In-sample fit and model comparison in favour of Gaussian model
- Smooth IER/IQR achieve Gaussian increments
- Negative MPR: producers pay a premium as insurance fee
- Positive MPR: investors pay premium for reduction of risk

- Lack of location specific future contracts and indices
    - retrieve information from seasonal productivity factor maps

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The model

Assume that the underlying $U(t)$ follows a mean-reverting additive process

$$U(t) = \Lambda(t) + Y(t)$$

$\Lambda(t)$ - deterministic trend-seasonal production level

$Y(t)$ - short term variation

CARMA(*p*, *q*)

Suppose $Y(t)$ follows a stable continuous time autoregressive moving average process

$$a(D)\mathbf{Y_t} = b(D)D\mathbf{B}(t), \quad D \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t},$$

where the autoregressive polynomial is given by

$$P(z) = z^p + a_1 z^{p-1} + \ldots + a_p$$

and the moving average polynomial by

$$Q(z) = b_0 + b_1 z^q + \ldots + b_{p-1} z^{p-1}.$$

$$\widetilde{U}_t = \Lambda_t + Y_t$$

$$Y_t = \mathbf{b}^\top \mathbf{X}_t \qquad \text{state equation}$$

$$d\mathbf{X}_t = (\mathbf{A}\mathbf{X}_t + \mathbf{e}_p \sigma_t \theta_t)dt + \mathbf{e}_p \sigma_t dB_t^\theta \qquad \text{observation equation}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\alpha_p & -\alpha_{p-1} & \cdots & \cdots & -\alpha_1 \end{pmatrix} \quad \mathbf{e}_p = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 1 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{pmatrix} \quad \mathbf{X}_t = \begin{pmatrix} X_t \\ X_t^{(1)} \\ \vdots \\ X_t^{(p-2)} \\ X_t^{(p-1)} \end{pmatrix}$$

Proof *A*

Future price dynamics

For the short term variation dynamics

$$Y_t = \mathbf{b}^\top \exp\{A(t-s)\}\mathbf{X}_s + \int_s^t \mathbf{b}^\top \exp\{A(t-u)\}\mathbf{e}_p \sigma_u \mathrm{d}B_u, \quad u \le s < t$$

we define the future price $F(t, \tau_1, \tau_2)$ at time *t* for a contract maturing at $\tau_2 > t$

$$F(t, \tau_1, \tau_2) = \mathsf{E}^{Q^\theta}\left[\frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} U(s)\mathrm{d}s \middle| \mathcal{F}_t\right], \quad 0 \le t \le \tau < \infty,$$

where Q^θ is the risk-neutral probability measure.

Brownian Motion

Lévy Process

Future prices

Consider a stochastic process $X(t)$ with dynamics under Q^θ

$$\mathbf{X}_t = \exp\{\mathbf{A}(t-s)\}\mathbf{X}_s + \int_s^t \exp\{\mathbf{A}(t-u)\}e_p\sigma_u\theta_u du$$
$$+ \int_s^t \exp\{\mathbf{A}(t-u)\}e_p\sigma_u dB_u^\theta$$

The mean of the stochastic process is then given by

$$\mu_\theta(s,t,\mathbf{X}_t) \triangleq e_1^\top \exp\{\mathbf{A}(t-s)\}\mathbf{X}_s + \int_s^t e_1^\top \exp\{\mathbf{A}(t-u)\}e_p\sigma_u\theta_u du$$

and the variance by

$$\Sigma^2(s,t) \triangleq \int_s^t \sigma_u \left[e_1^\top \exp\{\mathbf{A}(t-u)\}e_p\right]^2 du.$$

Future prices

$$\widehat{F}_{t,\tau_1,\tau_2} = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \left(1 + \exp\left[-\left\{\Lambda_t + \mu_\theta(s,t,\mathbf{X}_t) + \frac{1}{2}\Sigma^2(s,t)\right\}\right]\right)^{-1}$$

Then the back-transformed power utilisation is given by:

$$U_t = \left(1 + \exp\left[-\left\{\Lambda_t + \mu_\theta(s,t,\mathbf{X}_t) + \Sigma(s,t)Z\right\}\right]\right)^{-1}, Z \sim \mathsf{N}(0,1)$$

CARMA(p, q)-Lévy

$$
\begin{aligned}
U_t &= \Lambda_t + Y_t \\
d\Lambda_t &= \Lambda_t dt \\
Y_t &= \mathbf{b}^\top \mathbf{X}_t && \text{state equation} \\
d\mathbf{X}_t &= (\mathbf{A}\mathbf{X}_t)dt + \mathbf{e}_p d\mathbf{L}_t, && \text{observation equation}
\end{aligned}
$$

$$
\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\alpha_p & -\alpha_{p-1} & \cdots & \cdots & -\alpha_1 \end{pmatrix}
\quad \mathbf{e}_p = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
\quad \mathbf{b} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{pmatrix}
\quad \mathbf{X}_t = \begin{pmatrix} X_t \\ X_t^{(1)} \\ \vdots \\ X_t^{(p-2)} \\ X_t^{(p-1)} \end{pmatrix}
$$

Proof A

Let $\mathbf{L}_t = \int_0^t \int_0^\infty z N^L(ds, dz), t \in [0, T]$, the pure jump Lévy process with N^L a Poisson random measure with Lévy measure / that satisfies $\int_0^\infty z l(dz) < \infty$

If all eigenvalues of coefficient matrix **A** have negative real parts, then $(\mathbf{X}_t)_{t \in \mathbb{R}}$ is given by

$$\mathbf{X}_t = \int_{-\infty}^t \exp\{\mathbf{A}(t-s)\}\mathbf{e}_p d\mathbf{L}_s$$

$$Y_t = \mathbf{b}^\top \mathbf{X}_t = \int_{-\infty}^t \mathbf{b}^\top \exp\{\mathbf{A}(t-s)\}\mathbf{e}_p d\mathbf{L}_s$$

The future price is given by

$$F_t(T) = \Lambda_T + \left[\int_{-\infty}^t b^\top \exp\{A(T-s)\}\mathrm{d}\mathbf{L}_s + \mathsf{E}^{Q^\theta}[\mathbf{L}_1] \int_t^T b^\top \exp\{A(T-s)\}\theta(s) ds\right]$$

Logit-normal adjustment

For the logit-normal transformation define CAR(p) model for wind power

$$\widetilde{U}_t = \gamma(U) \stackrel{\text{def}}{=} \log\left(\frac{U_t}{1-U_t}\right) = \Lambda_t + Y_t, \quad U_t \in (0,1),$$

$$U_t = \gamma^{-1}(\widetilde{U}_t) \stackrel{\text{def}}{=} \{1+\exp(-\widetilde{U}_t)\}^{-1} = [1+\exp\{-(\Lambda_t + Y_t)\}]^{-1},$$

Pinsen (2012)

Risk Premium

Let the risk premium (RP) be:

$$RP^i_{\tau^i_1,\tau^i_2} \triangleq \int_{\tau^i_1}^{\tau^i_2} \theta_u \sigma_u \mathbf{e}_1^\top \mathbf{A}^{-1}[\exp\{\mathbf{A}(\tau^i_2 - u)\} - \mathbf{I}_p]\mathbf{e}_p du,$$

Constant MPR estimated by OLS:

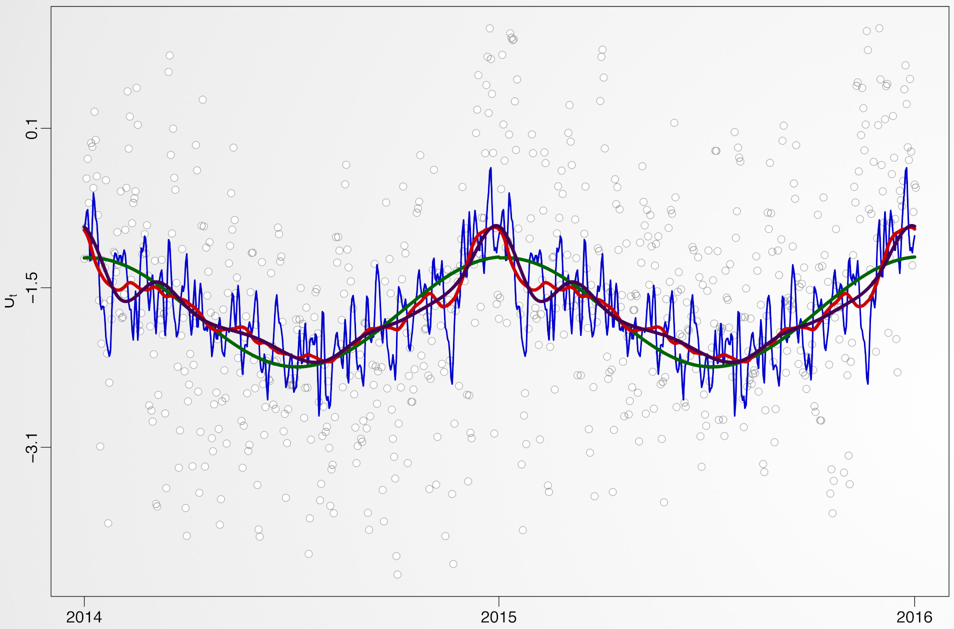
$$\hat{\theta}^i_t = \arg\min_{\theta^i_t} \left(F_{NAREX(t,\tau^i_1,\tau^i_2)} - \widehat{F}_{NAREX(t,\tau^i_1,\tau^i_2)} \right)^2,$$

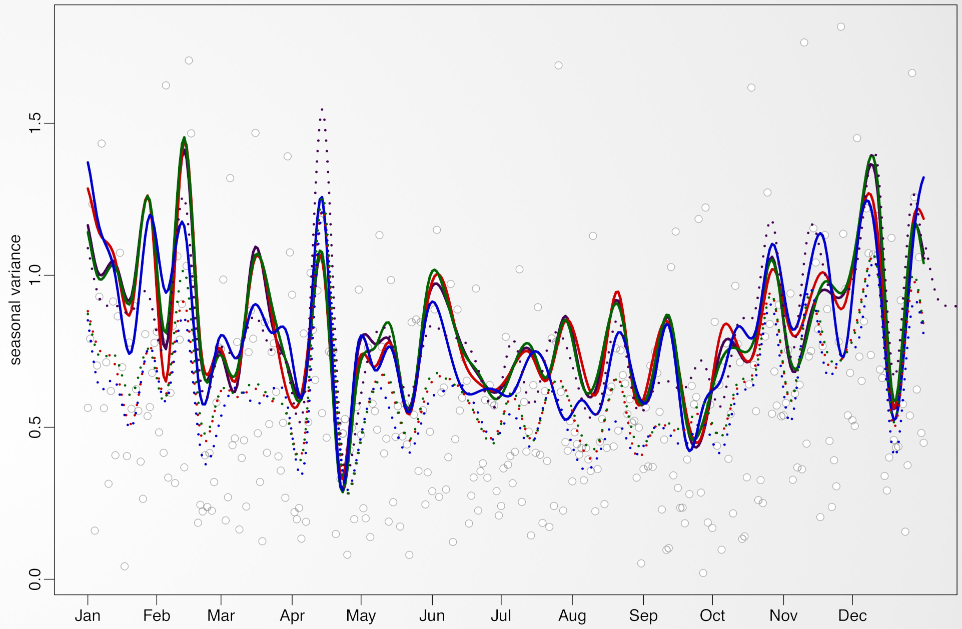
Smooth MPR estimated by smoothing-splines:

$$\arg \min_{\{f,\zeta\} \in \mathbb{R}} \sum_{t=1}^{n} \left\{ \widehat{\theta}_t - f(u_t) \right\}^2 + \zeta \int dt \left\{ \frac{\partial^2 f(u_t)}{\partial t^2} \right\}^2$$

where $i = 1, \ldots, I$ future contracts with measurement periods $[\tau^i_1, \tau^i_2], t \leq \tau^i_1 < \tau^i_2$

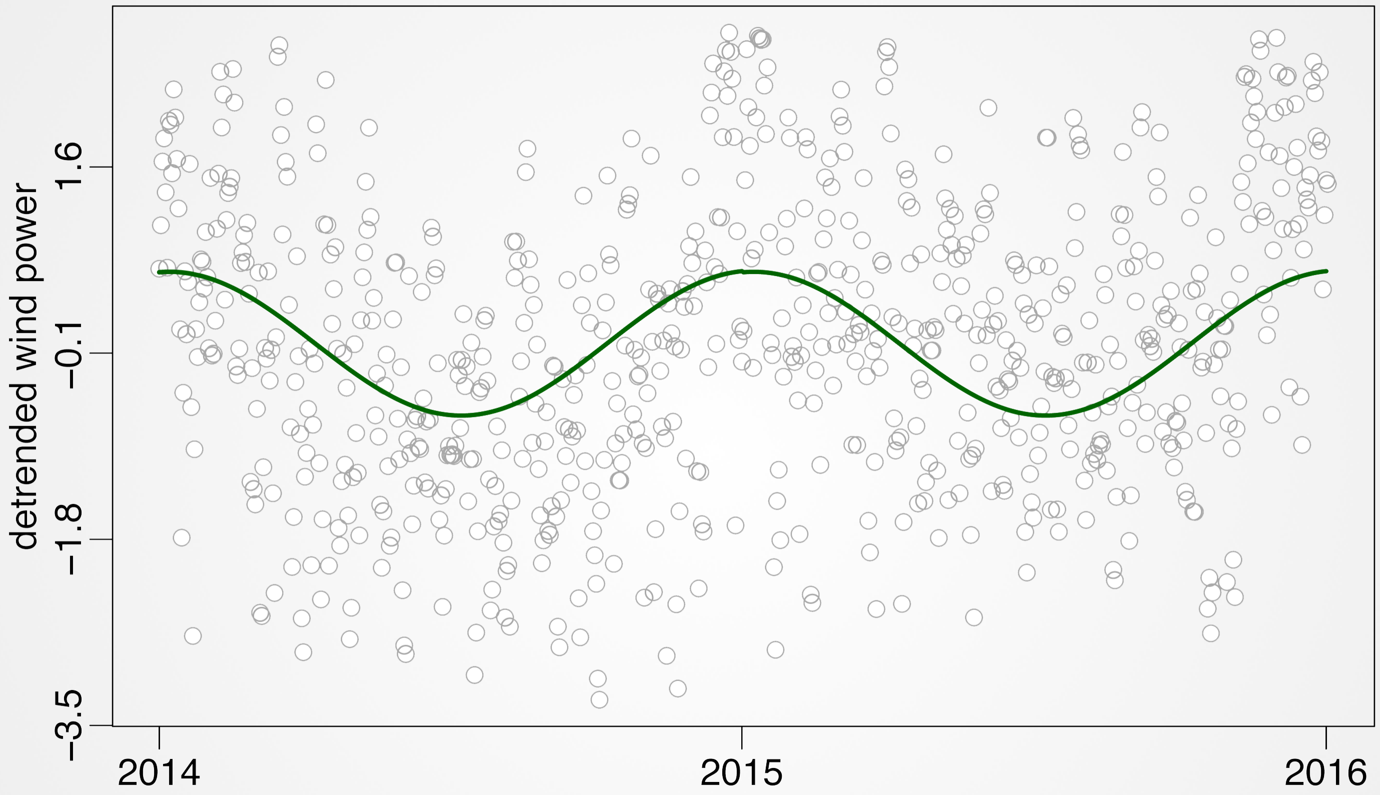
Seasonality and seasonal variance





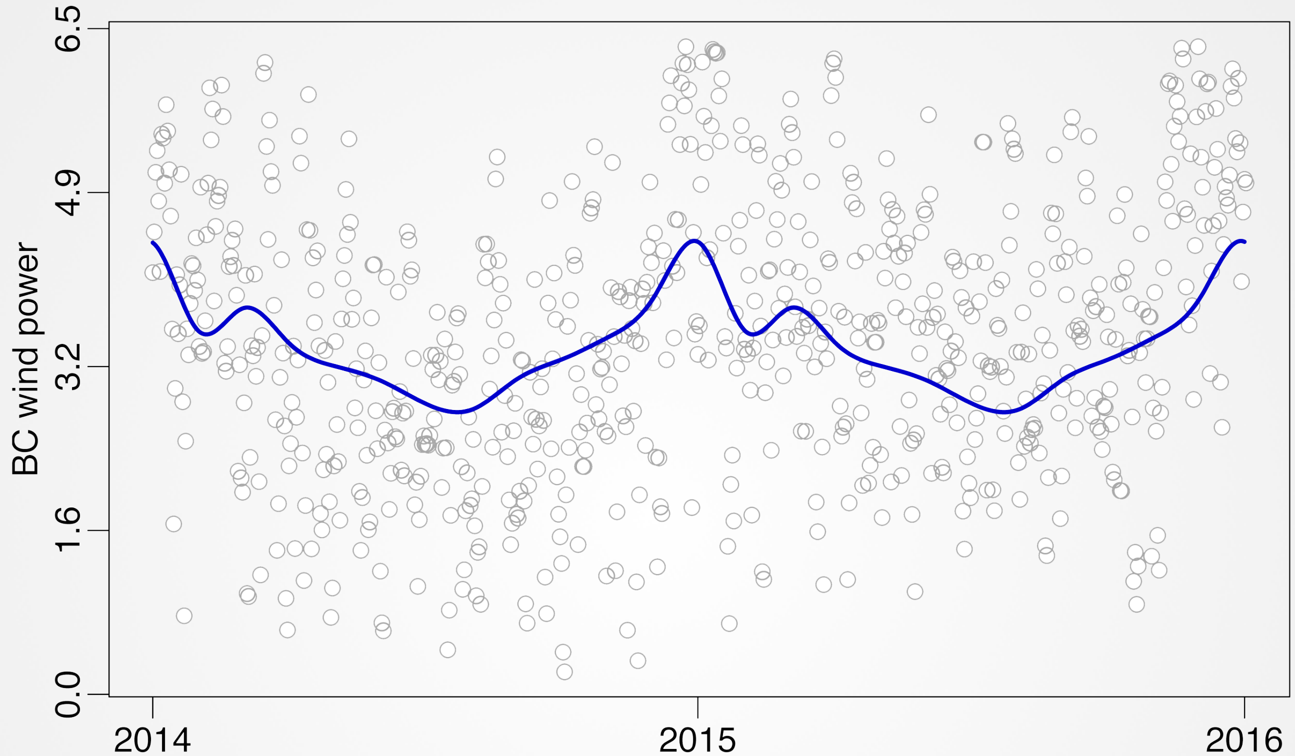
Left: Time series of logit-transformed daily average WP utilisation with different seasonality estimates. Right: Seasonal variance.

Truncated Fourier Series



$$\Lambda_t = a + b \cdot t + \sum_{l=1}^{L} c_l \cos\left\{\frac{2\pi(t - d_l)}{l \cdot 365}\right\}$$

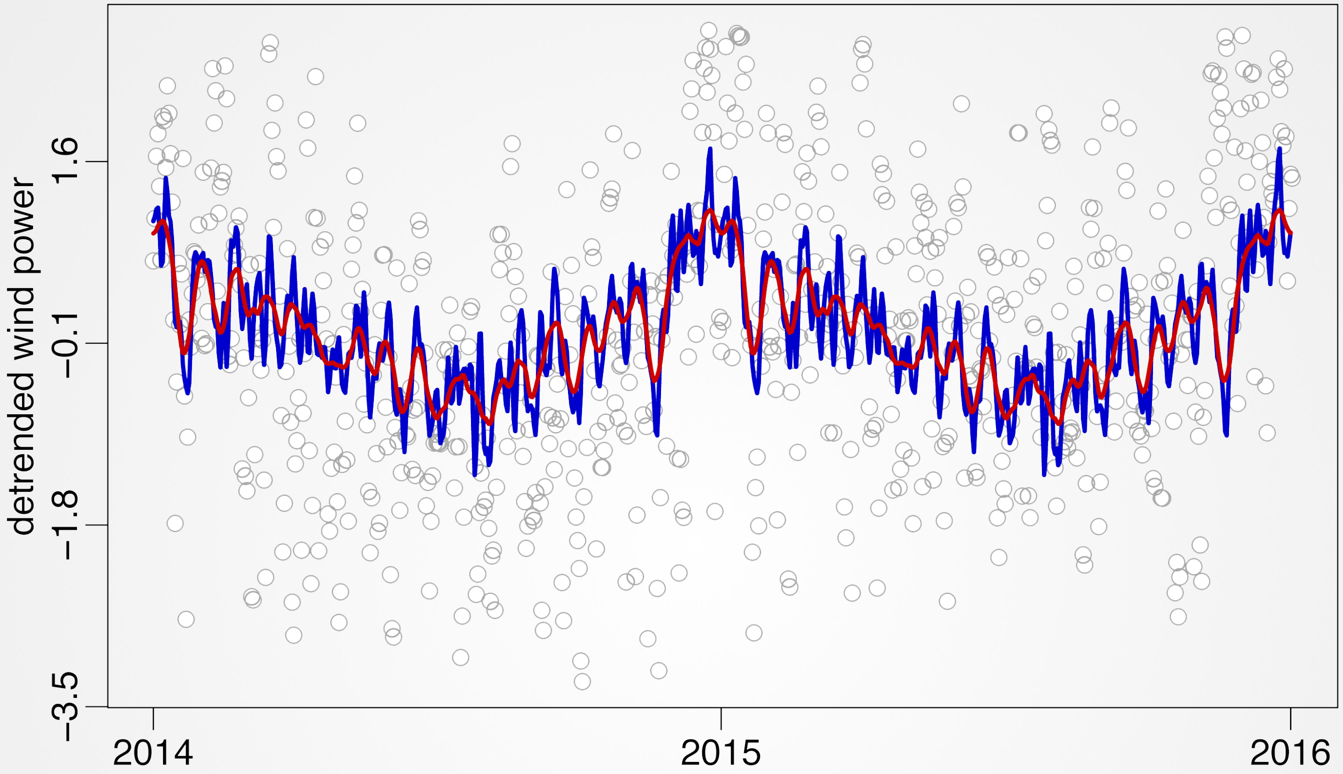
Periodic B-splines



$$\Lambda_t = \arg\min_{\alpha_j} \sum_{t=1}^{365} \left\{ \bar{U}_t - \sum_{j=1}^{J} \alpha_j \Psi_j(s_t) \right\}^2,$$

where $\Psi_j(s_t)$ is a vector of known basis functions, α_j are coefficients, J is the number of knots. Ziel et al. (2017)

Local Linear Smoothing



$$\arg\min_{e,f} \sum_{t=1}^{365} \left\{\bar{U}_t - e_s - f_s(t-s)\right\}^2 K\left(\frac{t-s}{h}\right),$$

with h selected via cross validation or rule of thumb

Approximating σ_t by IQR or IER

The quantile and expectile loss functions by Breckling and Chambers (1988) are defined as

$$\rho_{\tau,\alpha}(u) = |\tau - \mathbf{I}\{u < 0\}|\,|u|^{\alpha},$$

with quantile loss for $\alpha = 1$, and expectile loss for $\alpha = 2$,
A τ-level moment is given by the expectile

$$e(\tau|\mathbf{t}) \stackrel{\text{def}}{=} \arg\min_{\theta} \mathsf{E}[\rho_{\tau}(Y - \theta)\,|\mathbf{t}]$$

Then the normalised inter expectile range (IER) is defined

$$\sigma_{IER}(t) \stackrel{\text{def}}{=} \frac{\mathbf{e}(\alpha = 0.75|\mathbf{t}) - \mathbf{e}(\alpha = 0.25|\mathbf{t})}{2\mathbf{e}^{-1}(\alpha = 0.75|\Phi)}$$

Inter Quartile and Expectile Ranges

A robust approximation of volatility is given by the normalised Inter Quartile Range (IQR)

$$\sigma_{IQR}(t) \stackrel{\text{def}}{=} \frac{\mathbf{q}(\alpha = 0.75|\mathbf{t}) - \mathbf{q}(\alpha = 0.25|\mathbf{t})}{2\Phi^{-1}(\alpha = 0.75)}$$

and make use of this relationship between the variance and the standard normal cdf Φ

Bowman and Azzalini (1997)

$\widehat{\sigma}_t$ with smoothing splines

Minimise Anderson-Darling-test

$$\arg\min_{m,\zeta,\kappa} -T - \sum_{t=1}^{T} \frac{(2t-1)}{T} \Big[\log F(Y_t) + \log\Big\{1 - F(Y_{T+1-t})\Big\}\Big] + \left| \frac{1}{\kappa \cdot \sqrt{T}} \left(e^\top e\right)^{\frac{1}{2}} - 1 \right|,$$

conditional on smooth seasonal IER

$$\frac{1}{365} \sum_{t=1}^{365} \{\sigma_{t,k} - m(t)\}^2 + \zeta \int dt \left\{ \frac{\partial^2 m(t)}{\partial t^2} \right\}^2,$$

with $k = \{IQR, IER\}$

	ADT	JBT	SWT	CvM	KST
LL.RoT	0.052	0.152	0.118	0.055	0.095
LL.RoT-IER	0.064	0.501	0.199	0.084	0.113
LL.RoTos-IER	0.075	0.590	0.209	0.096	0.163
BS-IER	0.075	0.590	0.209	0.096	0.163
LL.CV-IER	0.073	0.569	0.361	0.085	0.083
LL.CV-IER.CV		0.138			0.051
TFS-IER	0.103	0.756	0.201	0.120	0.307
BS-IQR	0.166	0.208	0.064	0.210	0.307
LL.CV-IQR	0.098			0.179	0.346
LL.CV-IQR.CV		0.193			0.085
TFS-IQR	0.152			0.167	0.240

p-values of five different normality tests for deseasonalised data after processing seasonal variance. *p*-values below 0.05 are omitted.

$\mathbf{X}_t$ can be written as a Continuous-time AR(p) (CAR(p)):

For $p = 1$,

$$dX_{1t} = -\alpha_1 X_{1t} dt + \sigma_t dB_t$$

For $p = 2$,

$$\begin{aligned} X_{1(t+2)} &\approx (2 - \alpha_1) X_{1(t+1)} \\ &+ (\alpha_1 - \alpha_2 - 1) X_{1t} + \sigma_t (B_{t-1} - B_t) \end{aligned}$$

For $p = 3$,

$$\begin{aligned} X_{1(t+3)} &\approx (3 - \alpha_1) X_{1(t+2)} + (2\alpha_1 - \alpha_2 - 3) X_{1(t+1)} \\ &+ (-\alpha_1 + \alpha_2 - \alpha_3 + 1) X_{1t} + \sigma_t (B_{t-1} - B_t) \end{aligned}$$

Proof $CAR(3) \approx AR(3)$

Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{pmatrix}$$

- use $B_{t+1} - B_t = \varepsilon_t$
- assume a time step of length one $dt = 1$
- substitute iteratively into X_1 dynamics

Proof $CAR(3) \approx AR(3)$:

$$
\begin{aligned}
X_{1(t+1)} - X_{1(t)} &= X_{2(t)}dt \\
X_{2(t+1)} - X_{2(t)} &= X_{3(t)}dt \\
X_{3(t+1)} - X_{3(t)} &= -\alpha_1 X_{1(t)}dt - \alpha_2 X_{2(t)}dt - \alpha_3 X_{3(t)}dt + \sigma_t \varepsilon_t \\
X_{1(t+2)} - X_{1(t+1)} &= X_{2(t+1)}dt \\
X_{2(t+2)} - X_{2(t+1)} &= X_{3(t+1)}dt \\
X_{3(t+2)} - X_{3(t+1)} &= -\alpha_1 X_{1(t+1)}dt - \alpha_2 X_{2(t+1)}dt \\
&\quad -\alpha_3 X_{3(t+1)}dt + \sigma_{t+1}\varepsilon_{t+1} \\
X_{1(t+3)} - X_{1(t+2)} &= X_{2(t+2)}dt \\
X_{2(t+3)} - X_{2(t+2)} &= X_{3(t+2)}dt \\
X_{3(t+3)} - X_{3(t+2)} &= -\alpha_1 X_{1(t+2)}dt - \alpha_2 X_{2(t+2)}dt \\
&\quad -\alpha_3 X_{3(t+2)}dt + \sigma_{t+2}\varepsilon_{t+2}
\end{aligned}
$$

Gauss Lévy

Seasonal variance: LLE - mirroring observations

To avoid the boundary problem, use mirrored observations:

Assume $h_K < 365/2$, then the observations look like $\hat{\varepsilon}^2_{-364}, \hat{\varepsilon}^2_{-363}, \ldots, \hat{\varepsilon}^2_0, \hat{\varepsilon}^2_1, \ldots, \hat{\varepsilon}^2_{730}$, where

$$
\begin{aligned}
\hat{\varepsilon}^2_t &\stackrel{\text{def}}{=} \hat{\varepsilon}^2_{365+t}, \; -364 \leq t \leq 0 \\
\hat{\varepsilon}^2_t &\stackrel{\text{def}}{=} \hat{\varepsilon}^2_{t-365}, \; 366 \leq t \leq 730
\end{aligned}
$$

Risk-neutral probabilities - Brownian motion

The measure change is given by the Girsanov transform

$$\mathrm{d}B_i^{\theta}(t) = -\theta_{B,i}(t)\mathrm{d}t + \mathrm{d}B_i(t),$$

where $\{B_i(t)\}_{i=1}^{m}$ are Brownian motions

$\theta_{B,i}$ is the compensation for bearing the risk associated with non-extreme market variations, e.g. diffusion component

Suppose the Novikov condition (square-integrability) holds

$$\mathsf{E}\left[\exp\left\{\frac{1}{2}\int_0^T \theta^2(t)dt\right\}\right] < \infty$$

then B^θ is a Brownian motion under the the probability Q_B^θ with density of the Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}Q_B^\theta}{\mathrm{d}P}\right|_{\mathcal{F}_t} = \exp\left\{-\int_0^t \theta(s)\mathrm{d}B_s - \frac{1}{2}\int_0^t \theta^2(s)\mathrm{d}s\right\}$$

The Girsanov measure change gives the dynamics of Y(*t*)

$$Y_s = \mathbf{b}^\top \exp\{\mathbf{A}(s-t)\}\mathbf{X} + \int_t^s \mathbf{b}^\top \exp\{\mathbf{A}(s-u)\} e_p \sigma_u \theta_u du$$
$$+ \int_t^s \mathbf{b}^\top \exp\{\mathbf{A}(s-u)\} e_p \sigma_u \mathrm{d}\mathbf{B}_u^\theta$$

Risk-neutral probabilities - Lévy process

The measure change is achieved by the Girsanov transform, assuming $\theta(t)$ is a Borel measurable function, then the density of the Radon-Nikodym derivative is given by

$$\left.\frac{\mathrm{d}Q_L^\theta}{\mathrm{d}P}\right|_{\mathcal{F}_t} = \exp\left[\int_0^t -\theta(s)\mathrm{d}L_s - \int_0^t \psi_L\left\{\theta^2(s)_{Q_B^\theta}\right\}\mathrm{d}s\right]$$

$\theta(t)$ is real-valued function, integrable wrt the L\'evy process.

Applying Bayes theorem along density process of Q_θ we have

$$\log \mathsf{E}_{Q_s}\left[\exp\{iz^\top L(t)\}\middle|\mathcal{F}_s\right] = \{\psi(z - i\theta) - \psi(-i\theta)\}(t-s)$$