Common Functional Component Modelling

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Karhunen-Loève decomposition (FPCA)

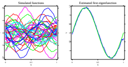
Task: construct a factor model for random sample $X_i, i = 1, \ldots, n$.
Popular solution motivated by Karhunen-Loève decomposition is:

$$X_i = \mu + \sum_{j=1}^{\infty} \beta_{ij}\gamma_j, \quad i = 1, \ldots, n, \tag{1}$$

μ $= \mathsf{E}(X_i)$ is the mean function

γ_r eigenfunctions of Covariance operator, corresponding to the r-th largest eigenvalue λ_r

β_{ir} $= \int X_i(t)\gamma_r(t)dt$ factor loadings, $\mathsf{Var}(\beta_{ir}) = \lambda_r$



Properties of KL-decomposition

best "empirical" basis:

$$\rho(v_1, \ldots, v_L) = \mathsf{E}(\| X_i - \mu - \sum_{j=1}^{L} \langle X_i - \mu, v_j \rangle v_j \|^2)$$

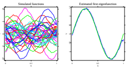
is minimized by $v_j = \gamma_j$,

linear transformation with max. variance:

$$\mathsf{Var}\langle v_r, X_i \rangle,$$

is maximized by $v_r = \gamma_r, r = 1, 2, \ldots, L$

for any choice of L orthonormal basis functions $v_1, \ldots, v_L$



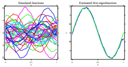
Two sample problem in principal components

We are mainly interested in two sample problems:

$$X_1^{(1)}, X_2^{(1)}, \dots, x_{n_1}^{(1)} \text{ and } X_1^{(2)}, X_2^{(2)}, \dots, X_{n_2}^{(2)} \tag{2}$$

Goal: Compare the distribution of functions in both samples by comparing their KL-decompositions:

$$X_i^{(p)} = \mu^{(p)} + \sum_{r=1} \beta_{ir}^{(p)} \gamma_r^{(p)}, \quad p = 1, 2. \tag{3}$$



Comparison of KL-decompositions

Common Eigenfunctions Hypothesis

- $H_{0_{2,r}} : \gamma_r^{(1)} = \gamma_r^{(2)}$

Common Eigenspaces Hypothesis

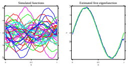
- $H_{0_{4,L}} : \mathcal{E}_L^{(1)} = \mathcal{E}_L^{(2)}$, $\mathcal{E}_L^{(p)} \stackrel{\text{def}}{=} \text{span}\{\gamma_1^{(p)}, \ldots, \gamma_L^{(p)}\},\ p = 1, 2$

Common Mean-functions hypothesis

- $H_{0_1} : \mu^{(1)} = \mu^{(2)}$

Common eigenvalues hypothesis

- $H_{0_{3,r}} : \lambda_r^{(1)} = \lambda_r^{(2)}$



Applications

- **Stability analysis** of KL factor (FPC) models

- Direct applications in finance:

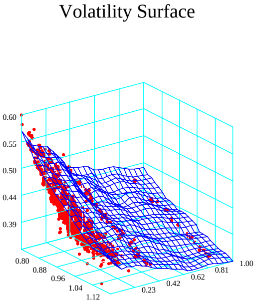
  Implied volatility (IV) analysis

  Analyze the dynamics and term structure of IV surface

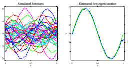
  Construct FPC model for slices of IV at fixed maturity

  Can we use same (common) model for different maturities ?

  $\longrightarrow$ Additional dim. reduction and precision



ODAX Implied Volatility Surface $\hat{\sigma}_t(\kappa, \tau)$, $t = 20030307$, $\kappa \in [0.8, 1.2]$ denotes moneyness (Standardized strike) and $\tau \in [0, 1]$ time to maturity (in years). Surface estimated using DSFM model, red points are raw IVs calculated from contract-data <film>



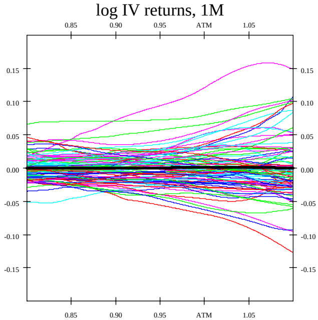
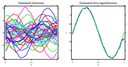
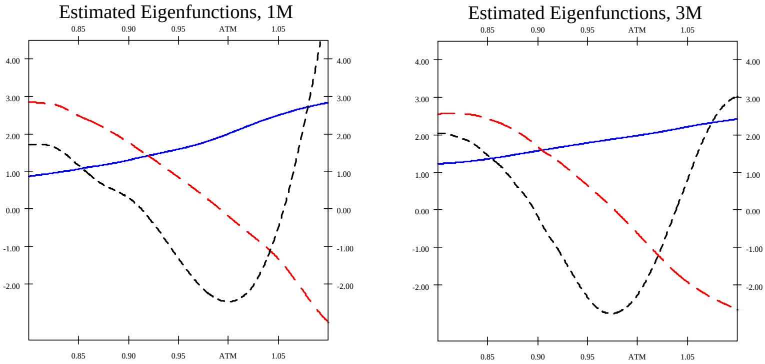


Figure 1: Two functional samples of interest: (estimated) daily log-returns of the "slices" of the IV Surface at 1-Month maturity , $X_i^{(1)}(\kappa) = \Delta log\sigma_i(\kappa, 1M)$ (left figure) and 3-Months maturity $X_i^{(2)}(\kappa) = \Delta log\sigma_i(\kappa, 3M)$ (right figure), Jan 2004-Jun 2004.

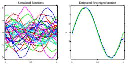




Estimated eigenfunctions 1M group (left plot) and 3M group (right plot), blue – first function, red – second function, black – third function

Outline of the talk

1. Introduction ✓
2. One sample inference
3. Two sample problem
4. Implied volatility analysis

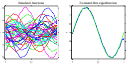


One sample inference

Given sample $X_i(t), \; i = 1, \ldots, n$
estimate γ_r, λ_r by eigenfunctions and eigenvalues $\hat{\gamma}_r, \hat{\lambda}_r$
of empirical covariance operator $\mathcal{C}_n$:

$$\mathcal{C}_n = \frac{1}{n} \sum_{i=1}^{n} \langle X_i - \bar{X}, \xi \rangle (X_i - \bar{X}) \tag{4}$$

[Dauxois, J., Pousse, A. and Romain, Y. (1982)]
give asymptotical results on $\hat{\gamma}_r, \hat{\lambda}_r$
i.e. $\| \gamma_r - \hat{\gamma}_r \| = \mathcal{O}_P(n^{-1/2})$



Dual approach

Focus on the matrix

$$M_{lk} = \langle X_l - \bar{X}, X_k - \bar{X} \rangle, \quad l, k = 1, \ldots, n. \tag{5}$$

Denote eigenvalues, eigenvectors of M by $l_1 \geq l_2 \ldots$, $p_1, p_2, \ldots$

Some calculations show

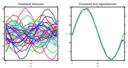
1. $\hat{\lambda}_r = l_r / n$
2. $\hat{\beta}_{ir} = \langle X_i - \bar{X}, \hat{\gamma}_r \rangle = \sqrt{l_r}\, p_{ir}$
3. $\hat{\gamma}_r = \left(\sqrt{l_r}\right)^{-1} \sum_{i=1}^{N} p_{ir} \left(X_i - \bar{X}\right) = \left(\sqrt{l_r}\right)^{-1} \sum_{i=1}^{n} p_{ir} X_i$

Estimation procedure – Model

In practice the functional values are observed on a discrete grid and possibly contaminated with additional error:

$$Y_{ik} = X_i(t_{ik}) + \varepsilon_{ik}, \quad k = 1, \ldots, T_i, \tag{6}$$

where ε_{ik} are independent noise terms with $\mathrm{E}(\varepsilon_{ik}) = 0$, $\mathrm{Var}(\varepsilon_{ik}) = \sigma_i^2$, t_{ik} is fixed or random design, $T \stackrel{\mathrm{def}}{=} \min T_i$.



Estimation of M

Define

$$\chi_i(t) = \sum_{j=1}^{T_i} Y_{i(j)} I\left(t \in \left[\frac{t_{i(j-1)} + t_{i(j)}}{2}, \frac{t_{i(j)} + t_{i(j+1)}}{2}\right]\right), \; t \in [0,1],$$

and estimate M_{ij} by

$$\widehat{M}_{ij} = \int_0^1 \{\chi_i(t) - \bar{\chi}(t)\} \{\chi_j(t) - \bar{\chi}(t))\} \, dt,$$

where $\bar{\chi}(t) \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n \chi_i(t)$, $I(\cdot)$ denotes indicator function, $t_{i(j)}, j = 1, \ldots T_i$ ordered sample of design points, $t_{i(0)} = -t_{i(1)}$, $t_{i(T_i+1)} = 2 - t_{i(T_i)}$ and $Y_{i(j)}$ observation belonging to $t_{i(j)}$.

Bias correction for M_{ii}

Redefine $t_{i(1)} := -t_{i(2)}$ and $t_{i(T_i+1)} = 2 - i(T_i)$

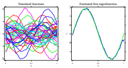
$$\chi_i^*(t) \stackrel{\text{def}}{=} \sum_{j=2}^{T_i} Y_{i(j-1)} I\left(t \in [\frac{t_{i(j-1)} + t_{i(j)}}{2}, \frac{t_{i(j)} + t_{i(j+1)}}{2}]\right)$$

.

Estimate the diagonal terms M_{ii} by

$$\widehat{M}_{ii} = \int_0^1 \{\chi_i(t) - \bar{\chi}(t)\} \{\chi_i^*(t) - \bar{\chi}(t))\} \, dt \qquad (7)$$

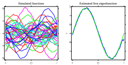
Aim: avoid additional bias implied by: $\mathrm{E}_\varepsilon(Y_{ik}^2) = X_i(t_{ij})^2 + \sigma_i^2$.



Estimation of $\hat{\gamma}_r$

$$\hat{\gamma}_r = \left(\sqrt{l_r}\right)^{-1} \sum_{i=1}^{n} p_{ir} X_i$$

1. Estimate M by $\hat{M}$ and calculate $\hat{l}_j$, $\hat{p}_j$, $j = 1, \ldots, n$
2. Estimate X_i by $\hat{X}_i$ (e.g. local smoothers with bandwidth b) (Choice of b: cross validation)
3. Calculate $\hat{\gamma}_{r,T} = \left(\sqrt{\hat{l}_r}\right)^{-1} \sum_{i=1}^{n} \hat{p}_{ir} \hat{X}_i$



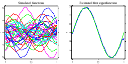
Choice of smoothing parameter

Cross-Validation (leave-one-out) argument:

$$b_{CV} = \arg\min_b \sum_i \sum_j \left\{ Y_{ij} - \hat{\mu}_{T,-i}(t_{ij}) - \sum_{r=1}^{s} \hat{\vartheta}_{ri} \hat{\gamma}_{r;T,-i}(t_{ij}) \right\}^2$$

For a fixed $s \in \mathbb{N}$

$\hat{\mu}_{T,-i}$ and $\hat{\gamma}_{r;T,-i}$, $r = 1, \ldots, s$ denote $\hat{\mu}$ and $\hat{\gamma}_r$ estimated from (Y_{kj}, t_{kj}), $k = 1, \ldots, i-1, i+1, \ldots, n$, $j = 1, \ldots, T_k$.
$\hat{\vartheta}_{ri}$ denote OLS estimates of $\hat{\beta}_{ij}$.



Asymptotic results – Basic assumptions

1. $X_1, \ldots, X_n$ is i.i.d. sample of a.s. twice continuously differentiable random functions on $L_2[0,1]$

2. The estimates $\hat{X}_i$ are determined by local linear or Nadaraya-Watson estimator with bandwith b

Asymptotical results - Estimation error

Under assumptions 1) and 2) and using some further regularity conditions we obtain

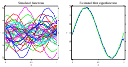
i) $n^{-1}\sum_{i=1}^{n}(\hat{\beta}_{ir}-\hat{\beta}_{ir;T})^2 = \mathcal{O}_P(T^{-1})$ and

$$|\hat{\lambda}_r - \frac{\hat{l}_r}{n}| = \mathcal{O}_P(T^{-1}+n^{-1}) \qquad (8)$$

ii) If additionally $(Tb^2)^{-1} \to 0$ as $n, T \to \infty$, then

$$|\hat{\gamma}_r(t) - \hat{\gamma}_{r;T}(t)| = \mathcal{O}_P(b^2 + (nTb)^{-1/2} + (Tb^{1/2})^{-1} + n^{-1}) \qquad (9)$$

for all $t \in [0,1]$



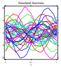
Asymptotic results - Eigenfunctions

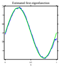
Under assumption 1) and using some further regularity conditions we obtain:

1) $\hat{\gamma}_r(t) - \gamma_r(t) = \sum_{s \neq r} \left\{ \frac{1}{n(\lambda_s - \lambda_r)} \sum_{i=1}^{n} \beta_{si}\beta_{ri} \right\} \gamma_s(t) + R_r(t),$

where $\|R_r\| = \mathcal{O}_p(n^{-1})$ for all $t \in [0,1]$ and

$$\sqrt{n} \sum_{s \neq r} \left\{ \frac{1}{n(\lambda_s - \lambda_r)} \sum_{i=1}^{n} \beta_{si}\beta_{ri} \right\} \gamma_s(t) \xrightarrow{\mathcal{L}} N\left(0,\ \sum_{s \neq r} \frac{\lambda_s \lambda_r}{(\lambda_s - \lambda_r)^2} \gamma_s(t)^2 \right)$$





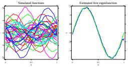
Asymptotic results - Mean and eigenvalues

2) If, furthermore, $\lambda_{r-1} > \lambda_r > \lambda_{r+1}$ for some r then

$$\sqrt{n}(\hat{\lambda}_r - \lambda_r) \xrightarrow{\mathcal{L}} N(0, 2(\lambda_r)^2), \tag{10}$$

3) For all $t \in [0,1]$

$$\sqrt{n}\{\bar{X}(t) - \mu(t)\} = \sum_r \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \beta_{ri} \right\} \gamma_r(t) \xrightarrow{\mathcal{L}} N\left(0, \ \sum_r \lambda_r \gamma_r(t)^2 \right),$$



When is the estimation error negligible ?

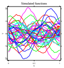
1) T is much larger than n, $n/T^{4/5} \to 0$,
and use "optimal" bandwidth $b \sim T^{-1/5}$.

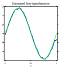
2) T is smaller than n but $n/T^2 \to 0$,
and use "undersmoothing" bandwidth $b \sim (nT)^{-1/5}$.

In both cases 1) and 2) the theorems imply that

$$|\hat{\lambda}_r - \frac{\hat{l}_r}{n}| = \mathcal{O}_p(|\hat{\lambda}_r - \lambda_r|) \text{ and } \|\hat{\gamma}_r - \hat{\gamma}_{r;T}\| = \mathcal{O}_p(\|\hat{\gamma}_r - \gamma_r\|).$$

Inference about FPC will then be first order equivalent to an inference based on known functions X_i.



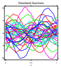


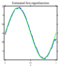
Simulated example

Random linear combinations of two Fourier functions:

$$X_i(t) = \beta_1 \frac{1}{\sqrt{(1/2)}} \sin(2\pi t) + \beta_2 \frac{1}{\sqrt{(1/2)}} \cos(2\pi t) + \varepsilon \qquad (11)$$

$\beta_1 \sim N(0,6)$, $\beta_2 \sim N(0,4)$, $\varepsilon \sim N(0,0.25)$
generated on the grid $U_{ij} \sim U[0,1]$,
$j = 1,\ldots,150$, $i = 1,\ldots,40$.





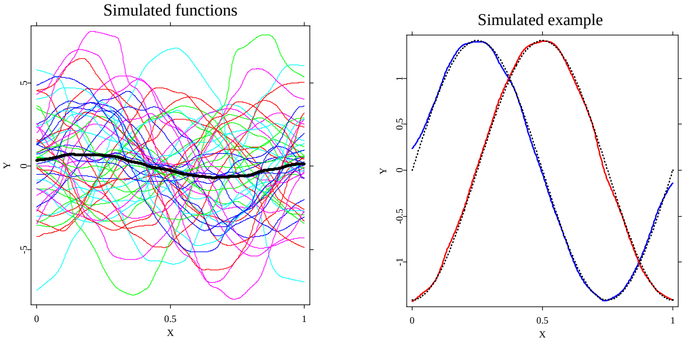
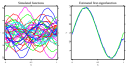


Figure 2: Simulated example, in the left picture the Nadaraya-Watson estimator of simulated functions are plotted (b=0.07) and estimated mean functions (black thick), in the right picture the estimated first (blue) and second (red) eigenfunction, true eigenfunctions: (first blue, second red dashed).



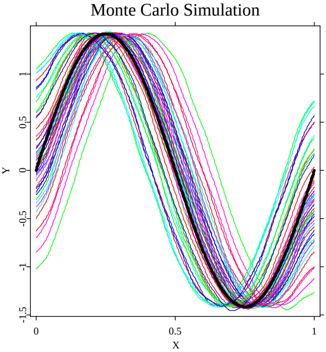
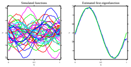


Figure 3: Monte Carlo Simulation, $\lambda_1 = 6$, $\lambda_2 = 4$, $N = 70$, $T = 100$, $\varepsilon \sim N(0, 0.1)$ 50 replications, thin lines are estimated first eigenfunctions, the bold black line is true eigenfunction



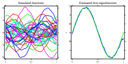
Two sample problem in principal components

We are mainly interested in two sample problems:

$$X_1^{(1)}, X_2^{(1)}, \ldots, X_{n_1}^{(1)} \text{ and } X_1^{(2)}, X_2^{(2)}, \ldots, X_{n_2}^{(2)} \tag{12}$$

Goal: Compare the distribution of functions in both samples by comparing their KL-decompositions:

$$X_i^{(p)} = \mu^{(p)} + \sum_{r=1} \beta_{ir}^{(p)} \gamma_r^{(p)}, \quad p = 1, 2. \tag{13}$$



Testing Common Eigenfunctions Hypothesis

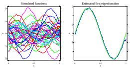
$$H_{0_{2,r}} : \gamma_r^{(1)} = \gamma_r^{(2)}$$

In case where $X_i^{(1)}(t), X_2^{(2)}(t)$ are directly observable we may use following test statistics:

$$D_{2,r} = \| \hat{\gamma}_r^1 - \hat{\gamma}_r^2 \| \,.$$

Reject $H_{0_{2,r}}$ if $D_{2,r} > \Delta_{2,r;1-\alpha}$ where $\Delta_{2,r;1-\alpha}$ is critical value of distribution of:

$$\Delta_{2,r} = ||\hat{\gamma}_{r,N}^{(1)} - \gamma_r^{(1)} - (\hat{\gamma}_{r,N}^{(2)} - \gamma_r^{(2)})||,$$

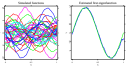


Bootstrap procedure

Distribution of $\Delta_{2,r}$ depends on unknown (true) eigenfunction, but can be approximated by bootstrap distribution of Δ_r^*:

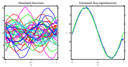
$$\Delta_{2,r}^* = ||\hat{\gamma}_r^{(1)*} - \hat{\gamma}_r^{(1)} - (\hat{\gamma}_r^{(2)*} - \hat{\gamma}_r^{(2)})||.$$

where $\hat{\gamma}_r^{1*}$ and $\hat{\gamma}_r^{2*}$ are estimates to be obtained from independent bootstrap samples $X_1^{(1)*}(t), X_2^{(1)*}(t), \dots, X_{n_1}^{(1)*}(t)$ and $X_1^{(2)*}(t), X_2^{(2)*}(t), \dots, X_{n_2}^{(2)*}(t)$.



Validity of the test

1) Under H_0: we have $\gamma_r^{(1)} = \gamma_r^{(2)}$ and, therefore, $D_{2,r} = \Delta_{2,r} \Rightarrow$ asymptotically correct level $(P(D_{2,r} > \Delta_{2,r;1-\alpha}) \approx \alpha)$.
2) Under H_1, then $D_{2,r} \neq \Delta_{2,r}$. The distribution of $\Delta_{2,r}$ the distribution of $D_{2,r}$ is shifted by the difference $\gamma_r^{(1)} - \gamma_r^{(2)} \Rightarrow$ Power.



Common Eigenspaces Test

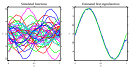
$$H_{0_{4,L}} : \mathcal{E}_L^{(1)} = \mathcal{E}_L^{(2)}$$

$H_{0_{4,L}}$ corresponds to:

$$\sum_{r=1}^{L} \gamma_r^{(1)}(t)\gamma_r^{(1)}(s) = \sum_{r=1}^{L} \gamma_r^{(2)}(t)\gamma_r^{(2)}(s) \quad \text{for all } t, s \in [0,1]$$

Test Statistics:

$$D_{4,L} \stackrel{\text{def}}{=} \int\int \left\{ \sum_{r=1}^{L} \hat{\gamma}_r^{(1)}(t)\hat{\gamma}_r^{(1)}(s) - \sum_{r=1}^{L} \hat{\gamma}_r^{(2)}(t)\hat{\gamma}_r^{(2)}(s) \right\}^2 dtds$$



Using same arguments as for $H_{0_{3,r}}$, the distribution under $H_{0_{4,L}}$ can be approximated by the bootstrap distribution of:

$$\Delta^*_{4,L} \stackrel{\text{def}}{=} \int\int \Bigg[\sum_{r=1}^{L} \{ \hat{\gamma}_r^{(1*)}(t)\hat{\gamma}_r^{(1)*}(s) - \hat{\gamma}_r^{(1)}(t)\hat{\gamma}_r^{(1)}(s) \} - \sum_{r=1}^{L} \{ \hat{\gamma}_r^{(2)*}(t)\hat{\gamma}_r^{(2)*}(s) - \hat{\gamma}_r^{(2)}(t)\hat{\gamma}_r^{(2)}(s) \} \Bigg]^2 dtds$$

Common Mean and Eigenvalues Tests

$$H_{0_1} : \mu^{(1)} = \mu^{(2)} \text{ and } H_{0_{3,r}} : \lambda_r^{(j)} = \lambda_r^{(2)},\ r = 1, 2, \ldots$$

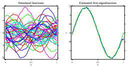
The corresponding test-statistics

$$D_1 \stackrel{\text{def}}{=} \|\hat{\mu}^{(1)} - \hat{\mu}^{(2)}\|^2, \text{ and } D_{3,r} \stackrel{\text{def}}{=} |\hat{\lambda}_r^{(1)} - \hat{\lambda}_r^{(2)}|^2$$

Again, the critical values can be approximated using bootstrap distributions of:

$$\Delta_1^* \stackrel{\text{def}}{=} \|\hat{\mu}^{(1)*} - \hat{\mu}^{(1)} - (\hat{\mu}^{(2)*} - \hat{\mu}^{(2)})\|^2,$$

$$\Delta_{3,r}^* \stackrel{\text{def}}{=} |\hat{\lambda}_r^{(1)*} - \hat{\lambda}_r^{(1)} - (\hat{\lambda}_r^{(2)*} - \hat{\lambda}_r^{(2)})|^2$$



Theoretical results

Assume two independent samples each fulfilling 1) and some further regularity conditions then we obtain for any $\delta > 0$

i) $|P(\Delta_1 \geq \delta) - P(\Delta_1^* \geq \delta | \mathcal{X}_1, \mathcal{X}_2)| = \mathcal{O}_P(1)$

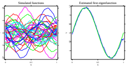
ii) If $\lambda_{r-1}^{(1)} > \lambda_r^{(1)} > \lambda_{r+1}^{(1)}$ and $\lambda_{r-1}^{(2)} > \lambda_r^{(2)} > \lambda_{r+1}^{(2)}$ hold for some $r = 1, 2, \dots,$ then

$$|P(\Delta_{k,r} \geq \delta) - P\left(\Delta_{k,r}^* \geq \delta | \mathcal{X}_1, \mathcal{X}_2\right)| = \mathcal{O}_P(1), \quad k = 2, 3$$

iii) If $\lambda_r^{(1)} > \lambda_{r+1}^{(1)}$ and $\lambda_r^{(2)} > \lambda_{r+1}^{(2)}$ holds for all $r = 1, \dots, L$, then

$$|P(\Delta_{4,L} \geq \delta) - P\left(\Delta_{4,L}^* \geq \delta | \mathcal{X}_1, \mathcal{X}_2\right)| = \mathcal{O}_P(1)$$

as $n \to \infty$.

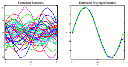


Simulated example – common eigenfunctions

Random linear combinations of Fourier functions on $[0,1]$:

$$
\begin{aligned}
X_i^{(1)}(t) &= \beta_1^{(1)}\phi_{r_1^{(1)}}(t) + \beta_2^{(1)}\phi_{r_2^{(1)}}(t) \\
X_i^{(2)}(t) &= \beta_1^{(2)}\phi_{r_1^{(2)}}(t+\text{shift}) + \beta_2^{(2)}\phi_{r_2^{(2)}}(t+\text{shift})
\end{aligned}
$$

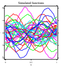
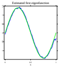
$\beta_1^{(p)} \sim N(0,\lambda_1^{(p)})$, $\beta_2^{(p)} \sim N(0,\lambda_2^{(p)})$
generated on the equidistant grid $t_{ik} = t_k, k = 1, \ldots T$,
$i = 1, \ldots, N$. $T = 100$, $n = 70$



Simulations results

$\lambda_1^{(1)}$	$\lambda_2^{(1)}$	$\lambda_2^{(2)}$	$\lambda_2^{(2)}$	shift 0	shift 0.05	shift 0.1	shift 0.15	shift 0.2	shift 0.25
10	5	8	4	0.13	0.41	0.85	0.96	1	1
4	2	2	1	0.12	0.48	0.87	0.96	1	1
2	1	1.5	2	0.14	0.372	0.704	0.872	0.92	0.9

Table 1: $\alpha = 0.1,\ n = 70,\ T = 100$

Implied volatility modeling - Data description

- German DAX Options IV, Jan 2004-Jun 2004
- Daily prices
- Analysis performed on log returns of $\sigma_t(\kappa, \tau_k)$, $\tau_k =$1M,3M
- $\sigma_t(\kappa, \tau_k)$,τ_k=1M,3M
  interpolated from the IV observable on the day t
  using arbitrage-free interpolation in total variance $\sigma^2(\kappa).\tau$

  For deeper discussion of financial aspects, see [Fengler (2005)]

Factor models for both maturity groups

Construct factor models for $\triangle log \widehat{\sigma}_t(\kappa, 1M)$:

$$\triangle log \widehat{\sigma}_t(\kappa, 1M) = \triangle log \overline{\sigma}_t(\kappa, 1M) + \sum_{j=1}^{K_{1M}} \beta_{tj}^{1M} \widehat{\gamma}_j^{1M}(\kappa), t = 1, \ldots, N_{1M}$$

$$\triangle log \widehat{\sigma}_t(\kappa, 3M) = \triangle log \overline{\sigma}_t(\kappa, 3M) + \sum_{j=1}^{K_{3M}} \beta_{tj}^{3M} \widehat{\gamma}_j^{3M}(\kappa), t = 1, \ldots, N_{3M}.$$

Is **common factor model** (with $\widehat{\gamma}_j^{1M} = \widehat{\gamma}_j^{3M}$) appropriate ?
$\longrightarrow$ Test the equality of the factor functions

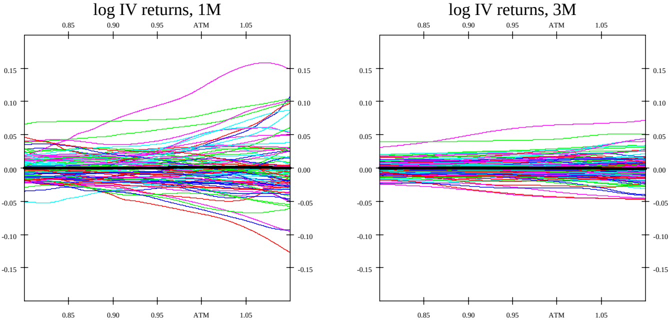


Figure 4: Nadaraya-Watson estimator of the IV-log-returns for maturity $\tau = 0.12$ (1M) in left figure and $\tau = 0.36$ (3M) in right figure. The bold line is the sample mean of corresponding group

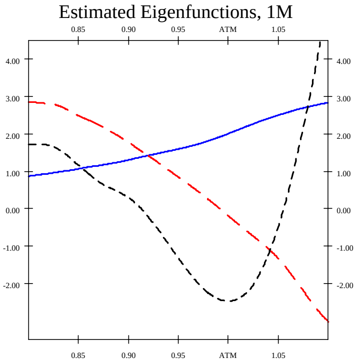
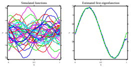


Figure 5: *Estimated eigenfunctions 1M group, blue – first function, red – second function, black – third function*



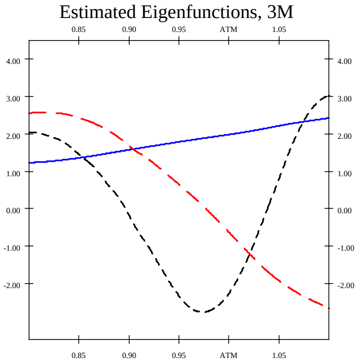
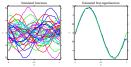


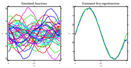
Figure 6: *Estimated eigenfunctions 3M group, blue – first function, red – second function, black – third function*



Variance explained by the eigenfunctions

	var. explained 1M	var. explained 3M
$\hat{\gamma}_1$	89.9%	93.0%
$\hat{\gamma}_2$	7.7%	4.2%
$\hat{\gamma}_3$	1.7%	1.0%
$\hat{\gamma}_4$	0.6%	0.4%

Table 2: Variance explained by the eigenfunctions for the group 1M (first column) and 3M group (second column)



Common Eigenfunctions Test

We tested the equality of eigenfunctions:
using the bootstrap test with $B = 2000$.

test	result
${\gamma_1}^{1M} = {\gamma_1}^{3M}$	rejected
${\gamma_2}^{1M} = {\gamma_2}^{3M}$	not rejected
${\gamma_3}^{1M} = {\gamma_3}^{3M}$	not rejected

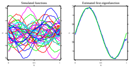
Table 3: Results of Pairwise Common Eigenfunctions Tests, $\alpha = 0.05$, using Bootstrap test with $B = 2000$.

Common Eigenspaces Test

We tested the equality of eigenspaces using the bootstrap test with $B = 2000$.

test	result	P-value
$\mathcal{E}_2^{1M} = \mathcal{E}_2^{3M}$	not rejected	0.61
$\mathcal{E}_3^{1M} = \mathcal{E}_3^{3M}$	not rejected	0.09

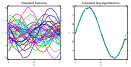
Table 4: Results of Common Eigenspaces Tests, $\alpha = 0.05$, using Bootstrap test with $B = 2000$.



Conclusions – IV analysis

+ We extracted the three factor functions for IV, already known from empirical finance
+ Eigenfunctions have similar shape
- The test rejected the equality the eigenfunction
+ The test doesn't reject the equality of the eigenspaces

OUTLOOK: Analysis of loadings-distribution
Adaptive estimation



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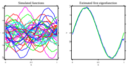

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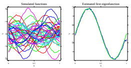

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