Empirical Pricing Kernels and Investor Preferences

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An investor observes the stock price and forms his subjective opinion about the future evolution.

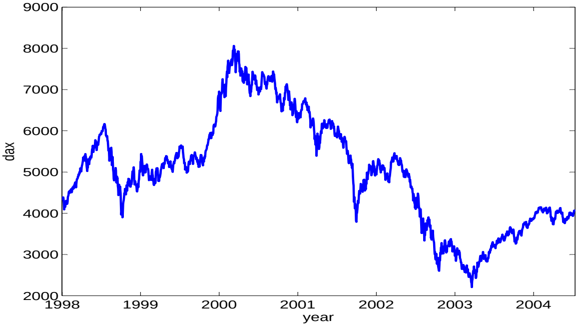


Figure 1: DAX, 1998 – 2004. Daily observations.

An opinion on the future value S_t can be described by a **subjective density** p (historical or physical density).

Examples:

- Black-Scholes model (Nobel prize 1997): log normal distribution
- GARCH model (Nobel prize 2003, Engle): stochastic volatility
- non-parametric diffusion model (Ait-Sahalia 2000)

Log returns $\{r_i\}$ are modeled with a GARCH-M (discrete Heston) model:

$$r_i = \mu - \frac{1}{2}V_i + \sqrt{V_i}Z_i$$
$$V_i = \omega + \beta V_{i-1} + \alpha(Z_{i-1} - \gamma\sqrt{V_{i-1}})$$

From the initial stock price S_0 the final stock price can be constructed:

$$S_t = S_0 \exp(\sum_{i=1}^{t} r_i).$$

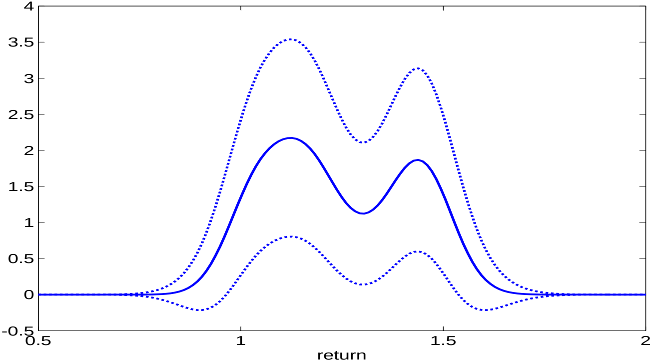


Figure 2: Subjective historical density with confidence bands on t=24 March 2000 for half a year returns, $(t-0.5, t)$, $\tau = 0.5$ (non-parametric kernel estimator)

There is also a **state-price density** (SPD) q implied by the market prices of options.

The SPD (a.k.a. **risk-neutral density**) differs from p because it corresponds to replication strategies (*martingale risk neutral measure*).

A person alone does not use in general a replication strategy but thinks in terms of his p density.

For SPD estimation a Heston continuous stochastic volatility model is used, which is an industry standard for option pricing models:

$$\frac{dS_t}{S_t} = rdt + \sqrt{V_t} dW_t^1$$

where the volatility process is modelled by a square-root process:

$$dV_t = \xi(\eta - V_t)dt + \theta\sqrt{V_t} dW_t^2,$$

and W^1 and W^2 are Wiener processes with correlation ρ.

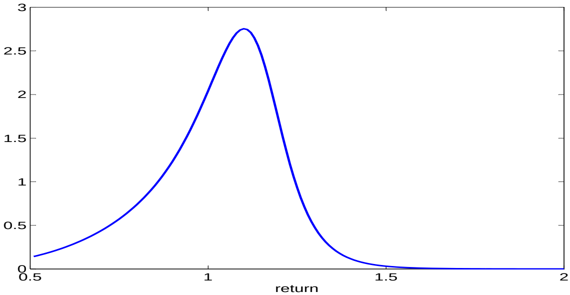
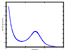


Figure 3: SPD on 24 March 2000, $r_{0.5} = 4.06\%$. Using option prices with time-to-maturity between 0.25 and 1 and moneyness between 0.5 and 1.5 we get the estimate for the SPD $\tau = 0.5$ years ahead.



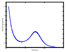
The **pricing kernel** $\mathcal{K}(x)$ is defined as:

$$\mathcal{K}(x) = \frac{q(x)}{p(x)}$$

An estimate of the pricing kernel is called **empirical pricing kernel** (EPK). We use the estimate:

$$\hat{\mathcal{K}}(x) = \frac{\hat{q}(x)}{\hat{p}(x)}$$

where $\hat{q}$ and $\hat{p}$ are the estimated risk-neutral and subjective densities.



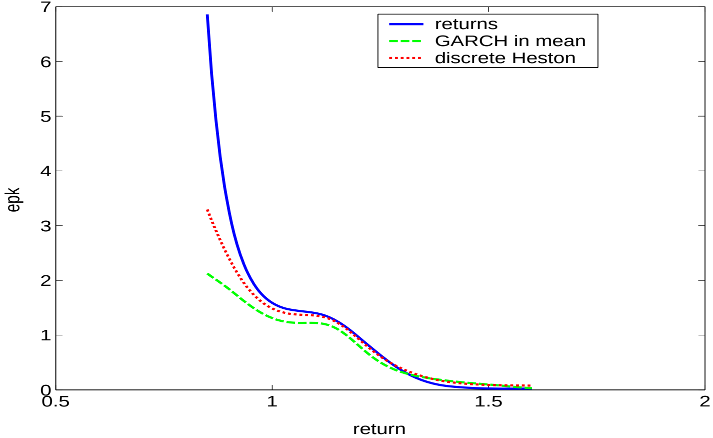


Figure 4: Empirical pricing kernel on 24 March 2000 for $\tau = 0.5$ year, $r_{0.5} = 4.06\%$.

Questions

- Is the EPK monotone?
- How to explain the non-monotonicity of the pricing kernel?
- What type of utility functions can generate observed pricing kernels and prices?
- What happens if the hypothesis of the existence of the representative investor is abandoned?

Outline

1. Motivation ✓
2. Pricing equation and pricing kernel (SDF)
3. Pricing kernel estimation and monotonicity test
4. Decomposition of the market utility function
5. Individual utility functions
6. Market aggregation mechanism
7. Estimation of the distribution of investor types
8. Outlook

Utility Maximisation Problem

$$\max_{\{\xi\}} U(C_0) + \beta \mathrm{E}^P \left[U(C_T) \right] \tag{1}$$

s.t. $C_0 = e_0 - P_0 \xi$

$C_T = e_T + \psi(S_T)\xi$

where $\psi(S_T)$	–	a pay-off profile contingent on S_T
P_0	–	the price of the asset at $t = 0$
ξ	–	portfolio position
β	–	subjective discount factor
e_0, e_T	–	wages at $t = 0$ and T
E^P	–	expectation w. r. to a historical measure P

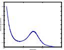
Pricing Equation

If the utility function depends only on state variables and $\beta = const$, then for **any** security paying $\psi(S_T)$:

$$P_0 = \mathrm{E}^P \left[\beta \frac{U'(C_T)}{U'(C_0)} \psi(S_T) \right] = \mathrm{E}^P \left[\tilde{m}(C_T) \psi(S_T) \right] \tag{2}$$

where the stochastic discount factor (SDF) is:

$$\tilde{m}(C_T) = \beta \frac{U'(C_T)}{U'(C_0)} = const \cdot U'(C_T)$$



Stochastic Discount Factor Projection

Pricing equation using the SDF projection onto asset prices S_T (a state variable alternative to C_T):

$$P_0 = \mathrm{E}^P \left[m(S_T) \psi(S_T) \right] = \int_0^\infty m(s) \; \psi(s) \; p(s) ds, \qquad (3)$$

where the projection:

$$m(S_T) = \mathrm{E}^P \left[\tilde{m}(C_T) | S_T \right]$$

Pricing with $\tilde{m}$ and m is equivalent if the projection is unique. The projection is **linear** if $\psi(S_T) = S_T$ (budget constraint).

Risk-neutral pricing equation:

$$\begin{aligned} P_0 &= e^{-r\tau}\mathrm{E}^Q\left[\psi(S_T)\right] = e^{-r\tau}\int_0^\infty \psi(s)\; q(s)\; ds = \qquad (4)\\ &= e^{-r\tau}\int_0^\infty \psi(S_T)\;\frac{q(s)}{p(s)}\;p(s)ds \qquad (5)\end{aligned}$$

where $p(s)$ and $q(s)$ are subjective and risk neutral pdf's

Since (3) and (5) are equivalent (hold for **any** $\psi(S_T)$), the pricing kernel is:

$$\mathcal{K}(S_T) = \frac{q(S_T)}{p(S_T)} = \frac{U'(S_T)}{U'(S_0)}$$

The Black-Scholes Model

Geometric Brownian motion process:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{6}$$

The historical density p is log-normal:

$$p(x) = \frac{1}{x}\frac{1}{\sqrt{2\pi}\tilde{\sigma}} \exp\left\{ -\frac{1}{2}\left(\frac{\log x - \tilde{\mu}}{\tilde{\sigma}}\right)^2 \right\}, \; x > 0$$

where $\tilde{\mu} = (\mu - \frac{\sigma^2}{2})t + \log S_0$ and $\tilde{\sigma} = \sigma\sqrt{t}$

$p(x)$ and $q(x)$ are both log-normal and the pricing kernel is

$$\mathcal{K}(x) = \left(\frac{x}{S_0}\right)^{-\frac{\mu-r}{\sigma^2}} \exp\left\{\frac{(\mu-r)(\mu+r-\sigma^2)T}{2\sigma^2}\right\}$$

Up to a linear transformation the utility function is a CRRA function:

$$U(S_T) = \left(1 - \frac{\mu - r}{\sigma^2}\right)^{-1} \; S_T^{(1-\frac{\mu-r}{\sigma^2})} \tag{7}$$

In terms of $R_T = \frac{S_T}{S_0}$:

$$U(R_T) = a\frac{R_T^{1-\gamma}}{1-\gamma}$$

Estimation of the Pricing Kernel

The empirical pricing kernel is:

$$\hat{\mathcal{K}}(S_T) = \frac{\hat{q}(S_T)}{\hat{p}(S_T)},$$

PK estimation:

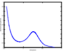
- the risk neutral density q from option prices with the Heston model
- the historical subjective density p from stock prices with the GARCH-M, discrete Heston and non-parametric kernel density models

Estimation of the Subjective Density p

Model	History
GARCH in mean	2.0y
discrete Heston	2.0y
non-parametric kernel	1.0y

Table 1: Models and the time periods used for their calibration.

The GARCH-M and discrete Heston is simulated $\tau = 0.5$y ahead with 2000 repetitions.



Estimation of the Risk Neutral Density q

Risk neutral density q is estimated from DAX option prices using the stochastic volatility Heston model:

$$\frac{dS_t}{S_t} = rdt + \sqrt{V_t} dW_t^1$$

where the volatility process is:

$$dV_t = \xi (\eta - V_t) dt + \theta \sqrt{V_t} dW_t^2$$

W_t^1, W_t^2 – Wiener processes with correlation ρ

The parameters in the Heston model can be interpreted as:

ξ	–	mean-reversion speed, $\xi = 2$ (Bergomi, 2005)
η	–	long-term variance
V_0	–	short-term variance
ρ	–	correlation
θ	–	volatility of volatility

η and V_0 control the term structure of the implied volatility surface (i.e. time to maturity direction).

ρ and θ control the smile/skew (i.e. moneyness direction).

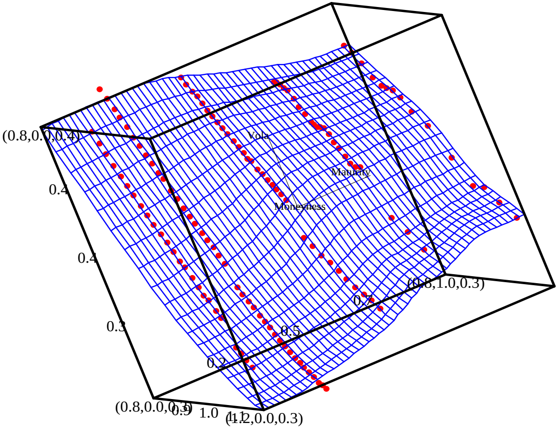


Figure 5: Implied volatility surface.

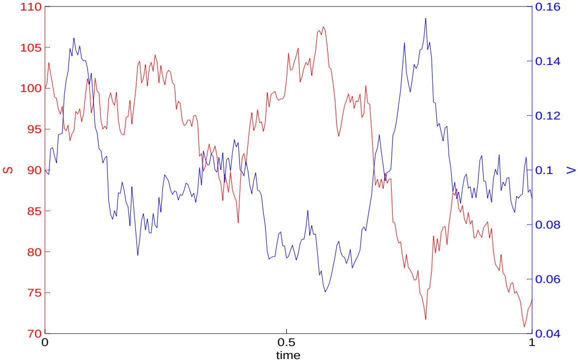


Figure 6: Simulated paths in the Heston model for the parameters $V_0 = 0.1$, $\eta = 0.08$, $\xi = 2$, $\theta = 0.3$, $\rho = -0.7$. S – stock process, V – variance process.

We estimate the parameters of the SPD by minimising the ASE of the implied volatilities:

$$\frac{1}{n}\sum_{i=1}^{n}(IV_i^{model} - IV_i^{market})^2$$

where IV^{model} and IV^{market} refer to model and market implied volatilities; n is the number of observations on the surface.

Typically, we observe option prices with time to maturity $\tau \in [0.25; 1]$ years and moneyness $K/S_0 \in [0.5; 1.5]$.

Plain vanilla call option prices are calculated by a method of Carr and Madan:

$$C(K,T) = \frac{\exp\{-\alpha \log(K)\}}{2\pi} \int_0^\infty \exp\{-\mathbf{i}v\log(K)\}\psi_T(v)dv$$

for a damping factor $\alpha > 0$. The function ψ_T is given by

$$\psi_T(v) = \frac{\exp(-rT)\phi_T\{v-(\alpha+1)\mathbf{i}\}}{\alpha^2+\alpha-v^2+\mathbf{i}(2\alpha+1)v}$$

where ϕ_T is the characteristic function of $\log(S_T)$.

The characteristic function:

$$\phi_T(z) = \exp\left\{\frac{-(z^2+\mathbf{i}z)V_0}{\gamma(z)\coth\frac{\gamma(z)T}{2}+\xi-\mathbf{i}\rho\theta z}\right\} \times \frac{\exp\left\{\frac{\xi\eta T(\xi-\mathbf{i}\rho\theta z)}{\theta^2}+\mathbf{i}zTr+\mathbf{i}z\log(S_0)\right\}}{\left(\cosh\frac{\gamma(z)T}{2}+\frac{\xi-\mathbf{i}\rho\theta z}{\gamma(z)}\sinh\frac{\gamma(z)T}{2}\right)^{\frac{2\xi\eta}{\theta^2}}} \qquad (8)$$

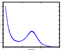
where $\gamma(z) \stackrel{\text{def}}{=} \sqrt{\theta^2(z^2+\mathbf{i}z)+(\xi-\mathbf{i}\rho\theta z)^2}$ see e.g. (Cizek et al., 2005).

The density $f(\log S_T)$ can be recovered with Fourier inversion:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathbf{i}tx} \phi_T(t) dt,$$

The risk neutral density $q(S_T)$ is given as a transformed density:

$$q(x) = \frac{1}{x} f\{\log(x)\}$$



Estimation of the Subjective Density p

The log-returns r_i of DAX for 0.5 year are modelled with the GARCH-M model:

$$
\begin{aligned}
r_i &= \mu + \sqrt{V_i} Z_i \\
V_i &= \omega + \beta V_{i-1} + \alpha r_{i-1}^2
\end{aligned}
$$

From S_0 we can construct S_t as:

$$S_t = S_0 \exp\left(\sum_{i=1}^{t} r_i\right)$$

- Fit the GARCH-M model for DAX returns
- Simulate N time series of the returns (N=2000)
- Compute the final N DAX prices
- Evaluate $\hat{p}$ using kernel density estimation

Other applied models:

- discrete Heston
- non-parametric kernel

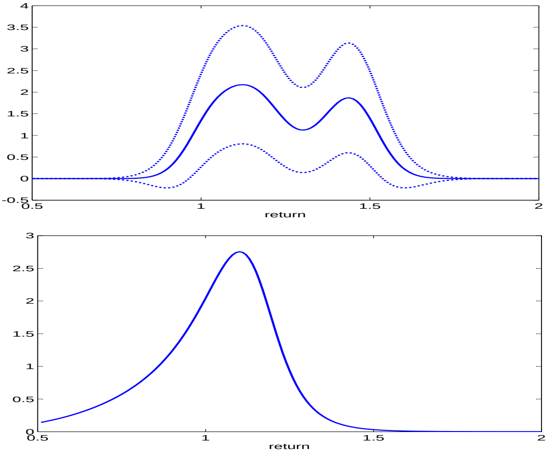


Figure 7: Empirical historical and risk neutral price densities, 24 March 2000.

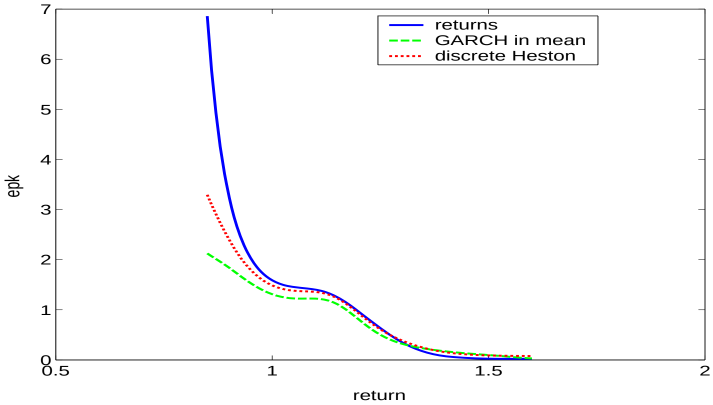


Figure 8: Empirical pricing kernels on 24 March 2000.

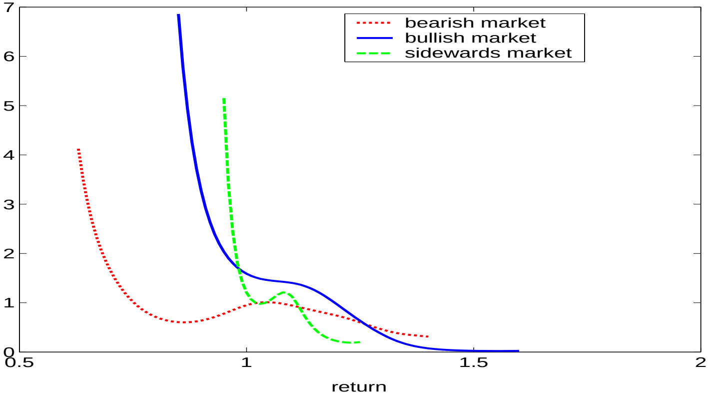


Figure 9: Empirical pricing kernel on 24 March 2000, 30 July 2002 and 30 June 2004.

Relative risk aversion coefficient:

$$RRA(S_T) = -S_T \frac{U''(S_T)}{U'(S_T)}.$$

RRA can be estimated directly from the risk neutral and historical densities:

$$\begin{aligned} RRA(S_T) &= -S_T \frac{q'(S_T)p(S_T) - q(S_T)p'(S_T)}{p^2(S_T)} \Big/ \frac{q(S_T)}{p(S_T)} = \\ &= S_T \left\{ \frac{p'(S_T)}{p(S_T)} - \frac{q'(S_T)}{q(S_T)} \right\}. \end{aligned}$$

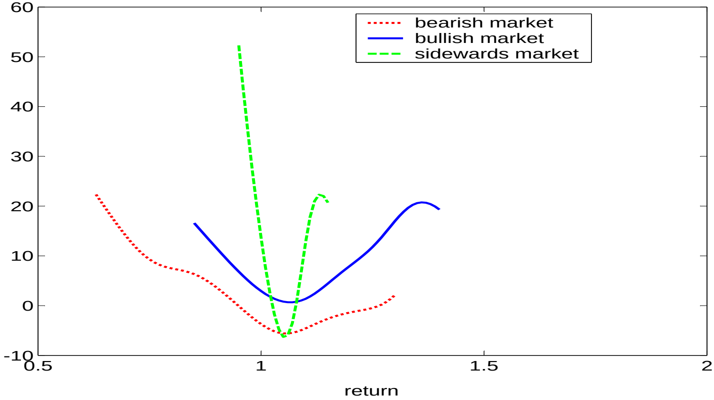
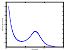


Figure 10: Relative risk aversion on 24 March 2000, 30 July 2002 and 30 June 2004.



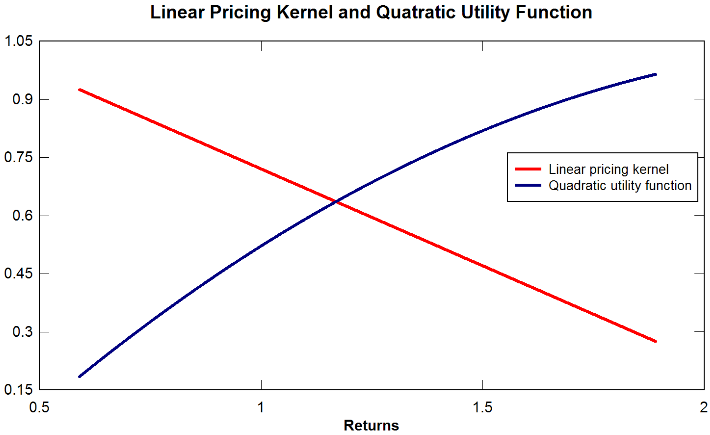
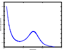


Figure 11: Linear pricing kernel and quadratic utility function (CAPM model). $U(S_T) = -aS_T^2 + bS_T + c$.



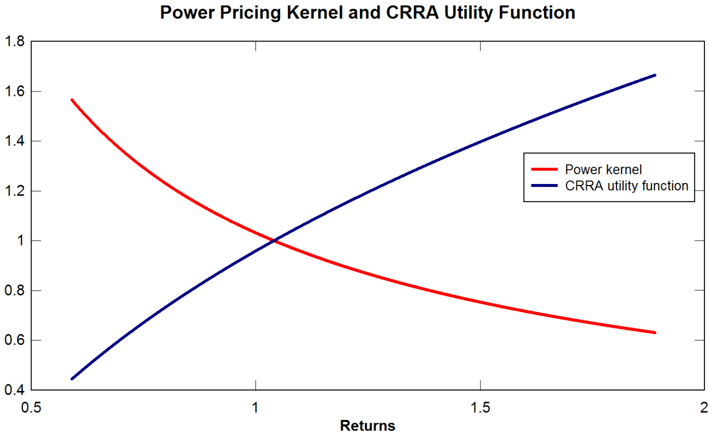
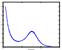


Figure 12: Power pricing kernel and CRRA utility function. $U(S_T) = a\frac{S_T^{1-\gamma}}{1-\gamma}$.



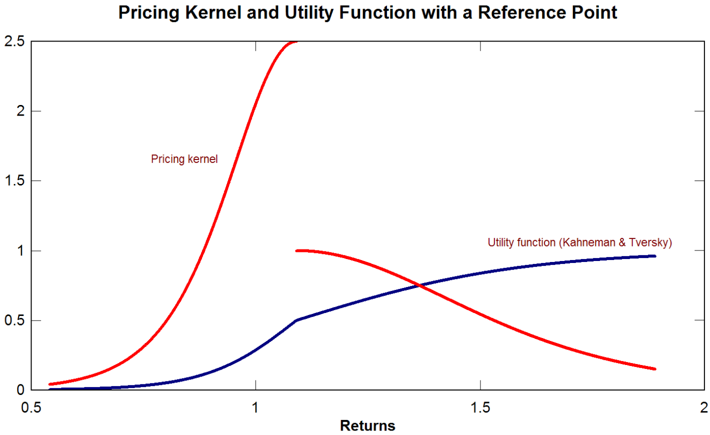
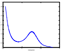


Figure 13: Pricing kernel and utility function suggested by Kahneman and Tversky based on behavioural experiments.



Pricing Kernel Monotonicity Test

$\{S_i\}_{i=1}^n \sim p$, historical subjective density

q, risk-neutral density; $S_{(k)}$ order statistic

$\mathcal{K}$, pricing kernel

$\mathcal{K}_k = \mathcal{K}(S_{(k)}) = \frac{q(S_{(k)})}{p(S_{(k)})}$, decreasing $\forall$ I and J, $I \leq k \leq J$

- spacing method to reduce to exp model
- ML test for monotonicity in (I, J)
- multiple testing to find $\hat{I}$ and $\hat{J}$

Pyke's theorem: Let i.i.d. $U_i \sim U(0,1)$ and i.i.d. $e_i \sim Exp(1)$, $i = 1, \ldots, n$.

$$\mathcal{L}\left(U_{(k+1)} - U_{(k)}\right) = \mathcal{L}\left(\frac{e_k}{\sum_{s=1}^{n} e_s}\right), \qquad 1 \leq k \leq n-1.$$

Hence:

$$n\left(U_{(k+1)} - U_{(k)}\right) \approx e_k. \tag{9}$$

With the cdf $P(x)$:

$$U_{(k+1)} - U_{(k)} = P\left(S_{(k+1)}\right) - P\left(S_{(k)}\right) \approx p\left(S_{(k)}\right)\left(S_{(k+1)} - S_{(k)}\right)$$

Hence from (9):

$$n\left(S_{(k+1)} - S_{(k)}\right) q(S_{(k)}) \approx \frac{q(S_{(k)})}{p(S_{(k)})} e_k = \mathcal{K}\left(S_{(k)}\right) e_k = \mathcal{K}_k e_k.$$

Test with observations

$$Z_k = \mathcal{K}_k e_k$$

whether $\mathcal{K}_k$ is monotone.

Maximum Likelihood Ratio Test

$$\mathcal{M}(I,J) = \{x_k \geq 0: \quad x_k \geq x_{k+1}, \quad I \leq k \leq J\}$$

For $Z = (Z_1, \ldots, Z_k)$ define the log-likelihood:

$$\log\{p(Z,\mathcal{K})\} = -\sum_{k=I}^{J} \frac{Z_k}{\mathcal{K}_k} - \sum_{k=I}^{J} \log \mathcal{K}_k,$$

Maximum log-likelihood:

$$\max_{\mathcal{K}} \log\{p(Z,\mathcal{K})\} = -n - \sum_{k=1}^{n} \log(Z_k).$$

The test statistic:

$$\xi(I,J) = \log \frac{\max_{\mathcal{K}\in\mathcal{M}(I,J)} p(Z,\mathcal{K})}{\max_{\mathcal{K}} p(Z,\mathcal{K})}$$

The critical value ($\mathcal{K}_k = 1$):

$$h_\alpha(I,J) = M(I,J) + t_\alpha V(I,J)$$

where $M(I,J) = E_0\xi(I,J)$, $V^2(I,J) = E_0\{\xi(I,J) - M(I,J)\}^2$.
t_α is calculated by Monte Carlo as the solution of

$$P_0\left[\max_{I=1,n}\max_{J=I+1,n}\{\xi(I,J) - M(I,J) - t_\alpha V(I,J) \geq 0\}\right] = \alpha$$

ML ratio monotonicity test:

- compute $Z_k = n\left(S_{(k+1)} - S_{(k)}\right) q(S_{(k)})$
- compute test statistic
  $\xi(I,J) = \max_{\mathcal{K}\in\mathcal{M}(I,J)} \log\{p(Z,\mathcal{K})\} - \max_{\mathcal{K}} \log\{p(Z,\mathcal{K})\}$
- H_0 is rejected if $\xi(I,J) - h_\alpha(I,J) < 0$

Estimation of the Market Utility Function

Utility function is derived from the market data under the **representative investor** assumption:

$$U(S_T) = \int_0^{S_T} m(x)dx$$

A cardinal utility function can be defined up to a linear transformation.

$$U(R_T) = \int_0^{R_T} \frac{q(S_0 x)}{p(S_0 x)} dx$$

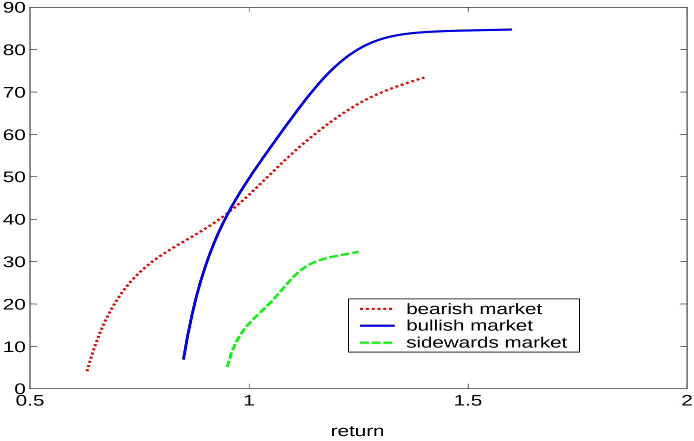


Figure 14: Market utility functions on 24 March 2000, 30 July 2002 and 30 June 2004.

Decomposition of the Utility Function

Observation: the portions of the utility function below $R_T = \frac{S_T}{S_0} = 1$ and above 1.15 are very well approximated with hyperbolic absolute risk aversion (shifted CRRA, Sharpe (2006)) functions:

$$U(x) = a(x - c)^\gamma + b, \tag{10}$$

The HARA function becomes infinitely negative for $x = c$ and is extended as $U(x) = -\infty$ for $x < c$. HARA($c = 0$)=CRRA.

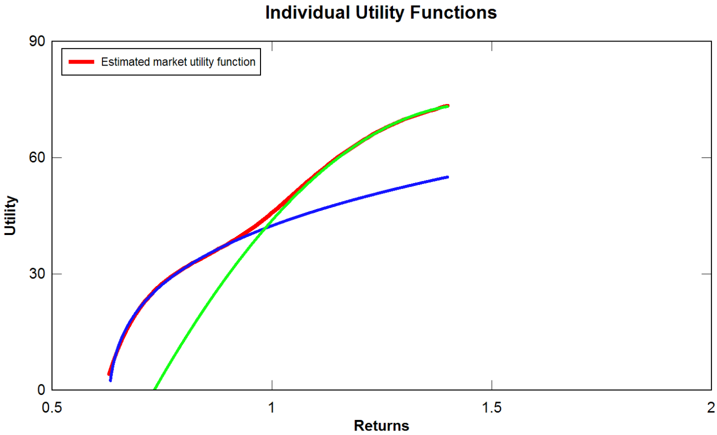
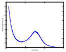


Figure 15: Decomposition of the utility function, $\tau = 0.5$ years, 30 July 2002.



Individual Utility Functions

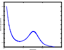
Investor i has utility comprising two HARA components:

$$U(x, c_{2,i}) = \begin{cases} \max\left\{U(x, \theta_1, c_1); U(x, \theta_2, c_{2,i})\right\}, & \text{if } \; x > c_1 \\ -\infty, & \text{if } \; x \leq c_1 \end{cases}$$

where $\theta = (a, b, \gamma)^\top$, $c_{2,i} > c_1$. Investors differ in the parameter $c_{2,i}$.

	a_i	b_i	γ_i	c_i
$i = 1$ (bearish market)	80.58	-20.57	0.25	0.626
$i = 2$ (bullish market)	-134.75	73.91	2.00	–

Table 2: θ estimated from upper/lower quantiles, 30 July 2002.



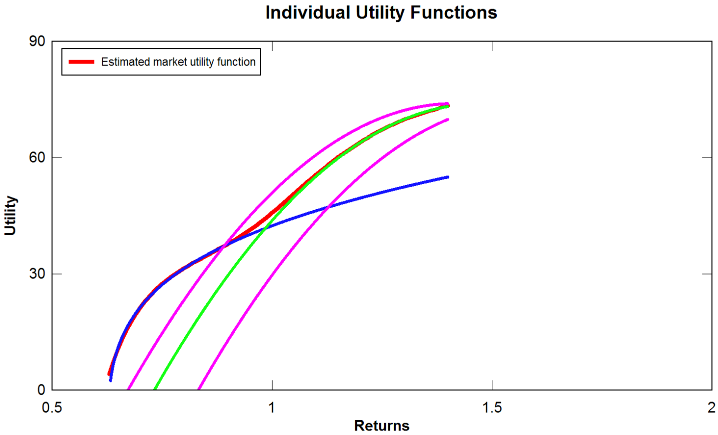
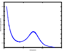


Figure 16: Individual and market utility functions with a switching point, $\tau = 0.5$ years, 30 July 2002.

Investor Types

- Switching from bearish to bullish happens at $z = z(c_{2,i})$
- Different investors have different perceptional boundaries between "good" and "bad" states
- Switching points are in $[0.95; 1.1]$, i.e. in the area that corresponds to present unit returns times half-year risk free interest rates
- There is a distribution of switching points (inverse problem)



Naive Utility Aggregation

- Specify the **observable** states of the world in the future by returns R_T
- Find a weighted average of the utility functions for each state. If the importance of the investors is the same, then the weights are equal
- **Problem**: utility functions of N different investors cannot be summed up since they are incomparable

Investor's Attitude Aggregation

- Specify **perceived** states of the world given by utility u
- Aggregate the outlooks concerning the **returns** in the future R_T for each perceived state
- Estimate the distribution of switching points
- Aggregation leads to an inverse problem

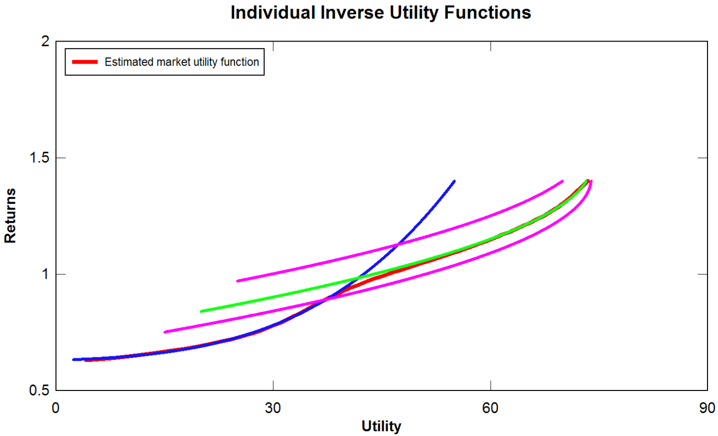


Figure 17: Inverse market and individual utility functions, $\tau = 0.5$ years, 30 July 2002.

For a **subjective** state described with utility u:

$$u = U^{(1)}(R_T^{(1)}, z_1) = U^{(2)}(R_T^{(2)}, z_2) = \ldots = U^{(N)}(R_T^{(N)}, z_N)$$

The aggregate estimate of the resulting return is

$$R_T^A(u) = N^{-1} \sum_{i=1}^{N} R_T^{(i)}(u) = N^{-1} \sum_{i=1}^{N} U^{-1}(u, z_i)$$

if all investors have the same market power.
Important property: the return aggregation procedure is invariant of *any* monotonic transformation

Distribution of Switching Points

The aggregate return in the **perceptional** state u is given by:

$$R^A(u) = \int U^{-1}(u, z) f(z) dz \tag{11}$$

In oder to solve (11) for $f(\cdot)$:

$$\min_{f(\cdot)\in\mathcal{F}} \int \left\{ R_f^A(u) - U_M^{-1}(u) \right\}^2 \tilde{P}(du), \tag{12}$$

where $U_M^{-1}(u)$ is the inverse of the estimated market utility function, $\tilde{P}$ is the distribution of utility levels.

Take

$$f \in \mathcal{F} = \left\{ f = \sum_{j=1}^{J} \theta_j I_{\{z \in B_j\}}, \theta_j \geq 0, \sum_{j=1}^{J} \theta_j h_j = 1, h_j = |B_j| \right\}.$$

The problem (12) becomes a quadratic programming problem:

$$\min_{\theta} \sum_{i=1}^{n} \left\{ R_f^A(u_i) - R_i \right\}^2$$

$$\theta_j \geq 0$$

$$\sum_{j=1}^{J} \theta_j h_j = 1$$

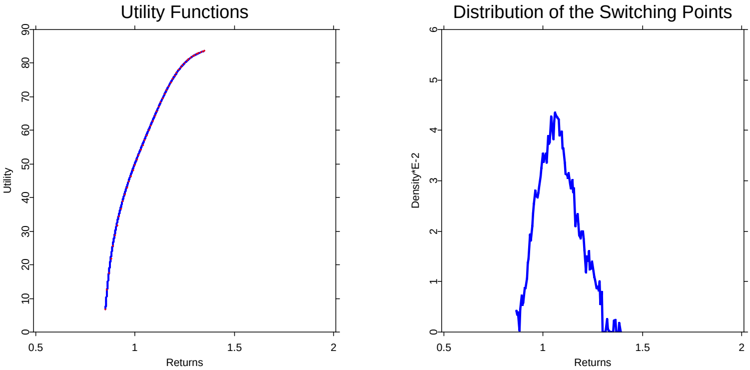


Figure 18: Left panel: the market utility function (red) and the fitted utility function (blue). Right panel: the distribution of the reference points. 24 March 2000, a bearish market.

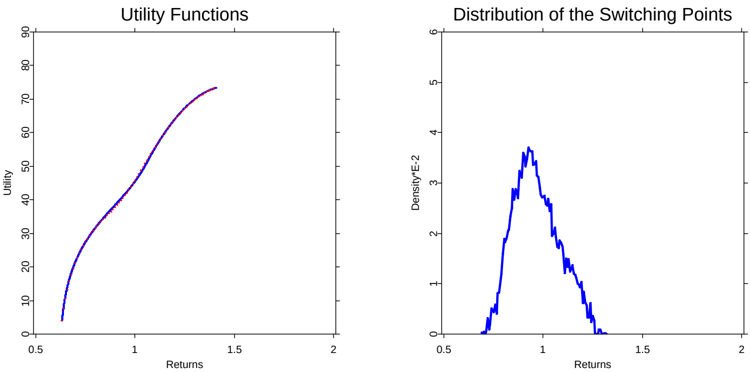


Figure 19: Left panel: the market utility function (red) and the fitted utility function (blue). Right panel: the distribution of the reference points. 30 July 2002, a stable market.

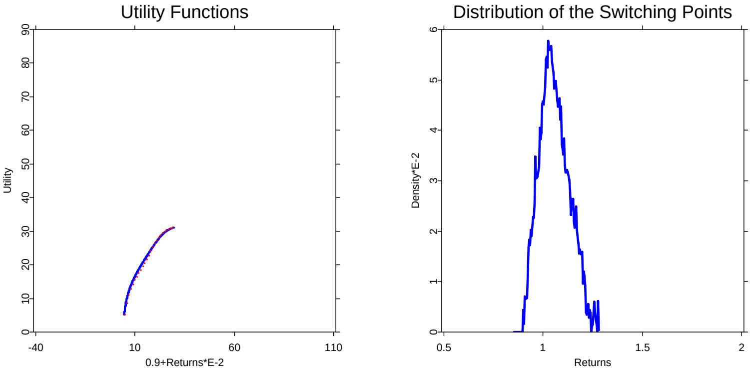


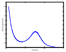
Figure 20: Left panel: the market utility function (red) and the fitted utility function (blue). Right panel: the distribution of the reference points. 30 June 2004, a bullish market.

Summary

- Representation of individual utility functions as consisting of two parts: for "good" and "bad" states of the world
- Investors behave as risk averse individuals in "good" and "bad" states but become risk seeking when switching occurs
- Utility function aggregation procedure based on subjective states of the world
- Formulation of an inverse problem for the estimation of the switching points distribution

Outlook

- Testing alternative utility function designs
- Refining the technique for estimating the distribution of switching points as an inverse problem
- Study of the dynamics of pricing kernels and individual utility functions (Giacomini et al., 2006)
- Testing the hypothesis of the local utility function non-concavity due to switching in a behavioural experiment



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