Risk Aversion and Pricing Kernel Dynamics

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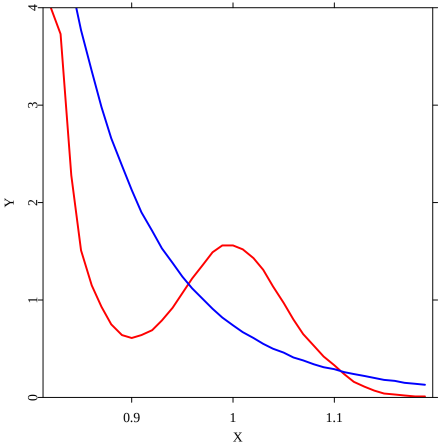


Figure 1: Empirical (red) and theoretical (blue) pricing kernel, DAX 19990205, $\tau = 10$ days.

Price at time t from a payoff $\psi(S_T)$

$$z_t = E\left[\frac{u'(S_T)}{u'(S_t)}\psi(S_T)\right]$$

where

1. S_t is value at time t from wealth, consumption, asset
2. $\psi(S_T)$ is a payoff dependent on S_T
3. $u(x)$ is utility function representing investors preferences

Pricing Kernel

Pricing Kernel (PK) at time t and maturity $\tau = T - t$

$$M_\tau(S_T) = \frac{u'(S_T)}{u'(S_t)}$$

under risk aversion

1. utility function $u(x)$ concave, increasing
2. pricing kernel $M(x)$ monotone decreasing

1. absolute risk aversion (ARA)

$$\alpha(x) = -\frac{u''(x)}{u'(x)}$$

2. relative risk aversion (RRA)

$$\rho(x) = -x\frac{u''(x)}{u'(x)}$$

CRRA / CARA Utility Functions

CARA utility

$$u(x) = -\frac{1}{\alpha} e^{-\alpha x}$$

$\alpha > 0$ is the absolute risk aversion coefficient.

CRRA utility (power utility)

$$u(x) = \frac{x^{1-\gamma} - 1}{1 - \gamma}$$

$\gamma \in (0, 1)$ is the relative risk aversion coefficient.

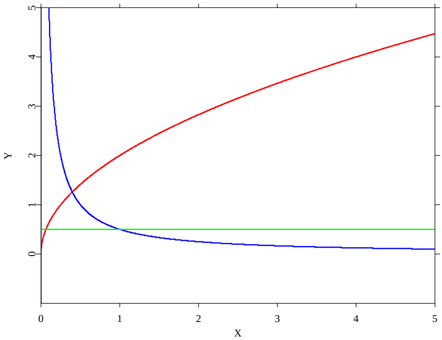


Figure 2: CRRA, $u(x) = \frac{x^\gamma}{\gamma}$ (red), $\alpha(x) = \frac{1-\gamma}{x}$ (blue), $\rho(x) = \gamma$ (green), $\gamma = 0.5$.

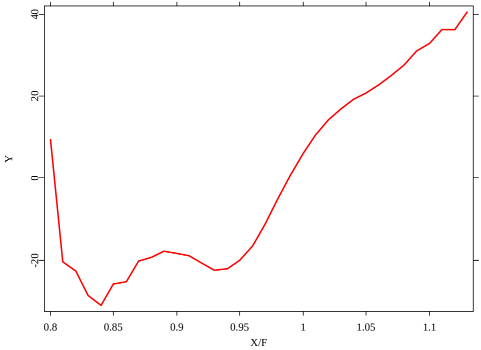


Figure 3: Empirical RRA, DAX 19990205, $\tau = 10$ days.

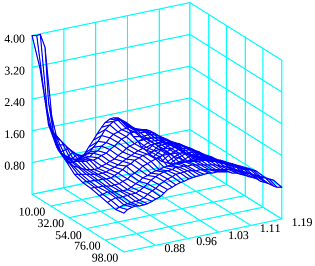


Figure 4: Empirical PK $M_\tau(\kappa)$ and RRA $\rho_\tau(\kappa)$ across maturities τ and future moneyness κ, DAX 19990225.

Empirical pricing kernels (EPK)

1. do not reflect risk aversion across all strikes
2. vary across time to maturity τ and time t

$$M(x) = M_{t,\tau}(x)$$

How to explain pricing kernel and risk aversion dynamics ?

Outline

1. Motivation ✓
2. Risk Measures
3. Pricing Kernels
4. Estimation
5. Empirical Results
6. References

Risk Measures

Preference order $\succeq$ on a set $\mathcal{P}$. For $\mu, \nu \in \mathcal{P}$:

$\mu \succ \nu$: μ is preferred to ν

$\mu \sim \nu$: indifference between μ and ν

$\mu \succeq \nu$: ν is not preferred to μ

Numerical representation $U : \mathcal{P} \rightarrow \mathbb{R}$ of $\succeq$

$$\mu \succeq \nu \Leftrightarrow U(\mu) \geq U(\nu)$$

U is unique up to affine transformations.

Von Neumann-Morgenstern representation

$$U(\mu) = \int u(x)\mu(dx)$$

1. $\mathcal{P}$ set of probability measures on $\mathbb{R}$
2. u elementary utility function.

Mean and Variance

$$m_\mu = \int x\mu(dx)$$

$$\sigma_\mu^2 = \int \{x - m_\mu\}^2 \mu(dx)$$

Example

$$\mu = \delta_a$$

$$m_\mu = \int x\delta_a(dx) = a$$

$$\sigma_\mu^2 = \int (x-a)^2 \delta_a(dx) = 0$$

where δ_a is the Dirac measure on a.

Definition: Preference order $\succeq$ is

1. **risk averse**

$$\mu \neq \delta_{m_\mu} : \delta_{m_\mu} \succ \mu$$

2. **risk proclive**

$$\mu \neq \delta_{m_\mu} : \mu \succ \delta_{m_\mu}$$

3. **risk neutral**

$$\delta_{m_\mu} \sim \mu$$

4. **monotone**

$$a, b \in \mathbb{R}, a > b : \delta_a \succ \delta_b$$

Preference order $\succeq$ on $\mathcal{P}$ with von Neumann-Morgenstern representation

$$U(\mu) = \int u(x)\mu(dx)$$

- $\succeq$ **risk averse** iff u **strictly concave**
- $\succeq$ **risk proclive** iff u **strictly convex**
- $\succeq$ **risk neutral** iff u **linear**
- $\succeq$ **monotone** iff u **strictly increasing**

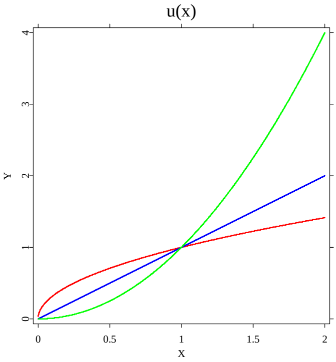


Figure 5: Monotone utility functions: risk averse $u(x) = \sqrt{x}$ (red), risk neutral $u(x) = x$ (blue) and risk proclive $u(x) = x^2$ (green).

Example

1. USD 4 or 9 with probability 0.5:

$$\begin{aligned} \mu &= 0.5(\delta_4 + \delta_9) \\ m_\mu &= 6.5 \end{aligned}$$

2. USD 6.5 with probability 1:

$$\nu = \delta_{m_\mu} = \delta_{6.5}$$

How do μ and ν relate under risk neutrality / aversion / proclivity ?

Linear utility function $u(x) = x$ (risk neutrality)

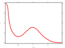
$$
\begin{aligned}
U(\mu) &= \int x\mu(dx) = \frac{4+9}{2} = 6.5 \\
U(\nu) &= \int x\delta_{6.5}(dx) = 6.5
\end{aligned}
$$

Therefore

$$U(\nu) = U(\mu)$$

hence,

$$\delta_{m_\mu} \sim \mu$$



Concave utility function $u(x) = \sqrt{x}$ (risk aversion)

$$
\begin{aligned}
U(\mu) &= \int \sqrt{x}\mu(dx) = \frac{\sqrt{4}+\sqrt{9}}{2} = 2.5 \\
U(\nu) &= \int \sqrt{x}\delta_{6.5}(dx) = \sqrt{6.5} = 2.5495
\end{aligned}
$$

Therefore

$$U(\nu) > U(\mu)$$

hence,

$$\delta_{m_\mu} \succ \mu$$

Convex utility function $u(x) = x^2$ (risk proclivity)

$$
\begin{aligned}
U(\mu) &= \int x^2 \mu(dx) = \frac{4^2 + 9^2}{2} = 48.5 \\
U(\nu) &= \int x^2 \delta_{6.5}(dx) = 6.5^2 = 42.25
\end{aligned}
$$

Therefore

$$U(\nu) < U(\mu)$$

hence,

$$\mu \succ \delta_{m_\mu}$$

Some Definitions

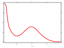
Certainty equivalent c_μ:

$$\delta_{c_\mu} \sim \mu$$

Risk premium π_μ

$$\pi_\mu = m_\mu - c_\mu$$

1. insurance buyer: π_μ maximal price to exchange μ for δ_{m_μ}
2. insurance seller: π_μ minimal price to exchange δ_{m_μ} for μ



Risk aversion: positive risk premium

$$
\begin{aligned}
\delta_{m_\mu} &\succ \mu \sim \delta_{c_\mu} \\
u(m_\mu) &> U(\mu) = u(c_\mu) \\
m_\mu &> c_\mu \\
\pi_\mu &> 0
\end{aligned}
$$

Risk neutrality: zero risk premium
Risk proclivity: negative risk premium

1. $u(x) = \sqrt{x}$ - risk aversion

$$\begin{aligned} U(\mu) &= 2.5 = U(\delta_{c_\mu}) = \sqrt{c_\mu} \\ c_\mu &= 2.5^2 = 6.25 \\ \pi_\mu &= 6.5 - 6.25 = 0.25 \end{aligned}$$

2. $u(x) = x$ - risk neutrality

$$\begin{aligned} U(\mu) &= 6.5 = U(\delta_{c_\mu}) = c_\mu \\ \pi_\mu &= 6.5 - 6.5 = 0 \end{aligned}$$

3. $u(x) = x^2$ - risk proclivity

$$\begin{aligned} U(\mu) &= 48.5 = U(\delta_{c_\mu}) = c_\mu^2 \\ c_\mu &= \sqrt{48.5} = 6.964 \\ \pi_\mu &= 6.5 - 6.964 = -0.4642 \end{aligned}$$

Measuring Risk Aversion

Taylor expansion at $c = c_\mu$ around $m = m_\mu$

$$u(c) \approx u(m) + u'(m)\pi$$

$$\begin{aligned} u(c) &= \int u(x)\mu(dx) \\ &\approx \int \{u(m) + u'(m)(x-m) + \frac{1}{2}u''(m)(x-m)^2\}\mu(dx) \\ &\approx u(m) + \frac{1}{2}u''(m)\sigma_\mu^2 \end{aligned}$$

Thus

$$\pi_\mu \approx -\frac{1}{2}\frac{u''(m)}{u'(m)}\sigma_\mu^2$$

Arrow-Pratt measures of

absolute risk aversion (ARA)

$$\alpha(x) = -\frac{u''(x)}{u'(x)}$$

relative risk aversion (RRA)

$$\rho(x) = -x\frac{u''(x)}{u'(x)}$$

where x is the expected value from a distribution.

Absolute Risk Aversion

- Constant (CARA): $\gamma \in \mathbb{R}$, $\alpha(x) = \gamma$

$$\Rightarrow \exists a, b \in \mathbb{R} : u(x) = a - be^{\gamma x}$$

- Decreasing (DARA): $x_1 > x_2 \Rightarrow \alpha(x_1) < \alpha(x_2)$
- Increasing (IARA): $x_1 > x_2 \Rightarrow \alpha(x_1) > \alpha(x_2)$
- Hyperbolic (HARA): $x \in (0, \infty)$

$$\alpha(x) = \frac{1-\gamma}{x} \Rightarrow u(x) = \begin{cases} \gamma^{-1} x^{\gamma} & , \gamma \in (0,1) \\ \log(x) & , \gamma = 0 \end{cases}$$

Relative Risk Aversion

- Constant (CRRA): $\rho \in \mathbb{R}$, $\rho(x) = \rho$
- Decreasing (DRRA): $x_1 > x_2 \Rightarrow \rho(x_1) < \rho(x_2)$
- Increasing (IRRA): $x_1 > x_2 \Rightarrow \rho(x_1) > \rho(x_2)$
- Special cases:

HARA utilities are CRRA
CARA utilities are IRRA.

Power Utility / HARA / CRRA

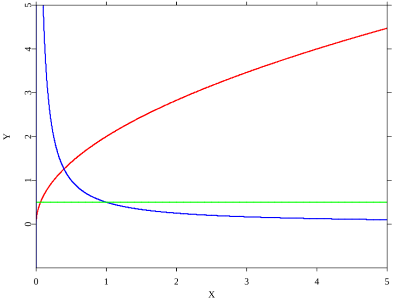
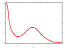


Figure 6: $u(x) = \frac{x^\gamma}{\gamma}$ (red), $\alpha(x) = \frac{1-\gamma}{x}$ HARA (blue), $\rho(x) = \gamma$ CRRA (green), $\gamma = 0.5$.



Quadratic Utility / IARA / IRRA

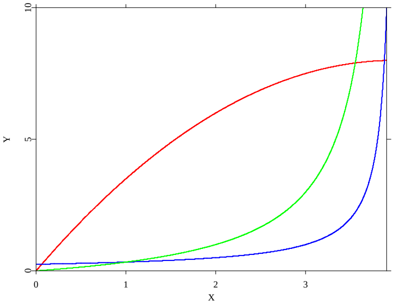


Figure 7: $u(x) = \beta x + \gamma x^2$ (red), $\alpha(x) = \frac{2\gamma}{\beta - 2\gamma}$ for $x \le \frac{\beta}{2\gamma}$, IARA (blue), $\rho(x) = x\frac{2\gamma}{\beta - 2\gamma}$, IRRA (green), $\gamma = -1.9$, $\beta = 4$.

Preference order $\succ^1$ more risk averse than $\succ^2$:

$$\begin{aligned} \pi_1(\mu) &> \pi_2(\mu), \forall \mu \in \mathcal{P} \\ \alpha_1(x) &> \alpha_2(x), \forall x \in I \\ u_1 &= F \circ u_2 \end{aligned}$$

where F is strictly increasing and concave.

Comparing Distributions

μ is *uniformly preferred* over ν

$$\mu \succeq^* \nu$$

if

$$\int u(x)\mu(dx) \geq \int u(x)\nu(dx)$$

for ALL strictly concave and increasing functions u.

Example

$$N(a, \sigma_1) \succeq^* N(b, \sigma_2) \Leftrightarrow a \geq b, \sigma_1 \leq \sigma_2$$

$$N(2,1) \succeq^* N(1,1) \succeq^* N(1,2)$$

$$N(1,1) \succeq^* N(0,1)$$

$$N(1,2) \not\succeq^* N(0,1)$$

Example

$$\log N(a_1, b_1) \succeq^* \log N(a_{,} b_2) \Leftrightarrow \begin{cases} a_1 + \frac{1}{2} b_1^2 \geq a_2 + \frac{1}{2} b_2^2 \\ b_1^2 \leq b_2^2 \end{cases}$$

$$\log N(5, 1) \succeq^* \log N(1.5, 2)$$
$$\log N(1, 1) \not\succeq^* \log N(1, 2)$$

Pricing Kernels

Stock prices follow

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

where

1. B_t is standard Browinan motion
2. $p(S_T)$ is the conditional distribution of S_T given information up to time t

Representative investor with

1. interest rates r,
2. utility function $u(x)$
3. wealth proces $\{W_s\}$
4. stock proces $\{S_s\}$
5. consumption process $\{C_s\}$, $C_s = 0, t < s < T$
6. adjusts amounts $\{q_s\}$ invested in S_s at times $t \leq s \leq T$
7. consumes all wealth at T, $C_T = W_T$

chooses q_s via

Merton Optimization Problem

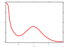
$$\max_{\{q_s, t \le s \le T\}} E[u(W_T)]$$

subject to

$$W_s \ge 0$$

$$dW_s = \{rW_s + q_s(\mu - r)\}ds + q_s \sigma dB_s$$

1. in equilibrium, $t \le s \le T$: $W_s = S_s$
2. at the end consume all wealth, i.e. $C_T = W_T = S_T$



Terminal condition at $s = T$:

$$\frac{u'(W_T)}{u'(W_t)} = e^{-r\tau}\zeta_T$$

where

$$\zeta_s = \exp\left[\int_t^s \left\{\frac{\mu - r}{\sigma}\right\} dB_u - \frac{1}{2}\int_t^s \left\{\frac{\mu - r}{\sigma}\right\}^2 du\right]$$

Defining state-price density as

$$q(S_T) = p(S_T)\zeta_T$$

Pricing Kernel

Price z_t of payoff $\psi(S_T)$

$$z_t = E\left[\frac{u'(S_T)}{u'(S_t)}\psi(S_T)\right]$$

Pricing kernel

$$M_t(S_T) = \frac{u'(S_T)}{u'(S_t)} = e^{-r\tau}\frac{q(S_T)}{p(S_T)}$$

Prices z_t can be written as

$$\begin{aligned} z_t &= E[M_t(S_T)\psi(S_T)] \\ &= \int M_t(S_T)\psi(S_T)p(S_T)dS_T \\ &= e^{-r\tau}\int \psi(S_T)q(S_T)dS_T \\ &= e^{-r\tau}E^*[\psi(S_T)] \end{aligned}$$

Utility Functions and Pricing Kernels

From the pricing kernel

$$M_t(S_T) = \frac{u'(S_T)}{u'(S_t)} = e^{-r\tau} \frac{q(S_T)}{p(S_T)}$$

we obtain the utility funtion

$$u(S_T) = e^{-r\tau} u'(S_t) \int \frac{q(S_T)}{p(S_T)} dS_T$$

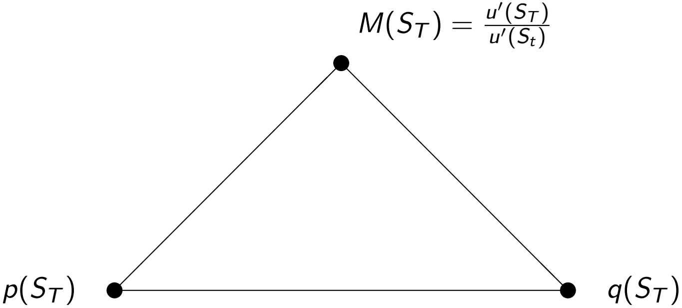
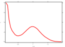


Figure 8: Pricing kernel, utility function, risk neutral and objective measures.

Example

Obtain $u(S_T)$ under Black-Scholes assumptions

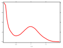
1. under objective measure P : $S_t \sim \log N\{\tau(\mu - \frac{\sigma^2}{2}), \sigma^2\tau\}$
2. under risk neutral measure Q : $S_t \sim \log N\{\tau(r - \frac{\sigma^2}{2}), \sigma^2\tau\}$
3. market price of risk $\lambda = \frac{\mu - r}{\sigma}$
4. $c = \frac{\lambda}{\sigma}$
5. $a = exp\left\{\frac{\lambda(\lambda-\sigma)\tau}{2}\right\}$
6. $b = e^{-r\tau}u'(S_t)$



$$
\begin{aligned}
\frac{q(S_T)}{p(S_T)} &= \left(\frac{S_T}{S_t}\right)^{-\frac{\mu-r}{\sigma^2}} exp\left\{\frac{(\mu-r)(\mu-r-\sigma^2)\tau}{2\sigma^2}\right\} \\
&= \left(\frac{S_T}{S_t}\right)^{-\frac{\lambda}{\sigma}} exp\left\{\frac{\lambda(\lambda-\sigma)\tau}{2}\right\} \\
&= a\left(\frac{S_T}{S_t}\right)^{-c}
\end{aligned}
$$

Hence

$$
u(S_T) = b\int a\left(\frac{S_T}{S_t}\right)^{-c} dS_T
$$



Therefore, up to a constant either we get

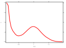
1. HARA / CRRA utility functions:

$$u(S_T) = \begin{cases} \frac{S_T^{1-c}}{1-c} & , \quad c \neq 1 \\ \log(S_T) & , \quad c = 1 \end{cases}$$

2. or risk neutral utility function

$$u(S_T) = kS_T, c = 0$$

where k is a constant.



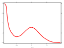
Example

Obtain $q(S_T)$ given power utility and $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$

1. under objective measure $P : S_t \sim \log N\{\tau(\mu - \frac{\sigma^2}{2}), \sigma^2\tau\}$
2. power utility function $u(x) = ka\frac{x^{1-c}}{1-c}$
3. $\lambda = \mu - r$, $c = \frac{\lambda}{\sigma}$
4. $a = exp\left\{\frac{\lambda(\lambda-\sigma)\tau}{2}\right\}$
5. $k = u'(S_t)\frac{e^{-r\tau}}{S_t{}^{-c}}$
6. $\frac{u'(S_T)}{u'(S_t)} = e^{-r\tau} a \left(\frac{S_T}{S_t}\right)^{-c}$

$$
\begin{aligned}
e^{r\tau}\frac{u'(S_T)}{u'(S_t)}p(S_T) &= \frac{a}{S_T\sqrt{\pi}\sigma^2\tau}\left(\frac{S_T}{S_t}\right)^{-\frac{\lambda}{\sigma}} exp\left[-\frac{\left\{\log\left(\frac{S_T}{S_t}\right)-(\mu-\frac{\sigma^2}{2})\right\}^2}{2\sigma^2\tau}\right] \\
&= \frac{a}{S_T\sqrt{\pi}\sigma^2\tau} exp\left[-\frac{\lambda}{\sigma}\log\left(\frac{S_T}{S_t}\right)-\frac{\left\{\log\left(\frac{S_T}{S_t}\right)-(\mu-\frac{\sigma^2}{2})\right\}^2}{2\sigma^2\tau}\right] \\
&= \frac{1}{S_T\sqrt{\pi}\sigma^2\tau} exp\left[-\frac{\left\{\log\left(\frac{S_T}{S_t}\right)-(r-\frac{\sigma^2}{2})\right\}^2}{2\sigma^2\tau}\right] \\
&= q(S_T)
\end{aligned}
$$

Hence under $Q : S_T \sim \log N\{\tau(r-\frac{\sigma^2}{2}), \sigma^2\tau\}$



Relative Risk Aversion

From $p(S_T)$ and $q(S_T)$ we obtain the relative risk aversion

$$\rho(S_T) = -S_T \frac{u''(S_T)}{u'(S_T)} = S_T \left\{ \frac{p'(S_T)}{p(S_T)} - \frac{q'(S_T)}{q(S_T)} \right\}$$

Black Scholes

Asset S_t, call option $\psi(S_T) = (S_T - K)^+$, time to maturity $\tau = T - t$, interest rate r_t, $q(S_T)$ risk neutral density:

$$\begin{aligned} C(S_t, K, \tau, r_t, \sigma_t) &= e^{-r\tau} \int_0^\infty \psi(S_T) q(S_T) dS_T \\ &= S_t \Phi(d_1) - K e^{-r\tau} \Phi(d_2) \end{aligned}$$

where

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

and

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

The risk-neutral density may be obtained as

$$
\begin{aligned}
e^{r\tau} \left| \frac{\partial^2 C}{\partial K^2} \right|_{K=S_T} &= \frac{1}{S_T \sqrt{\pi} \sigma^2 \tau} exp \left[- \frac{\left\{ \log\left(\frac{S_T}{S_t} \right) - (r - \frac{\sigma^2}{2}) \right\}^2}{2\sigma^2 \tau} \right] \\
&= q(S_T)
\end{aligned}
$$

Estimation

The following function are estimated by

1. implied volatility

$$\hat{\sigma}_t(S,\tau) = \frac{\sum_{i=1}^{n} K_S\left(\frac{S-S_i}{h_S}\right) K_\tau\left(\frac{\tau-\tau_i}{h_\tau}\right)\hat{\sigma}_i}{\sum_{i=1}^{n} K_S\left(\frac{S-S_i}{h_S}\right) K_\tau\left(\frac{\tau-\tau_i}{h_\tau}\right)}$$

2. call prices

$$\hat{C}(S_t, K, \tau, r_t, \hat{\sigma}_t)$$

3. state-price density

$$\hat{q}_{t,\tau}(S_T) = e^{r\tau}\left|\frac{\partial \hat{C}^2(S_t, K, \tau, r_t, \hat{\sigma}_t)}{\partial^2 K}\right|_{K=S_T}$$

1. objective distribution $\hat{p}_{t,\tau}(S_T)$

$$\text{GARCH } (1,1) \text{ or } \mu dt + \sigma dW_t$$

2. pricing kernel

$$\hat{M}(S_T) = e^{-r\tau} \frac{\hat{q}(S_T)}{\hat{p}(S_T)}$$

3. relative risk aversion

$$\hat{\rho}_{t,\tau}(S_T) = S_T \left\{ \frac{\hat{p}'(S_T)}{\hat{p}(S_T)} - \frac{\hat{q}'(S_T)}{\hat{q}(S_T)} \right\}$$

Empirical Results

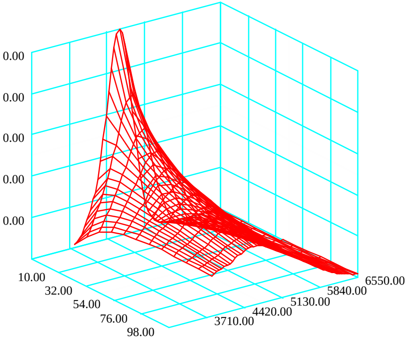
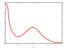


Figure 9: Implied distribution $\hat{q}_{t,\tau}(S_T)$ (red) for different strikes and maturities at date $t = 19990303$, $S_t = 4746$.



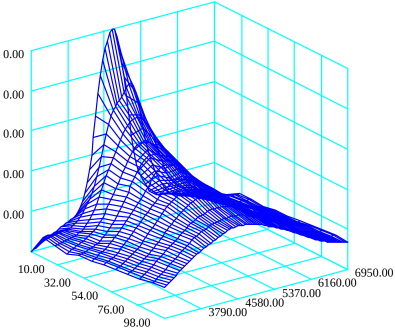
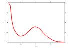


Figure 10: Subjective distribution $\hat{p}_{t,\tau}(S_T)$ (blue) for different strikes and maturities at date $t = 19990303$, $S_t = 4746$.



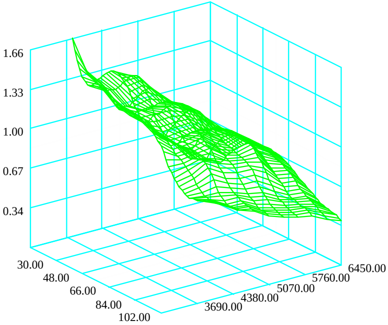


Figure 11: Pricing kernel $\hat{m}_{t,\tau}(S_T)$ (green) for different strikes and maturities at date $t = 19990303$, $S_t = 4746$.

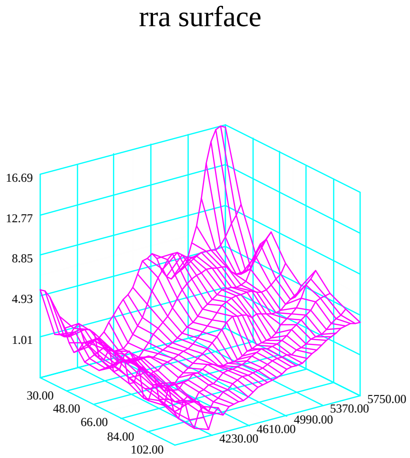


Figure 12: Implied relative risk aversion $\hat{\rho}_{t,\tau}(S_T)$ (magenta) for different strikes and maturities at date $t = 19990303$, $S_t = 4746$.

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