

Adaptive Interest Rate Modelling

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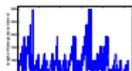
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Interest Rate

- Essential for pricing derivatives and hedging corresponding risk.
- A signal of macroeconomic activity.
- Influenced by macroeconomic variables.
- Follow unstable dynamic process.



Classical One-factor Short Rate Models

Vasicek Model

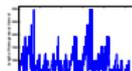
$$dr(t) = a\{b - r(t)\}dt + \sigma dW_t$$

CIR Model

$$dr(t) = a\{b - r(t)\}dt + \sigma\sqrt{r(t)}dW_t$$

- ▣ $r_t \geq 0$
- ▣ Mean reversion

Adaptive Interest Rate



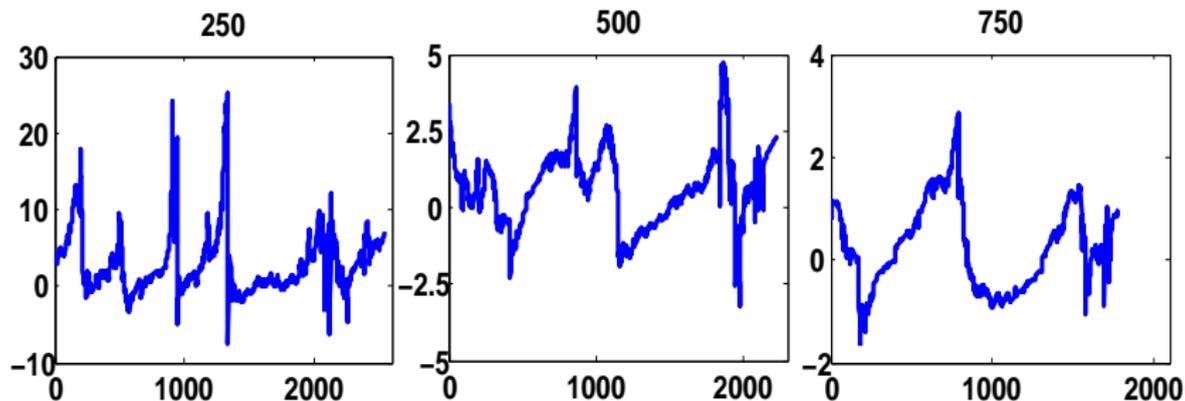
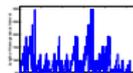


Figure 1: Moving window estimator \hat{a} with window sizes 250, 500 and 750



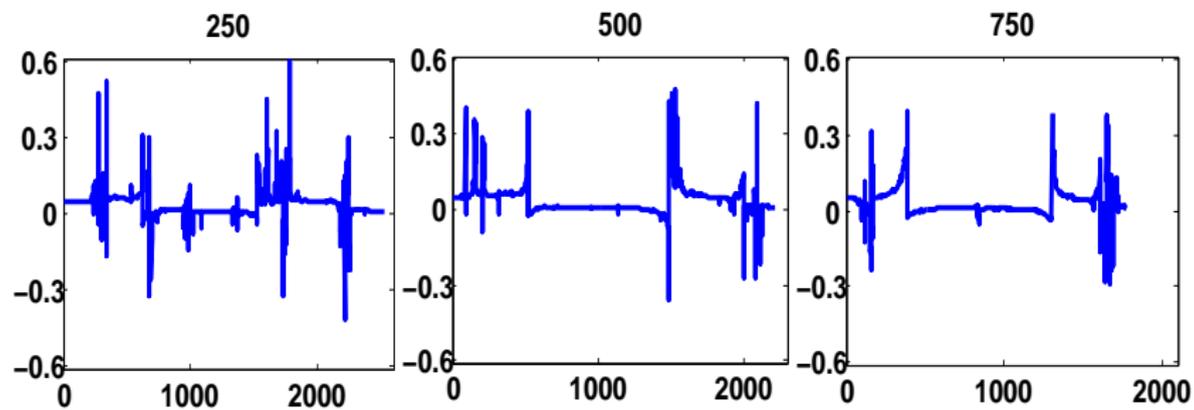
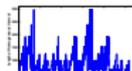


Figure 2: Moving window estimator \hat{b} with window sizes 250, 500 and 750



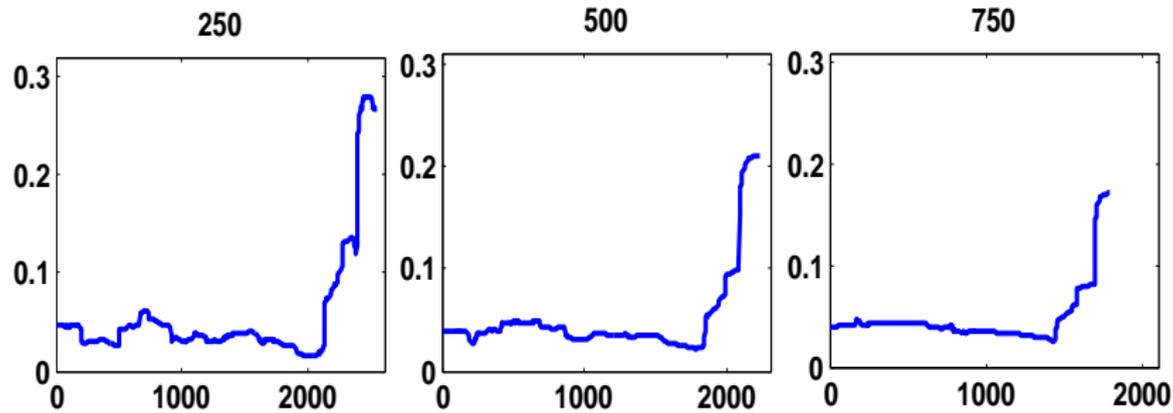
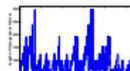


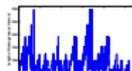
Figure 3: Moving window estimator $\hat{\sigma}$ with window sizes 250, 500 and 750



Extended Short Rate Models

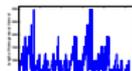
Three strands of extended interest rate models:

- Jump-Diffusion Models
- Regime-Switching Models
- Time-varying Coefficients Models.



Outline

1. Motivation ✓
2. Local Adaptive Approach for CIR model
3. Simulations
4. Empirical Study
5. Conclusion



Model

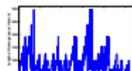
The time-varying CIR model with $\theta_t = (a_t, b_t, \sigma_t)^\top$:

$$dr(t) = a_t \{b_t - r(t)\} dt + \sigma_t \sqrt{r(t)} dW_t \quad (1)$$

Discretization:

$$r_{t_{i+1}} - r_{t_i} = a_t \{b_t - r_{t_i}\} \Delta t + \sigma_t \sqrt{r_{t_i}} Z_i \quad (2)$$

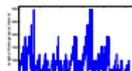
$$\{Z_i\} \sim N(0, \Delta t)$$



Local Parametric Approach (LPA)

Given time point t , go back and split time series into K intervals, $I_0 \subset I_1 \subset \dots \subset I_K$, with $I_k = [t - m_k + 1, t]$.

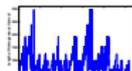
- Accept the smallest interval I_0 without change point (i.e. homogeneous interval).
- Sequentially check the historical intervals to search a change point.
- Two methods of LPA: Local model selection(LMS) and Local change point (LPC).



Algorithm of LPC

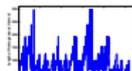
Goal: Find an unknown change point τ in the interval I_k .

1. Determine $\hat{\theta} = \tilde{\theta}_0$.
2. Increase interval to I_k , $k \geq 1$. Get $\tilde{\theta}_{I_k}$.
3. Compare test statistics with critical value. If test statistic is accepted go to step 4, otherwise go to step 5.
4. Let $\hat{\theta} = \tilde{\theta}_{I_k}$, and set $k = k + 1$, repeat step 2.
5. Detect the change point τ in I_k , $I_{\hat{k}} = I_{k-1}$ without change point.



Why we use LPA?

- Find the longest stable interval for each t .
- Allow for structural breaks and jumps in parameter values.
- Distinguish blooming and declining regimes.



Test Statistic

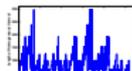
Test statistics $T_{I_{k+1}, \tau}$:

$$T_{I_{k+1}, \tau} = L_J(\tilde{\theta}_J) + L_{J^c}(\tilde{\theta}_{J^c}) - L_{I_{k+1}}(\tilde{\theta}_{I_{k+1}}) \quad (3)$$

where $J = [\tau + 1, t]$, and $J^c = [t - m_{k+1}, \tau]$, and $\tau \in J_k = I_k \setminus I_{k-1}$.

Consider the supremum of the test statistics over interval J_k :

$$T_k = \sup_{\tau \in J_k} T_{I_{k+1}, \tau} \quad (4)$$



Test Algorithm

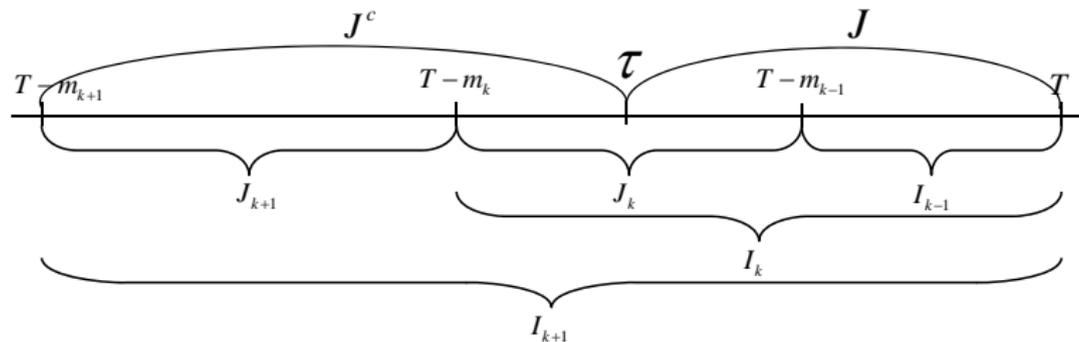
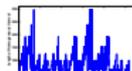


Figure 4: Construction of the Test Statistics in the Local Change Point Test

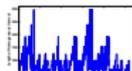


- The criteria for testing homogeneous intervals:

$$T_k \leq \mathfrak{z}_k, \quad \text{for } k \leq \hat{k} \quad (5)$$

and $T_{k+1} > \mathfrak{z}_{\hat{k}+1}$.

- $I_{\hat{k}}$ is the longest time homogeneous interval for time point t ,
- $\hat{\theta}_t = \tilde{\theta}_{I_{\hat{k}}}$.
- \mathfrak{z}_k is the critical value, obtained by Monte-Carlo simulations.



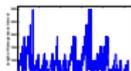
Risk Bound

Parametric risk bound $\mathfrak{R}_r(\theta_0)$, given the true value θ_0 , for any interval I_k ,

$$E_{\theta_0} |L_{I_k}(\tilde{\theta}_{I_k}, \theta_0)|^r \leq \mathfrak{R}_r(\theta_0) \quad (6)$$

where

$L_{I_k}(\tilde{\theta}_{I_k}, \theta_0) = L_{I_k}(\tilde{\theta}_{I_k}) - L_{I_k}(\theta_0)$ is the likelihood ratio between the two parameters.



Small Modeling Bias Condition

The SMB condition for the interval I_k , and given some $\theta \in \Theta$:

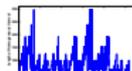
$$\mathbb{E} \Delta_{I_k}(\theta) \leq \Delta \quad (7)$$

and

$$\Delta_{I_k} = \sum_{t \in I_k} \mathcal{K}\{r(t), r(t; \theta)\}$$

$\mathcal{K}\{r_t, r_t(\theta)\}$: Kullback-Leibler divergence between $P_{r(t)}$ and $P_{r(t; \theta)}$.

Oracle Choice k^* : the largest I_{k^*} s.t. (7) holds.



Propagation Condition and Stability

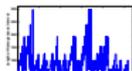
Propagation:

$$E_{\theta_0} |L_{I_k}(\tilde{\theta}_{I_k}, \hat{\theta}_{I_k})|^r \leq \rho \mathfrak{R}_r(\theta_0) \quad (8)$$

Stability: I_k is accepted interval, then $\hat{\theta}_{I_k} = \tilde{\theta}_{I_k}$

$$L(\hat{\theta}_{I_k}, \hat{\theta}_{I_{k+1}}) \leq \beta k \quad (9)$$

For fixed Δ , the loss $|L_{I_k}(\tilde{\theta}_{I_k}, \theta_0)|^r$ stochastically bounded by a constant proportional to e^{Δ}

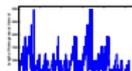


Critical Value

- Sequential choice of critical values \mathfrak{z}_k .
- Change point detected at step $\ell \leq k$
- \mathfrak{B}_ℓ : rejection at step ℓ .

$$\mathfrak{B}_\ell = \{T_1 \leq \mathfrak{z}_1, \dots, T_{\ell-1} \leq \mathfrak{z}_{\ell-1}, T_\ell > \mathfrak{z}_\ell\}$$

and $\hat{\theta}_{I_k} = \tilde{\theta}_{I_{\ell-1}}$ on \mathfrak{B}_ℓ , $\ell = 1, 2, \dots, k$.



Critical Value

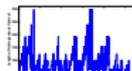
To determine $\hat{\beta}_1$,

$$\max_{k=1, \dots, K} E_{\theta_0} |L(\tilde{\theta}_{I_k}, \tilde{\theta}_{I_0})|^r \mathbf{1}(\mathfrak{B}_1) \leq \rho \mathfrak{R}_r(\theta_0) / K \quad (10)$$

\mathfrak{B}_ℓ only depends on $\hat{\beta}_1, \dots, \hat{\beta}_\ell$, controlled by $\hat{\beta}_\ell$. The minimal value ensures

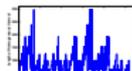
$$\max_{k \geq l} E_{\theta_0} |L(\tilde{\theta}_{I_k}, \tilde{\theta}_{I_{l-1}})|^r \mathbf{1}(\mathfrak{B}_l) = \rho \mathfrak{R}_r(\theta_0) / K \quad (11)$$

$\mathfrak{R}_r(\theta_0)$ is parametric risk bound.



Choice of the Length of Interval

- I_0 with length m_0 .
- Interval I_k : $m_k = [m_0 a^k]$ with $a > 1$.
- Results not sensitive to a .
- $r=0.5$, power of the loss function.
- $\rho= 0.2$, level of the test.
- $m_0 = 40$, $a = 1.25$, and $K= 15$, $m_K = 1136$.



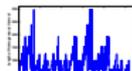
Oracle Property

I_{k^*} is the oracle interval, $E \Delta_{I_{k^*}}(\theta) \leq \Delta$, $k \leq k^*$, and $\hat{\theta}_{I_{\hat{k}}}$ close to the oracle estimate $\tilde{\theta}_{I_{k^*}}$

$$E \log \left\{ 1 + \frac{|L_{I_{k^*}}(\tilde{\theta}_{I_{k^*}}, \hat{\theta}_{I_{\hat{k}}})|^r}{\mathfrak{R}_r(\theta)} \right\} \leq \rho + \Delta \quad (12)$$

For $k > k^*$, the adaptive estimator $\hat{\theta}_{I_{\hat{k}}}$ satisfies

$$E \log \left\{ 1 + \frac{|L_{I_{k^*}}(\tilde{\theta}_{I_{k^*}}, \hat{\theta}_{I_{\hat{k}}})|^r}{\mathfrak{R}_r(\theta)} \right\} \leq \rho + \Delta + \log \left\{ 1 + \frac{\mathfrak{z}_{k^*}^r}{\mathfrak{R}_r(\theta)} \right\} \quad (13)$$

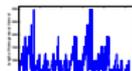


Simulation Setup

We simulate CIR process with 1500 observations and 100 times.

t	a	b	σ
$t \in [1, 500]$	0.2	0.04	0.03
$t \in [501, 1000]$	0.5	0.06	0.1
$t \in [1001, 1500]$	0.8	0.01	0.07

Table 1: The parameter settings for simulations of the CIR process



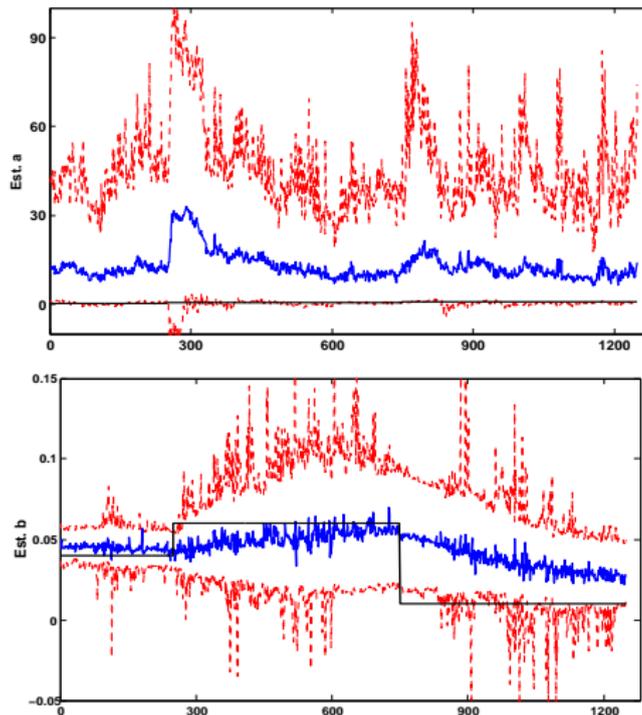
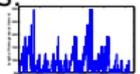


Figure 5: LPA estimator \hat{a} and \hat{b} with simulated CIR paths.



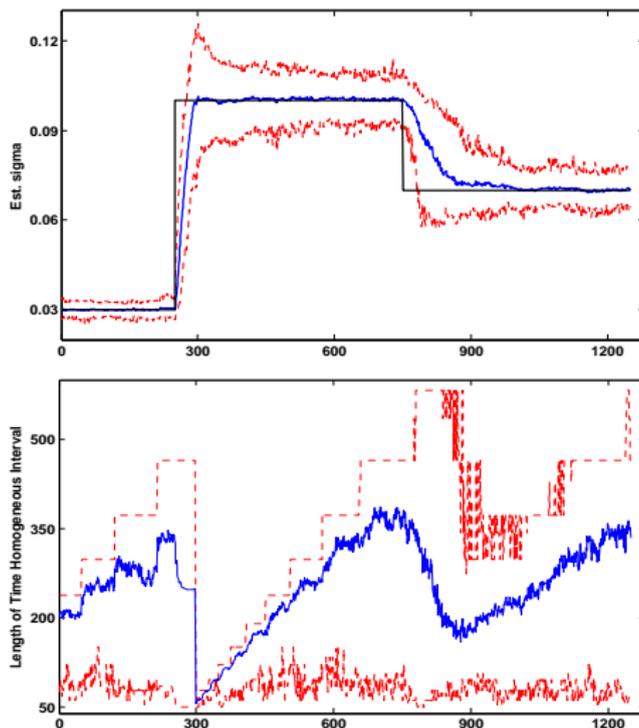
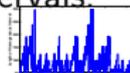


Figure 6: LPA estimator $\hat{\sigma}$ and selected time homogeneous intervals
Adaptive Interest Rate

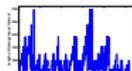


Data

Yield of 3M US T-Bill from the Federal Reserve Bank of St. Louis from 19980102 to 20090513.

	<i>Mean</i>	<i>SD</i>	<i>Skewness</i>	<i>Kurtosis</i>
r_t	0.0319	0.0176	-0.1159	-1.4104
dr_t	-1.764×10^{-5}	0.0006	-0.7467	34.4856

Table 2: Statistical Summary of 3-month T-Bill



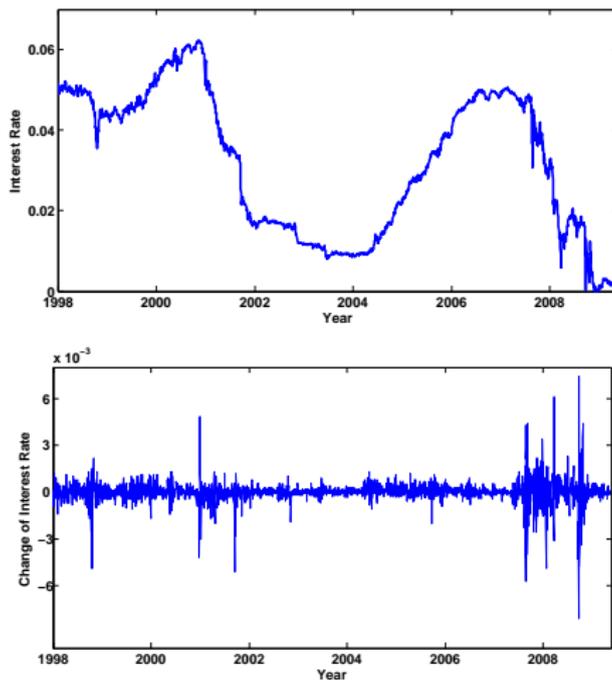
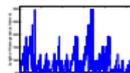


Figure 7: 3-month Treasury Bill Rate: 19980102—20090513. Top panel: Daily yields. Bottom panel: Changes of daily yields.

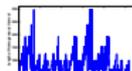
Adaptive Interest Rate



MLE Estimator of CIR model

Sample Size	\hat{a}	\hat{b}	$\hat{\sigma}$
19980102–20090513	0.2657	0.0153	0.0944
19980102–20070731	0.1424	0.0252	0.0428
20070801–20090513	3.6792	0.0081	0.2280

Table 3: Estimated parameters of CIR model using MLE



Critical Value

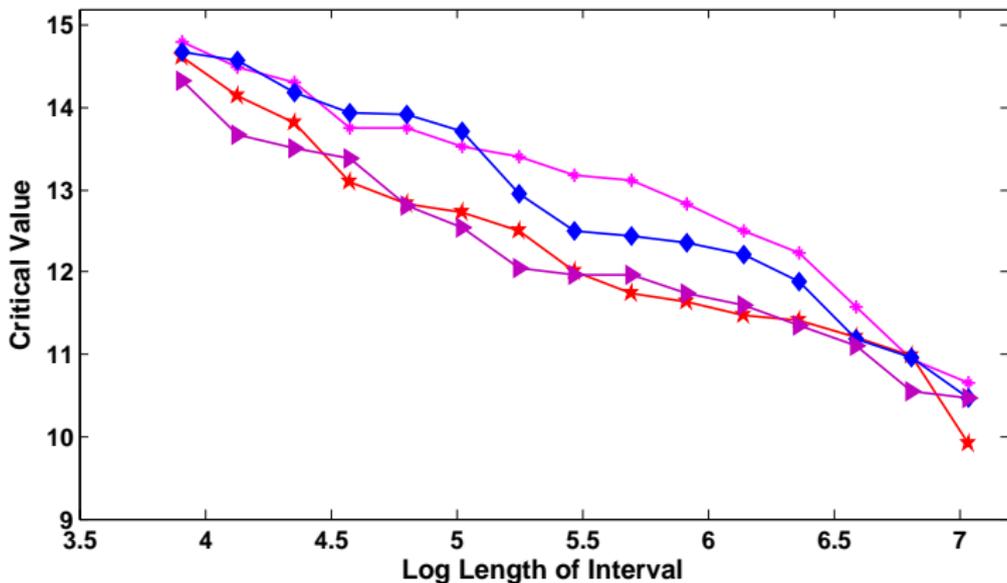
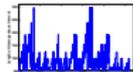


Figure 8: Critical values with $m_0 = 40$, $K=15$



Estimator \hat{a}

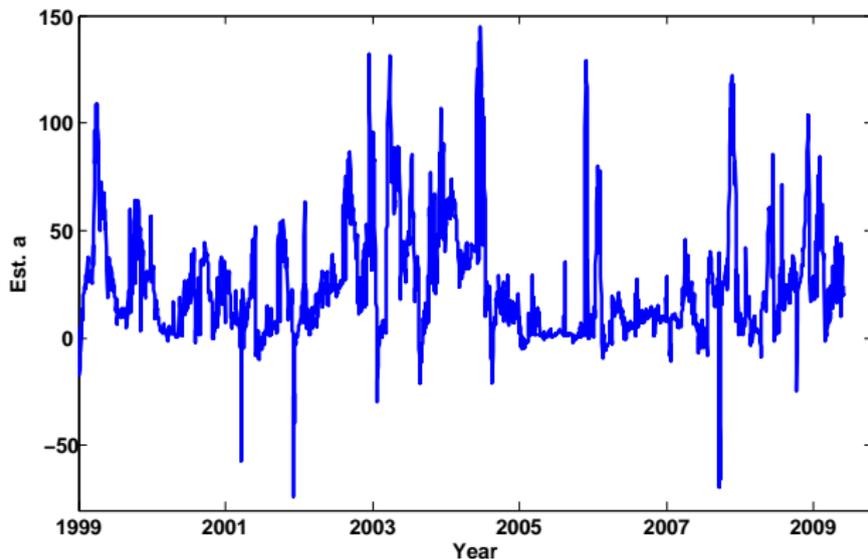
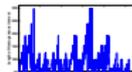


Figure 9: Estimated \hat{a} by LPA



Estimator \hat{b}

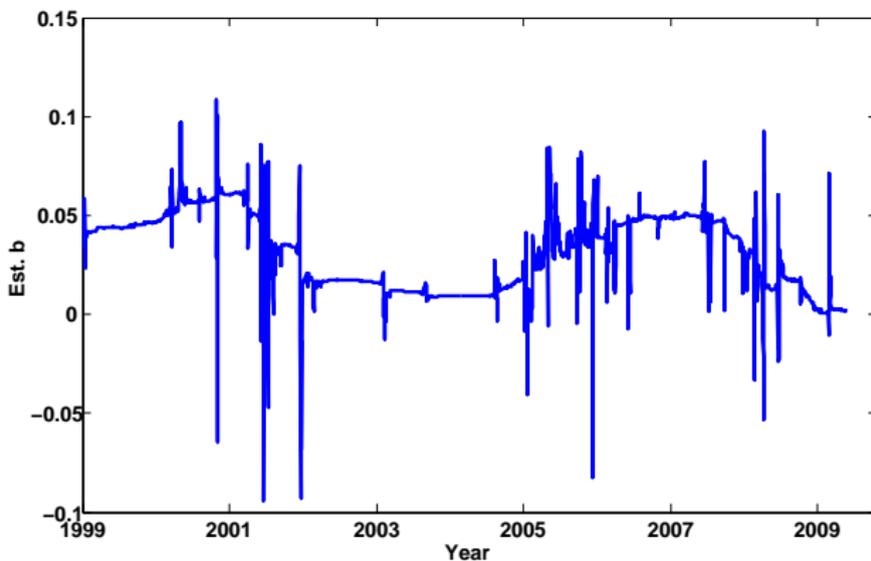
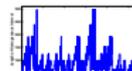


Figure 10: Estimated \hat{b} by LPA



Estimator $\hat{\sigma}$

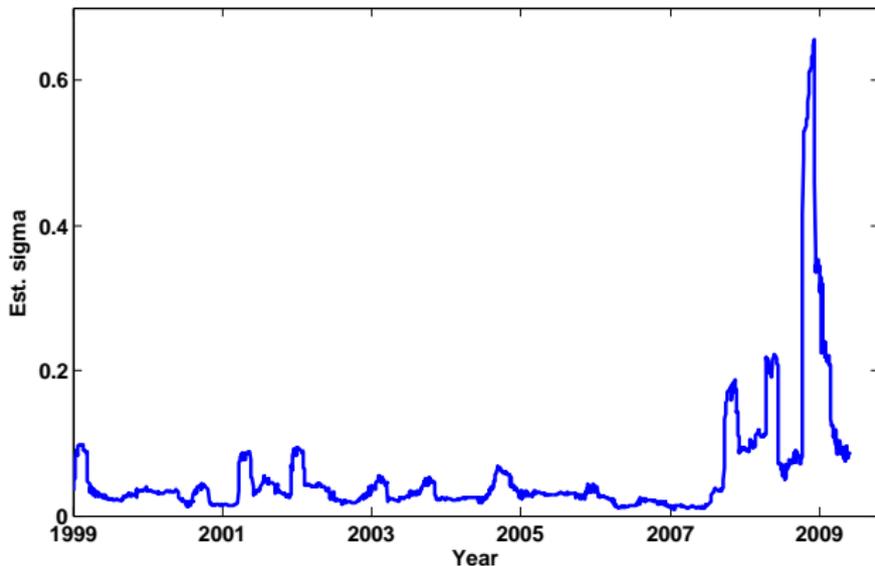
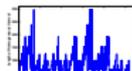


Figure 11: Estimated $\hat{\sigma}$ by LPA



Homogeneous Intervals

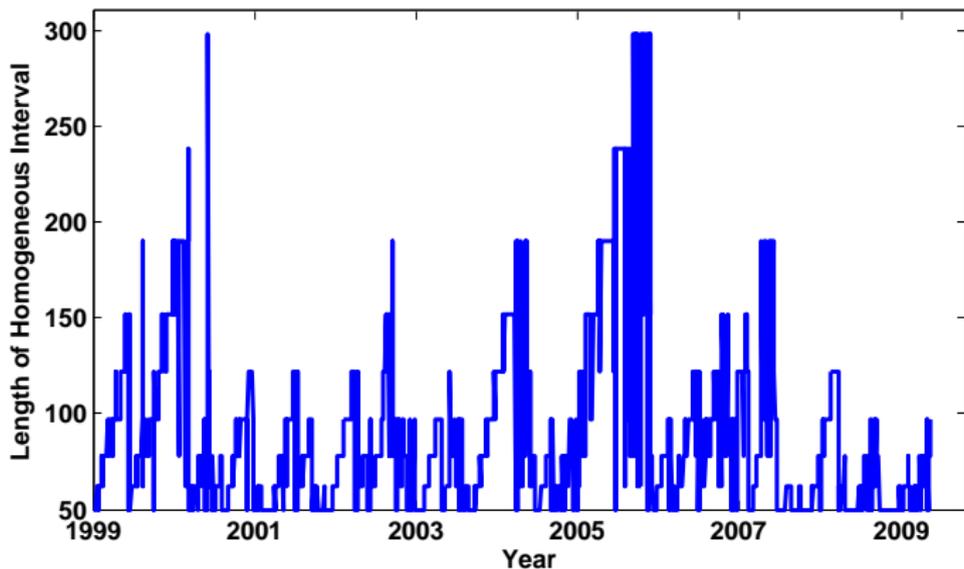
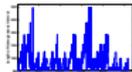


Figure 12: Selected time homogeneous intervals with $\rho = 0.2$, and $r = 0.5$

Adaptive Interest Rate



In Sample Fitting

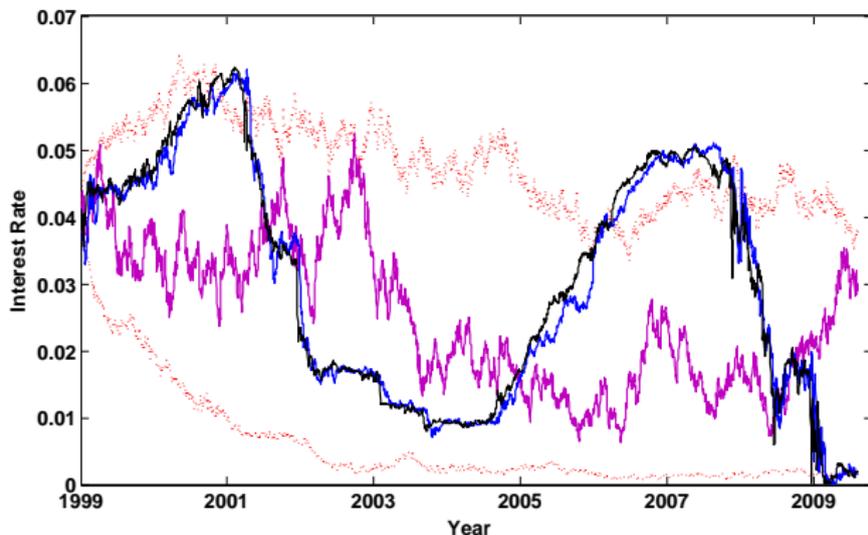
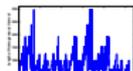


Figure 13: Confidence Interval (Red); Real Data (Black); LPA CIR (Blue);
CIR (Purple)

Adaptive Interest Rate



Forecasting

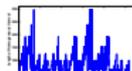
\mathcal{H} : the prediction period horizon, then the absolute prediction error(APE):

$$APE(t) = \sum_{h \in \mathcal{H}} |r_{t+h} - \hat{r}_{t+h|t}| / |\mathcal{H}| \quad (14)$$

$$APE \text{ Ratio} = \frac{APE_{LPA}(t)}{APE_{MW}(t)}$$

LPA: Local Parametric Approach.

MW: Moving Window Estimation.



Forecasting

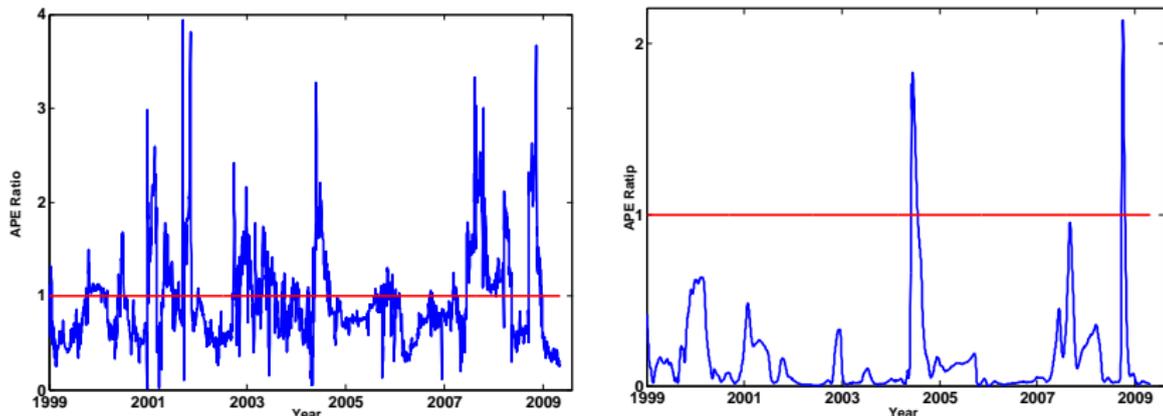
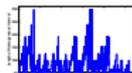


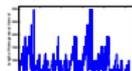
Figure 14: The APE ratio between LPA and MW with window size 250. Left: 1-day ahead forecasting; Right: 10-day ahead forecasting.



Performance of Forecasting

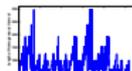
Horizon		MAE		
		$l = 250$	$l = 500$	$l = 750$
One Day	LPA	4.74×10^{-4}	4.85×10^{-4}	4.96×10^{-4}
	MW	4.78×10^{-4}	4.41×10^{-4}	4.16×10^{-4}
Ten Days	LPA	0.0201	0.0215	0.0232
	MW	0.1868	1.0032	1.8054

Table 4: The MAE of 1 day and 10 days ahead forecasting of the short rate based on the LPA and MW.



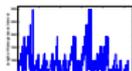
Conclusion

- Interest rate in recession is more volatile.
- The selected time homogeneous intervals can not last long due to the complexities of macroeconomy.
- The LPA can detect jumps and structural break points in the interest rate dynamics.
- The LPA outperforms the moving window estimation especially in long horizon forecasting.



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V. Spokoiny

Multiscale Local Change Point Detection with Applications to Value-at-Risk

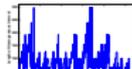
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O. Vasicek

An Equilibrium Characterization of the Term Structure

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Adaptive Interest Rate Modelling

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