VAR - DSFM Modelling for Implied Volatility String Dynamics

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Aims

Dynamic Semiparametric Factor Models (DSFM) for Implied Volatility (IV) Dynamics yield time dependent factor loadings. Loading series

- explain the nature of volatility risk
- allow to hedge positions of 'volatility derivatives'

Vector autoregressive (VAR) modelling of loading series

- How do the factor loadings jointly evolve over time?
- How are they related to macroeconomic indicators?

Improve assessment of market risk

An Implied Volatility Surface

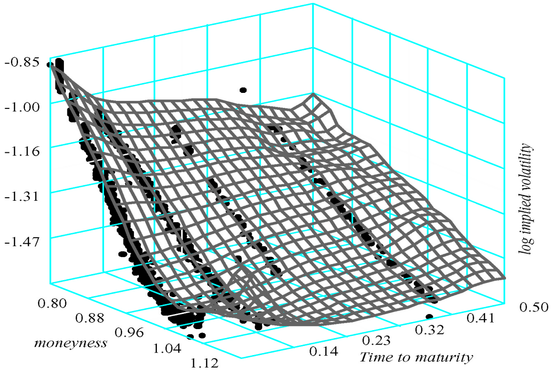


Figure 1: Implied volatility surface from DSFM fit for the DAX-Option on 20000502 (2 May 2000)

The semiparametric factor model

Log-implied volatility $Y_{t,j}$

$$Y_{t,j} = \sum_{l=0}^{L} z_{tl} m_l(X_{t,j}) + \varepsilon_{t,j} \quad (1)$$

$z_{t0} = 1, \quad j = 1, \ldots, J_t \ (t = 1, \ldots, T)$ is number of IV observations on day t, L is number of basis functions
$X_{t,j}$ is two-dimensional containing moneyness and maturity
z_{tl} are time dependent loadings or weights of the smooth basis function m_l, for $(l = 0, \ldots, L)$.

[Borak, Härdle and Fengler (2005)]

The semiparametric factor model

The estimates $\widehat{z}_{tl}$ and $\widehat{m}_l(.)$ are obtained by minimizing w.r.t (z,m):

$$\sum_{t=1}^{I}\sum_{j=1}^{J_t}\int\left\{Y_{t,j}-\sum_{k=0}^{L} z_{tl}m_l(u)\right\}^2 K_h(u-X_{t,j})\;du, \qquad (2)$$

$z_{t0}=1$

$K_h(u)=k_{h_1}(u_1)\times k_{h_2}(u_2)$

$k_h(v)=h^{-1}K(v/h)$ is a one-dimensional kernel function

$h=(h_1,h_2)^\top$ are bandwidths

Literature review

- [Skiadopoulos et al. (1999)] analyzed the IVS of S&P 500 and reported that at least two and at most six factors are necessary to capture the dynamics.
- [Cont and Fonseca (2002)], on dynamics of the S&P 500 implied volatility reported that the first three principal components account for 95% of the daily variance.

Literature review cont.

- [Fengler, M.R. (2005)] indicated three factors are sufficient to capture 95% variation in DAX implied volatilities.
- [Hafner (2004)] with a parametric approach, uses a four-factor model for DAX implied volatilities.
- [Borak, Härdle and Fengler (2005)] identified three loading series after fitting a DSFM for European DAX options.

Overview

1. Motivation ✓
2. Factor loadings series from DSFM for DAX options
3. Integration analysis and unit root tests
4. VAR modelling and dynamic interaction between loadings
5. Loadings and macroeconomic indicators
6. Conclusion

Data

- time series data on factor loadings are from a DSFM model on European DAX options
- $T = 1052$ observations on z_t from 04.01.1999 to 25.02.2003, excluding days with no option trades

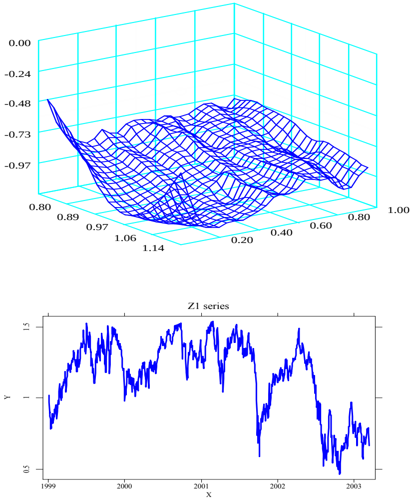


Figure 2: Basis function, $\hat{m}_1$ and corresponding loading series, z_{t1}

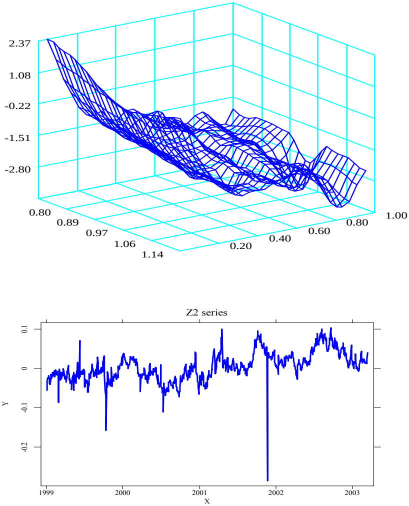
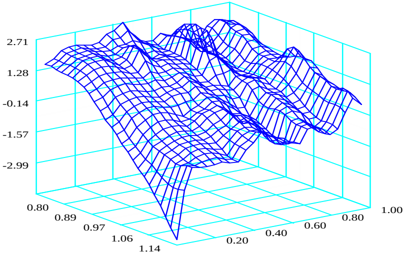


Figure 3: Basis function, $\hat{m}_2$ and corresponding loading series, z_{t2}



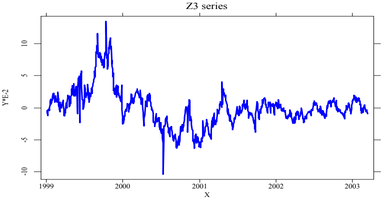


Figure 4: Basis function, $\hat{m}_3$ and corresponding loading series, z_{t3}

Factor loadings series

Factor loadings determine the movements of the Implied Volatility Surface (IVS)

- z_{t1} may be interpreted as representing the overall shift (up and down movement) of the IVS
- z_{t2} terms structure change of the IVS
- z_{t3} represent changes in moneyness slope of the IVS
- effect of factor loadings on IVS for 251 days (19990104 - 19991229)

Unit root tests

- z_t is investigated for unit roots
- stationarity $I(0)$: VAR model for levels
- integration $I(1)$: VAR model in first differences
- application of ADF test and ERS test

Unit root test results

Series	ADF-AIC	$\hat{p}$	ADF-HQ	$\hat{p}$	ERS-AIC	$\hat{b}$	ERS-HQ	$\hat{b}$
z_{t1}	-1.982 [0.295]	6	-2.241 [0.192]	2	3.787^{*}	6	2.953^{**}	6
Δz_{t1}	-15.199^{***} [0.000]	5	-23.582^{***} [0.000]	1	0.007^{***}	5	0.075^{***}	2
z_{t2}	-3.361^{**} [0.013]	8	-4.219^{***} [0.001]	4	5.295	8	3.338^{*}	4
Δz_{t2}	-15.646^{***} [0.000]	7	-15.646^{***} [0.000]	7	0.663^{***}	7	0.663^{***}	7
z_{t3}	-2.874^{**} [0.049]	7	-2.874^{**} [0.049]	7	1.446^{***}	7	1.446^{***}	7
Δz_{t3}	-13.855^{***} [0.000]	6	-13.855^{***} [0.000]	6	0.005^{***}	6	0.005^{***}	6

Table 1: ADF-AIC and ADF-HQ refer to ADF tests using AIC and HQ criteria respectively to estimate lag length p. ERS-AIC and ERS-SC criteria used, refer to the lag length b chosen for the estimation regression of the autoregressive spectral density estimator. ***, ** and * denote significance at the 1%, 5%, and 10% level respectively

Unit root test results

Test results do not agree in all cases but suggest

- stationarity of the three loading series

For possible structural breaks, unit root tests on subsample 04.01.1999 – 31.07.2001 (655 obs.) are applied for each series

- ADF and ERS tests confirm stationarity for z_{t2} , z_{t3} and nonstationarity for z_{t1}

Models for levels is analyzed to avoid over differencing

Robustness check by analyzing model in first differences

Models for Loadings Dynamics

The dynamics underlying z_t is modelled by a VAR(p) process

- in level $z_t = \nu + A_1 z_{t-1} + \cdots + A_p z_{t-p} + u_t$
- in first difference $\Delta z_t = z_t - z_{t-1}$,
  $\Delta z_t = \nu + A_1 \Delta z_{t-1} + \cdots + A_p \Delta z_{t-p} + u_t$

ν is $L \times 1$ vector of intercept parameters
$A_i, \;\; i = 1, \ldots, p$ are $L \times L$ parameter matrices
unobservable error term $u_t = (u_{t1}, \ldots, u_{tL})^\top$ with mean zero, time-invariant and non-singular covariance matrix $\Sigma_u = E[u_t u_t^\top]$

VAR Models diagnostics

Full sample (04.01.1999 – 25.02.2003)

- $p = 7$ for z_t and $p = 6$ for Δz_t reveal no autocorrelation

Sub-sample (04.01.1999 – 31.07.2001)

- lag length $p = 3$ reveals residuals with autocorrelation.
- lag length $p = 8$ reveals residuals free of autocorrelation

Evidence for non-normality and ARCH in the residuals is observed but left for further analysis

VAR Models diagnostics

Model	Sample	p	Q(20)	LMF(4)	LMF(8)	LBJ	ARCH(1)
z_t	full	7	0.22	0.09	0.38	0.00	0.00
z_t	sub	3	0.01	0.13	0.00	0.00	0.00
z_t	sub	8	0.16	0.18	0.27	0.00	0.00
Δz_t	full	6	0.22	0.13	0.16	0.00	0.00
Δz_t	sub	8	0.18	0.53	0.49	0.00	0.00

Table 2: Diagnostic tests for full sample: 1999/4/1-2003/2/25 and sub-sample 1999-2001/7/31. Lag order p of diagnostic tests. Adjusted portmanteau test Q(20) involving 20 autocorrelation matrices, LM tests for autocorrelation of order 4 and 8. Multivariate Lomnicki-Jarque-Bera tests for nonnormality (LJB) and multivariate first order ARCH test

Impulse Response Analysis

Effect of a shock in one variable at time t on variables in VAR system

- Impulse e.g. in $u_{1,t}$ while $u_{j,t} = 0$ for $j = 2, \ldots, L$ and $u_{t+h} = 0$ for $h > 0$

Response in z_{t+n} where n is forecast horizon.
Unreasonable analysis if error terms are strongly correlated.
Orthogonalization of error terms (Cholesky factorization):
orthogonalized impulse responses

$$\widehat{P}_u = \begin{pmatrix} 1 & -0.49 & -0.23 \\ -0.49 & 1 & -0.10 \\ -0.23 & -0.10 & 1 \end{pmatrix} \tag{3}$$

Interpretation of Shocks

- positive shock in z_{t1}: higher overall risk
- positive shock in z_{t2}: risk of longer maturities decrease relative to shorter maturities
- positive shock in z_{t3}: raises relative risk of options with lower moneyness values (lower strike)

Impulse Response Analysis

Starting with a fairly general VAR(7) model (Figure 5):

- innovation in z_{t1} has permanent negative effect on z_{t2} and a small positive effect on z_{t3}, which becomes insignificant after about 6 periods
- innovation in z_{t2} has permanent positive effect on itself but no significant effect with other variables.
  Similar result is obtained for a shock in z_{t3}

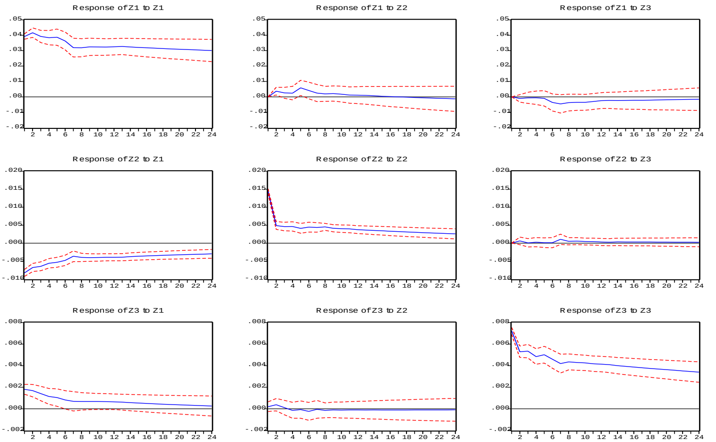


Figure 5: Impulse-Responses: VAR(7) for $z_t = (z_{t1}, z_{t2}, z_{t3})^\top$
Sample period: 04.01.1999 – 25.02.2003

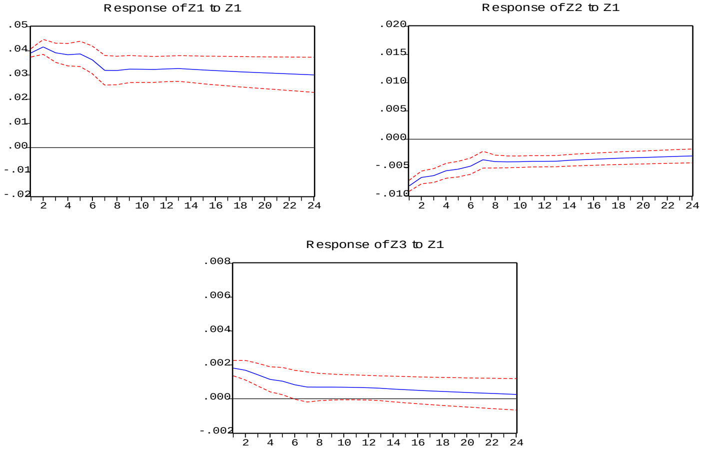


Figure 5b: Impulse-Responses to shocks in z_{t1}: VAR(7)
Sample period: 04.01.1999 – 25.02.2003

Generalized Impulse Response Analysis

Orthogonalized IRs depend on ordering of variables

- GIRF are unique and invariant to orderings of variables [Pesaran, M.H. & Shin,Y. (1998)]:
  the difference of conditional expectation given a one time shock occurs in series z_t.

Linear model: GIRF independent of observed hisory.
Results are similar to orthogonalized IRs; exceptions: Figure 6

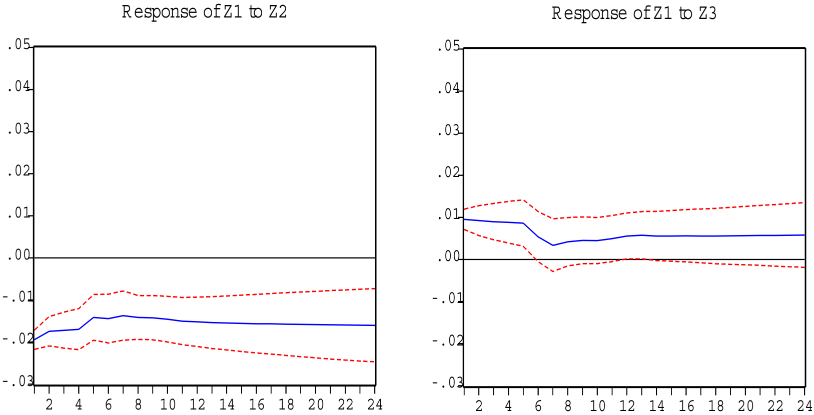


Figure 6: Generalized Impulse-Responses: VAR(7) for $z_t = (z_{t1}, z_{t2}, z_{t3})^\top$. Period: 04.01.1999 – 25.02.2003

Granger causality

Addresses the usefulness of each loading series in forecasting the others. Application of Granger causality tests

- testing zero restrictions of some VAR coefficients
- overfitting the VAR model by one lag to remove the singularity of the coefficient covariance matrix

[Granger (1969)]

Granger causality tests

H_0	Test result
$z_{t1} \nrightarrow z_{t2}, z_{t3}$	$F(14,3072) = 4.53\ (0.00)$
$z_{t2} \nrightarrow z_{t1}, z_{t3}$	$F(14,3072) = 1.66\ (0.06)$
$z_{t3} \nrightarrow z_{t1}, z_{t2}$	$F(14,3072) = 0.86\ (0.60)$
$z_{t3} \nrightarrow z_{t1}$	$\chi^2(7) = 5.04\ (0.65)$
$z_{t3} \nrightarrow z_{t2}$	$\chi^2(7) = 6.84\ (0.45)$
$z_{t1} \nrightarrow z_{t3}$	$\chi^2(7) = 8.02\ (0.33)$
$z_{t2} \nrightarrow z_{t3}$	$\chi^2(7) = 6.44\ (0.49)$
$z_{t1}, z_{t2} \nrightarrow z_{t3}$	$\chi^2(14) = 12.41\ (0.57)$

Table 2: $\nrightarrow$ denotes ‘does not Granger cause’. Results are based on model for z_t using $p = 7$ and full sample period 04.01.1999 - 25.02.2003. p-values in square brackets.

Granger causality tests

- Granger non-causality of z_{t1} for z_{t2} and z_{t3} and non-causality of z_{t2} for z_{t1} and z_{t3} is rejected at the 10% significance level
- z_{t3} is neither Granger-caused by z_{t1} nor z_{t2} and Granger non-causality from z_{t1} to z_{t3} and from z_{t2} to z_{t3} cannot be rejected

z_{t3} does not influence the dynamics of z_{t1} and z_{t2} in terms of the VAR model

Loadings and macroeconomic variables

- benchmark specification VAR model is extended to include macrovariables, log of euro/us exchange rate (LEX), log of oil prices (LPOIL) and interest rates (R12M) for the German stock market from 1.04.1999 – 2.25.2003.
- model and analyze a VAR(8) for $(z_{t1}, z_{t2}, z_{t3}, LEX, LPOIL, R12M)^\top$
- examine impulse response analysis of the system with possible economic intepretation of results

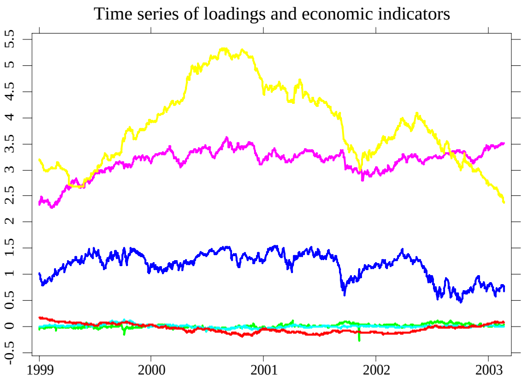


Figure 7: Factor loadings and economic indicators: z_{t1}(*blue*), z_{t2}(*green*), z_{t3}(*cyan*), LEX(*red*), $LPOIL$(*magenta*), $R12M$(*yellow*).

Impulse responses in $R12M$

- response in $R12M$: changes in risk compensation
- positive shock in z_{t1}: significant positive response in $R12M$
- positive shock in z_{t2}: (significant) negative response in $R12M$
- positive shock in z_{t3}: (significant) positive response in $R12M$

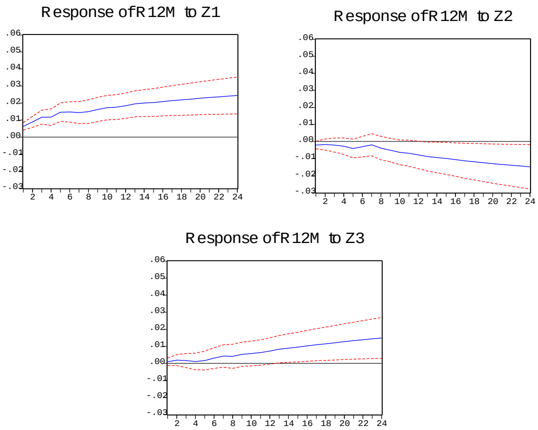


Figure 8: Generalized impulse responses in $R12M$: from VAR(8) for $z_t = (z_{t1}, z_{t2}, z_{t3})^\top$.

Responses to impulses in $R12M$

- positive shock in $R12M$: worse economic outlook and rising inflation expectations
- significant positive response in z_{t1}
- (signficant) negative response in z_{t2}
- no signficant response in z_{t3}:
  (macro)economic effects seem to feed into financial market risk via the maturity channel rather than via the moneyness dimension

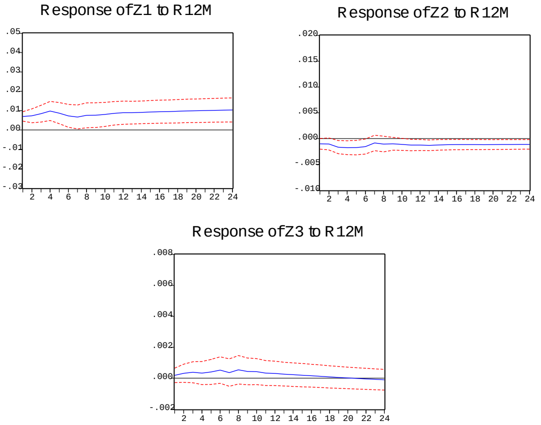


Figure 9: Generalized impulse responses in loading series to shocks in $R12M$: from VAR(8) for $z_t = (z_{t1}, z_{t2}, z_{t3})^\top$.

Further Results: *LPOIL* and *LEX*

- no significant links between *LPOIL* and loading series
- significant impulse-response relationship between *LEX* and z_{t1}: appreciation of Euro reduces volatility of DAX-options and higher volatility (financial market risk) induces EURO depreciation

Conclusion

- the VAR - DSFM Modelling framework provides a fairly good description of the IV dynamics and interrelations between the loadings that determine the movements of the IVS
- a VAR model reveals significant interaction between first and second loading series
- 12-month interest rate is significantly linked to volatility risk factors of German stock market
- interest rate channel seems to be most important for relation of macro and financial market risks
- an important outlook is to develop useful strategies for hedging against IV risk factors

Unit root tests

The Augmented Dickey-Fuller (ADF) test refers to the regression equation

$$\Delta z_{t,k} = \phi z_{t-1,k} + \sum_{i=1}^{p} a_i \Delta z_{t-i,k} + u_{t,k}, \tag{4}$$

where p is the number of lags of $\Delta z_{t,k}$ by which the regression equation (4) is augmented in order to get residuals free of autocorrelation.
Under H_0, the unit root the parameter ϕ should be zero. Hence, the t-statistic of the OLS estimator of ϕ is used as the ADF test statistic.

The limiting distribution of the test statistic is nonstandard. Critical or p-values have to be derived by the help of simulation methods.

The critical values used for the ADF test are -2.57 (10%), -2.86 (5%), and -3.44 (1%) (see, [Mackinnon, J.G (1991)])

Lag order p is determined by the AIC, HQ, and SC information criteria and test decisions may depend on the suggested order.

- ADF test suffers from low power and therefore may fail to detect a stationary time series

Point-optimal unit root test: (ERS) Elliot, Rothenberg and Stock (1996).
Superior to ADF procedure also in case of processes affected by conditional heteroscedasticity.
Test is based on quasi-differences of $z_{t,k}$ which are defined by

$$d(z_{t,k}|a) = \begin{cases} 1 & \text{if } t = 1 \\ z_{t,k} - a z_{t-1,k} & \text{if } t > 1, \end{cases}$$

a is the point alternative against which the null of a unit root is tested. Following the suggestion of Elliot et al. (1996), we use $a = \bar{a} = 1 - 7/T$ since only a constant term is considered.

Let $\hat{e}_t$ be the residuals from a regression of the time series on a quasi-differenced constant and let $S(\bar{a})$ and $S(1)$ be the sums of squared residuals for the cases $a = \bar{a}$ and $a = 1$ respectively. Then the test is defined by

$$ERS = (S(a) - aS(1))/\hat{\omega}_b, \tag{5}$$

where $\hat{\omega}_b$ is the spectral density estimator of $\hat{e}_t$ at frequency zero. We apply the autoregressive spectral density estimator as proposed by Elliot et al. (1996).
Critical values for the ERS test
(see,[Elliot, G., Rothenberg, T. J & Stock, J. H(1996)]) are 4.48 (10%), 3.26 (5%) and 1.99 (1%).

Residual correlation matrix

$\hat{P}_u$ is estimated residual correlation matrix for VAR(7): benchmark model

$$\hat{P}_u = \begin{pmatrix} 1 & -0.49 & -0.23 \\ -0.49 & 1 & -0.10 \\ -0.23 & -0.10 & 1 \end{pmatrix}. \tag{6}$$

Components of $\hat{P}_u$ are contemporaneously correlated, meaning that they have overlapping information to some extend.

To single out the individual effects, $\hat{P}_u$ is orthogonalized to be contemporaneously uncorrelated.
Cholesky decomposition provide a lower triangular matrix with positive main diagonals.

- Unfortunately, orthogonalization is not unique

Results of the IR analysis may depend to some extent on the ordering of the variables in the system. All possible variable orderings have been tried in computing the impulse responses.

Impulse Response Function (IRF)

Tracing the effect of a shock of size δ hitting the VAR system at time t on the state of the system at time $t+n$ given that no other shock hit the system. n is forecast horizon, ω_{t-1} is information set.

$$\begin{aligned} IRF(n,\delta,\omega_{t-1}) &= E\left[z_{t+n}|\varepsilon_t=\delta,\varepsilon_{t+1}=0,\ldots,\varepsilon_{t+0}=0,\omega_{t-1}\right] \\ &- E\left[z_{t+n}|\varepsilon_t=0,\varepsilon_{t+1}=0,\ldots,\varepsilon_{t+0}=0,\omega_{t-1}\right] \end{aligned}$$

To single out the individual effects, $\hat{P}_u$ is orthogonalized to be contemporaneously uncorrelated.

Generalized Impulse Response Function (GIRF)

The difference of conditional expectation given a one time shock occurs in series z_t.
n is forecast horizon, $\tilde{\omega}_{t-1}$ is the observed history and $\varepsilon_{j,t}$ is the chosen shock

$$GIRF(n, \delta, \omega_{t-1}) = E\left[z_{t+n} | \varepsilon_{j,t}, \omega_{t-1}\right] - E\left[z_{t+n} |, \tilde{\omega}_{t-1}\right]$$

GIR are unique and invariant to orderings of variables.
GIR coincide with orthogonalized IR if the residual covariance matrix is diagonal.

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