Estimating Pricing Kernel via Series Methods

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The Financial Market

In an arbitrage-free market, the European call price is given by

$$C_t(X) = e^{-r_{t,\tau}\tau} \int_0^\infty (S_T - X)^+ q_t(S_T)\, dS_T. \qquad (1)$$

- S_t the underlying asset price at time t
- X the strike price
- τ the time to maturity
- $T = t + \tau$ the expiration date
- $r_{t,\tau}$ deterministic risk free interest rate
- $q_t(S_T)$ RND of S_T conditional on information at t

The Financial Market

Under the subjective measure P_t, s.t. $dP_t(S_T) = p_t(S_T)dS_T$

$$\begin{aligned} C_t(X) &= e^{-r_{t,\tau}\tau} \int_0^\infty (S_T - X)^+ \frac{q_t(S_T)}{p_t(S_T)} p_t(S_T)\, dS_T \qquad (2) \\ &= e^{-r_{t,\tau}\tau} \int_0^\infty (S_T - X)^+ m_t(S_T) p_t(S_T)\, dS_T \end{aligned}$$

- $m_t(S_T)$ pricing kernel at time t for discounting payoffs occuring at T

Empirical Pricing Kernel (EPK)

- estimation of pricing kernel crucially depends on underlying distributional assumptions and data generating process of the underlying asset
- in practice, parametric stock price specifications do not hold (e.g. GBM in Black-Scholes model, Heston model)
- employ nonparametric methods

Empirical Pricing Kernel (1)

- estimate p and q separately, see e.g. Aït-Sahalia and Lo (2000), Brown and Jackwerth (2004), Grith et al. (2009)

$$\hat{m}_t(S_T) = \frac{\hat{q}_t(S_T)}{\hat{p}_t(S_T)} \tag{3}$$

  - q nonparametric second derivative of C w.r.t. X based on intraday option prices (kernel, local polynomial, splines, basis expansion)
  - p simpler methods (e.g. kernel density estimator) based on daily stock prices

Empirical Pricing Kernel (2)

- one step estimation of m in Engle and Rosenberg (2002) by series expansion

$$m_t(S_T) \approx \sum_{l=1}^{L} \alpha_{tl} g_l(S_T). \tag{4}$$

where $\{g_l\}_{l=1}^{L}$ are known basis functions and $\alpha_t = (\alpha_{t1}, \cdots, \alpha_{tL})^\top$ are time varying coefficients vectors

- better interpretability of the curves in dynamics

Outline

The Model (1)

For the i.i.d. call/strike data $\{Y_i, X_i\}_{i=1}^{n}$ observed at time t consider a nonparametric regression model

$$Y_i = C(X_i) + \varepsilon_i, \quad \mathsf{E}[\varepsilon_i | X_i] = 0 \tag{5}$$

where we assume that the call price $C(X) : \mathbb{R} \rightarrow \mathbb{R}$ is a function in X only given by (2)

$$C(X) = e^{-r_{t,\tau}\tau} \int_0^\infty (S_T - X)^+ m(S_T) p_t(S_T)\, dS_T$$

We are interested in estimating the function $m(S_T) : \mathbb{R} \rightarrow \mathbb{R}$

The Model (2)

Rewrite (5) using the series approximation for m in (4)

$$Y_i = \tilde{C}(X_i) + u_i,$$

where $u_i = (C(X_i) - \tilde{C}(X_i)) + \varepsilon_i$ and

$$\begin{aligned} \tilde{C}(X) &= e^{-r_{t,\tau}\tau} \int_0^\infty (S_T - X)^+ \sum_{l=1}^{L} \alpha_l g_l(S_T) p_t(S_T) dS_T \qquad (6) \\ &= \sum_{l=1}^{L} \alpha_l \left\{ e^{-r_{t,\tau}\tau} \int_0^\infty (S_T - X)^+ g_l(S_T) p_t(S_T) dS_T \right\} \end{aligned}$$

Estimation of α

For known basis functions and fixed L, the vector $\alpha = (\alpha_1, ..., \alpha_L)^\top$ is estimated using the following linear least square criteria

$$\hat{\alpha} = \arg\min_{\alpha} \sum_{i=1}^{n} \left\{ Y_i - \tilde{C}(X_i) \right\}^2 \qquad (7)$$

In (7), define

$$\psi_{il} = e^{-r\tau} \int_0^\infty (S_T - X_i)^+ g_l(S_T) p_t(S_T) dS_T. \qquad (8)$$

s.t. $\tilde{C}_i(X) = \sum_{l=1}^{L} \alpha_l \psi_{il}$.

Estimation of α, $\tilde{C}$ and m

Then

$$\hat{\alpha} = (\Psi^\top \Psi)^- \Psi^\top Y, \tag{9}$$

where $\Psi_{(n \times L)} = (\psi_{il})$ and $Y = (Y_1, \cdots, Y_n)^\top$,

$$\hat{\tilde{C}}(X) = \psi^L(X)^\top \hat{\alpha}, \tag{10}$$

where $\psi^L(X) = (\psi_1(X), \ldots, \psi_L(X))^\top$ and

$$\hat{m}(S_T) = g^L(S_T)^\top \hat{\alpha}, \tag{11}$$

where $g^L(S_T) = (g_1(S_T), \ldots, g_L(S_T))^\top$.

Simulation of S_T

In practice the integral in (8) is replaced by the sum

$$J^{-1}\sum_{s=1}^{J}(S_T^s - X_i)^+ g_l(S_T^s). \tag{12}$$

where $\{S_T^s\}_{s=1}^J$ is a simulated sample from the historical returns

$$r_{t-s,\tau} = log(S_{t-s}/S_{t-(s+1)}).$$

so that

$$S_T^s = S_t e^{r_{t-s,\tau}}.$$

Choice of the Tuning Parameter L (1)

Optimal selection L: the resulting MISE equals the smallest possible integrated square error Li and Racine (2007)

- Mallows's C_L (or C_p), Mallows (1973)

$$C_L = n^{-1} \sum_{i=1}^{n} \left\{ Y_i - \hat{\hat{C}}(X_i) \right\}^2 + 2\sigma^2 (L/n)$$

where σ^2 is the variance of u_i. One can estimate σ^2 by

$$\hat{\sigma}^2 = n^{-1} \sum_{i=1}^{n} \hat{u}_i^2, \quad \text{with} \quad \hat{u}_i = Y_i - \hat{\hat{C}}(X_i).$$

Choice of the Tuning Parameter L (2)

- Generalized cross-validation, Craven and Wahba (1979)

$$CV_L^G = \frac{n^{-1}\sum_{i=1}^{n}\left\{Y_i - \hat{\tilde{C}}(X_i)\right\}^2}{\{1-(L/n)\}^2}.$$

- Leave one out cross-validation, Stone (1974)

$$CV_L = \sum_{i=1}^{n}\left\{Y_i - \hat{\tilde{C}}_{-i}(X_i)\right\}^2$$

where $\hat{\tilde{C}}_{-i}(X_i)$ is the leave one estimate of $\tilde{C}(X_i)$ obtained by removing (X_i, Y_i) from the sample.

Assumptions: Newey (1997)

Assumption 1. $\{X_i, Y_i\}$ is i.i.d. as (X, Y), $var(Y|x)$ is bounded on S, the compact support of X.

Assumption 2. For every L there is a nonsinguar matrix of constants B such that, for $G^L(S_T) = g^L(S_T)$.

(i) The smallest eigenvalue of $\mathrm{E}[G^L(S_T^s)G^L(S_T^s)^\top]$ is bounded away from zero uniformly in L.

Assumptions: Newey (1997)

(ii) There exists a sequence of constants $\xi_0(L)$ that satisfy the condition $sup_{x \in S} ||G^L(S_T)|| \leq \xi_0(L)$, where $L = L(n)$ such that $\xi_0(L)^2/n \longrightarrow 0$ as $n \longrightarrow \infty$.

(iii) As $n \longrightarrow \infty$, $L \longrightarrow \infty$ and $L/n \longrightarrow 0$

Assumption 3. There exists $\theta > 0$, such that

$$\sup_{\theta} |m(S_T) - g^L(S_T)^\top \alpha| = \mathcal{O}(L^{-\theta}) \quad \text{as} \quad L \longrightarrow \infty. \qquad (13)$$

Convergence of $\hat{m}$

Under Assumptions 1, 2 and 3 one can show that

$$
\begin{aligned}
&\int_0^\infty \{\hat{m}(S_T) - m(S_T)\}^2 dP(S_T) = \\
&\quad = \int_0^\infty \{g^L(S_T)^\top(\hat{\alpha} - \alpha) + g^L(S_T)^\top\alpha - m(S_T)\}^2 dP(S_T) \\
&\quad \le ||\hat{\alpha} - \alpha||^2 + \int_0^\infty \{g^L(S_T)^\top\alpha - m(S_T)\}^2 dP(S_T) \\
&\quad = \mathcal{O}_p(L/n + L^{-2\theta}) + \mathcal{O}(L^{-2\theta}) = \mathcal{O}_p(L/n + L^{-2\theta}).
\end{aligned}
$$

Data

- **Source**: Reseach Data Center (RDC)
  http://sfb649.wiwi.hu-berlin.de

- Datastream DAX 30 Price Index;
  5000 overlapping monthly returns

- EUREX European Option Data; tick observations;
  Cross-sectional data: 20040121

Series Functions

- Use Laguerre polynomials, with the first two polynomials

$$g_1(x) = 1$$
$$g_2(x) = 1 - x$$

- Recurrence relation for $l = 2, \cdots, L$

$$g_{l+1}(x) = \frac{1}{l}\left((2l - 1 - x)g_{l-1}(x) - (l - 1)g_{l-2}(x)\right).$$

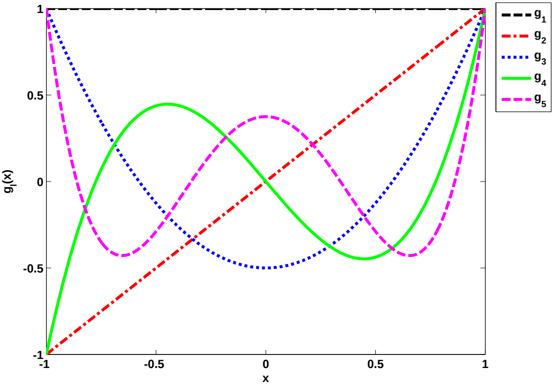
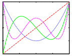


Figure 1: First five terms of the Laguerre polynomials



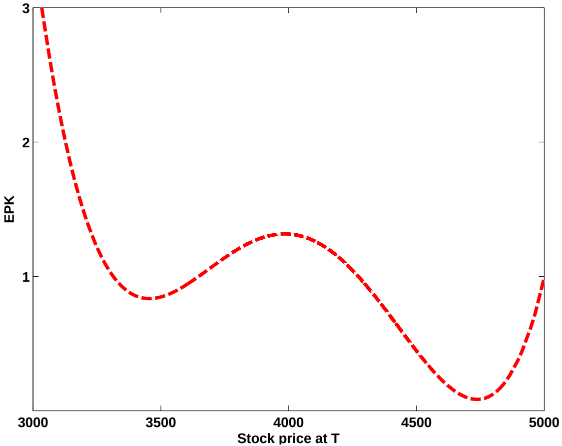
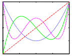


Figure 2: EPK on 20040121 using Laguerre polynomials as basis functions and $L = 5$ given by CV; $S_t = 4133$



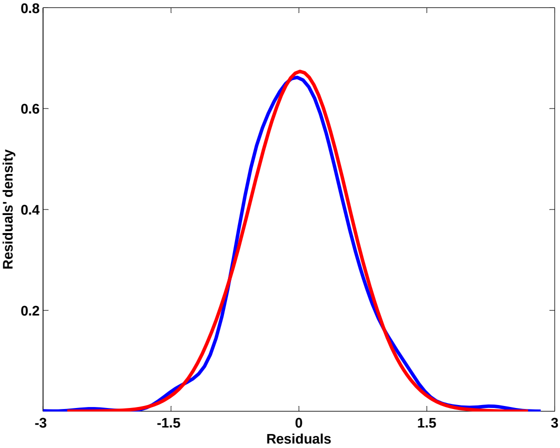


Figure 3: Kernel density estimators for the residuals of the fitted call curves (h by plug-in method) (blue curve) against a simulated normal density (red curve)

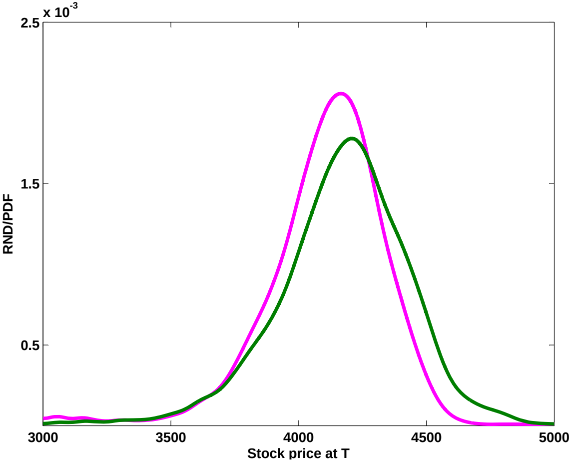


Figure 4: Simulated PDF for S_T (green curve) and estimated RND (magenta curve)

Further Improvements

- individual estimation of PK curves underutilize the available information
- FDA methods that assume a common curve structure seem to perform better
  - e.g. FPCA, DSFM
- a structural model for the coefficients can improve estimation
  - functional time series

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