Pricing Chinese Rain

a Multi-Site, Multi-Period Equilibrium Model

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No rain, no grain...

Weather risks

- source of uncertainty in crop production,
- variability of rain, temperature, snowfall, etc.
- Weather derivatives (WDs) are financial instruments that permit trade and insurance of weather risks,
- crop insurance issuer can transfer weather risks to financial markets via WDs,
- aim: make crop insurance affordable for farmers (China).

Rain does not fall on one roof alone...

- Agriculture, ✓
- tourism, entertainment, hydropower generation...
- diversification of financial portfolio.

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Baskets of WDs

- complicated structure of weather exposure:
  - multiple dependent sites,
  - multiple dependent underlyings: temperature and rainfall.
- Need baskets of WDs to cover weather risks,
- have to account for the underlying spatial dependency.

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Importance of Millimeters

Figure 1: Main agricultural areas in China (highlighted). Rice and wheat cultivation is important in Hunan and Hubei.

Rainfall Derivatives

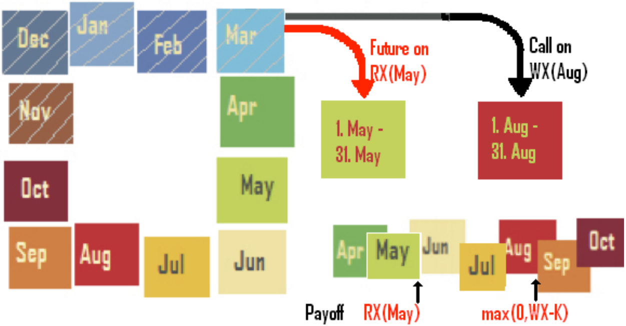
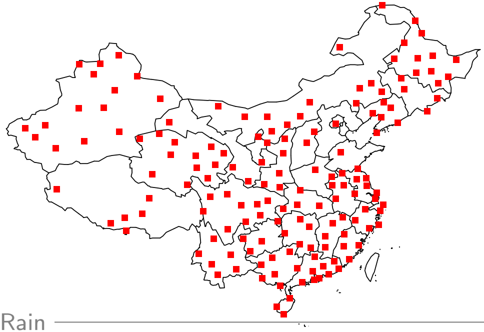


Figure 2: Future on cumulated rainfall (RX) in May and call on wet-day-index (WX) in August.

Rainfall Data

- Daily rainfall data in 0.1mm,
- 29 Provinces, 156 stations in China ,
- from 19510101 to 20091130,
- data available in RDC.



Measuring Rain...

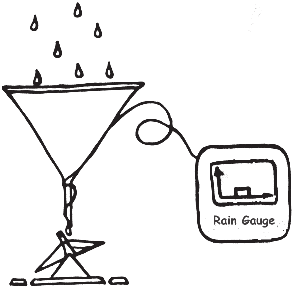
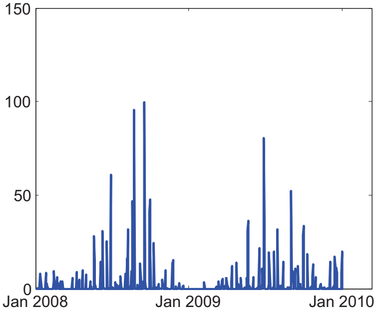


Figure 3: Rain gauge: defines "rain" as precipitation amount ≥0.1mm.



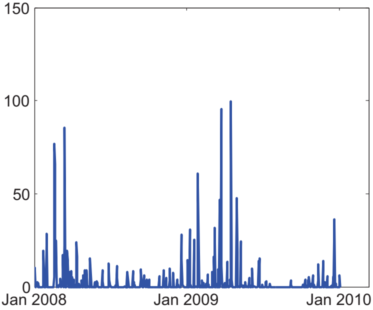


Figure 4: Daily precipitation amount in 0.1 mm for Enshi 恩施市 (left) and Yichang 宜昌市 (right), province of Hubei 湖北.

Pricing Chinese rain

- development of appropriate pricing approach
  - customized WDs,
- statistical modelling of relevant weather indices
  - account for spatial dependence,
- quantification of the relationship between weather and production (income).

Aim: reduce income uncertainty of farmers due to weather conditions ▸ income boxplots

Outline

1. Motivation ✓
2. Pricing Model
3. Statistical model for rainfall
4. Income-rainfall relationship
5. Market scenarios
6. Outlook

- Set of geographical sites $\mathcal{S}$,
- planing periods $t = 0, 1, \ldots, T$,
- set of agents J contains buyers (crop insurance), investor.

Portfolios: $\alpha_{j,t} = (\alpha_{j,t,s_1}, \ldots, \alpha_{j,t,s_n})^\top$, $s_i \in \mathcal{S}$, $i \leq n$ shares of **WDs** and $\beta_{j,t}$ shares of **risk free assets** B_t.

Price of the sth WD $W_{t,s}$, $s \in \mathcal{S}$, at $t = 0, \ldots, T$

Portfolio value of agent j at t: $\alpha_{j,t} W_t + \beta_{j,t} B_t$.

Example

- Farmer with income I_1 correlated to the cumulative rainfall in May in station s $(R_{\text{May},s})_{s\in\mathcal{S}_1}$,
- $\mathcal{S}_1$ set of neighbor stations $\subset \mathcal{S} = \{a, b, c\}$,
- correlations $\rho_a, \rho_b \neq 0$,
- hedge by holding a portfolio of WDs with payoff $\alpha_{1,a} R_{\text{May},a} + \alpha_{1,b} R_{\text{May},b}$.

Agents on the Market

Buyer(Crop insurer) j

- rainfall exposed income I_j
- portfolio: WDs + Bond B_t
- exponential utility with risk aversion a_j

Investor m

- specializes on issue of WDs
- portfolio: WDs + Bond B_t
- exponential utility with risk aversion a_m

Buyer's optimization problem

- Profit of Buyer j

$$\Pi_{j,T} = \underbrace{I_j\{(W_{T,s})_{s\in\mathcal{S}_j}, P_{j,T}\}}_{\text{weather dependent income}} + \underbrace{\overbrace{\sum_{s\in\mathcal{S}_j} \alpha_{j,T,s} W_{T,s}}^{\text{payoffs of: WDs}} + \overbrace{\beta_{j,T} B_T}^{\text{risk-free asset}}}_{\text{portfolio payoff } V_{j,T} \text{ with WDs}}$$

  $P_{j,T}$ production price, $(W_{T,s})_{s\in\mathcal{S}_j}$ weather events in sites $\mathcal{S}_j$.
- Utility maximization

$$\max_{\{\alpha_{j,t+1,s}\}_{s\in\mathcal{S}_j}} \mathsf{E}_t\left\{U_j\left(\Pi_{j,T}\right)\right\}$$

$$\text{s.t.} \sum_{s\in\mathcal{S}_j} \alpha_{j,t+1,s} W_{t,s} + \beta_{j,t+1} B_t - V_{j,t,s} = 0.$$

Investor's optimization problem

- Profit of investor m

$$\Pi_{m,T} = \underbrace{-\sum_{s\in\mathcal{S}} \alpha_{m,T,s} W_{T,s} + \beta_{m,T} B_T}_{\text{portfolio payoff } V_{m,T} \text{ with WDs and a risk-free asset}}$$

with $\mathcal{S}$ set of all traded stations.

- Utility maximization

$$\max_{\{\alpha_{m,t+1,s}\}_{s\in\mathcal{S}}} \mathsf{E}_t \{U_m(\Pi_{m,T})\}$$

$$\text{s.t. } \sum_{s\in\mathcal{S}} \alpha_{m,t+1,s} W_{t,s} - \beta_{m,t+1} B_t + V_{m,t} = 0.$$

Solution via dynamic programming

time	state variables	control variable
0	$(W_{0,s})_{s\in S}, (V_{0,k})_{k=j,m}$	$(\alpha_{1,j,s})_{s\in\mathcal{S}_j}, (\alpha_{1,m,s})_{s\in\mathcal{S}}$
...		
T-1	$(W_{T-1,s})_{s\in\mathcal{S}}, (V_{T-1,k})_{k=j,m}$	$(\alpha_{T,j,s})_{s\in\mathcal{S}_j}, (\alpha_{T,m,s})_{s\in\mathcal{S}}$
T	$(W_{T,s})_{s\in\mathcal{S}}, \{I_j(W_{T,s}, P_T)\}_{s\in\mathcal{S}_j}$	–

- start in $T-1$ and maximize the expected utility of T choosing $(\alpha_{k,T,s})_{s\in\mathcal{S},\ k=j,m}$ given info in $T-1$,
- under utility indifference derive demand/supply functions for $T-1$,
- move to the previous period and the maximize the corresponding expectation, continue to the present period.

Buyer's Demand

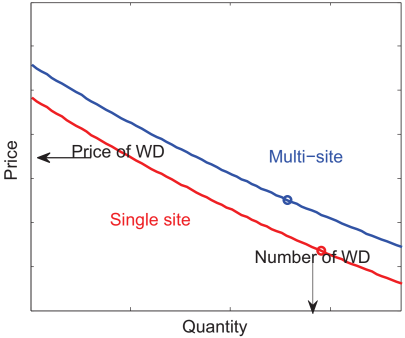


Figure 5: Solution to utility maximization: demand of a weather dependent buyer as a monotonic relationship bw price and quantity of WD.

▸ formal solution

Investor's Supply

Figure 6: Solution to utility maximization: supply of the investor as a monotonic relationship bw price and quantity of WD. ▸ formal solution

Single site vs multi-site

- investor: + (-) **dependencies** in underlying weather risks then ↓ (↑) **supply** due to higher (lower) risks she bears.
- buyer: ↑↓ **demand** depending on ▸ condition: magnitude of the covariance matters. This condition can be checked for a concrete application.

▸ back to results

Market Clearance

$$\sum_{j\in\mathcal{J}} \alpha^*_{j,t,s} = \alpha^*_{m,t,s}, \quad \text{for} \quad 0 \leq t \leq T$$

- equilibrium prices $(W^*_{t,s})_{s\in\mathcal{S}}$,
- equilibrium quantities $(\alpha^*_{k,t,s})_{s\in\mathcal{S}}$,
- these clear the market for agents $k \in \mathcal{J}$, and a set of stations $\mathcal{S}$.

A multi-site rainfall model

Wilks (1998)
Rainfall amount $R_{s,t}$ at time t in station s:

$$R_{s,t} = r_{s,t} X_{s,t}, \tag{1}$$

- $X_{s,t}$ rainfall occurrence at t in s

$$X_{s,t} = \begin{cases} 1 \text{ (wet, } \geq \ X_{s,min}), \\ 0 \text{ (dry, } < \ X_{s,min}), \end{cases}$$

  and $X_{s,min} > 0$ is a threshold defining the state "wet".
- $r_{s,t}$ is positive rainfall amount.

Spatial dependence of $\{X_{s,t}\}_{s\in\mathcal{S},t=1,\ldots,T}$

Threshold probability

$$p_{crit,s,t} = \begin{cases} p_{01,s,t} \text{ if } X_{s,t-1} = 0, \\ p_{11,s,t} \text{ if } X_{s,t-1} = 1, \end{cases},$$

where

$$p_{01,s,t} = P(X_{s,t} = 1 | X_{s,t-1} = 0),$$

$$p_{11,s,t} = P(X_{s,t} = 1 | X_{s,t-1} = 1).$$

Spatial dependence of $\{X_{s,t}\}_{s\in\mathcal{S},t=1,\dots,T}$

$X_{s,t}$ generated as

$$X_{s,t} = \begin{cases} 1 \text{ if } w_{s,t} \leq \Phi^{-1}(p_{crit,s,t}), \\ 0 \text{ if } w_{s,t} > \Phi^{-1}(p_{crit,s,t}), \end{cases}$$

$\Phi(\cdot)$ cdf of N(0,1), $\{w_{s,t}\}_{s\in\mathcal{S}} \sim \mathrm{N}(0_{|\mathcal{S}|}, \Sigma)$, with $\Sigma_{s,s'} = \mathrm{Corr}(w_{s,t}, w_{s',t})$ such that the empirical correlations $\mathrm{Corr}(X_{s,t}, X_{s',t})$ of the rainfall occurrences are mimicked in the generated rainfall occurrence series. continue to 3.9

Spatial dependence of $\{r_{s,t}\}_{s\in\mathcal{S},t=1,\dots,T}$

The multi-site rainfall amount $r_{s,t}|X_{s,t}=1$ follows a mixture of two exponential distributions with mixing parameter $\gamma_{s,t}$ and means $\beta_{1,s,t},\beta_{2,s,t}$ with pdf

$$
\begin{aligned}
& f_t(r_{s,t}=r|X_{s,t}=1,\beta_{1,s,t},\beta_{2,s,t},\gamma_{s,t}) \\
&\quad = \frac{\gamma_{s,t}}{\underset{\substack{\uparrow \\ \text{greater mean}}}{\beta_{1,s,t}}}\exp\left(-r/\beta_{1,s,t}\right)+\frac{(1-\gamma_{s,t})}{\underset{\substack{\uparrow \\ \text{smaller mean}}}{\beta_{2,s,t}}}\exp\left(-r/\beta_{2,s,t}\right)
\end{aligned}
$$

continue to 3.10

Spatial dependence of $\{r_{s,t}\}_{s\in\mathcal{S},t=1,\ldots,T}$

Rainfall amount is generated as

$$r_{s,t} = r_{min} - \beta_{s,t} \log \Phi(v_{s,t}) \tag{2}$$

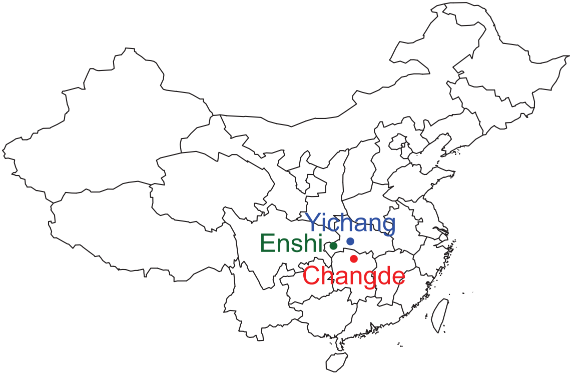
where r_{min} is the minimum measured rainfall amount and

$$\beta_{s,t} = \begin{cases} \beta_{1,s,t} \text{ if } \Phi(w_{s,t})/p_{s,crit} \leq \alpha_{s,t}, \\ \beta_{2,s,t} \text{ if } \Phi(w_{s,t})/p_{s,crit} > \alpha_{s,t}, \end{cases} , \tag{3}$$

and $v_{s,t}$ are normal covariates correlated such that the generated rainfall time series mimic sample correlations in the rainfall data.

▸ continue to 3.11

Stations



continue to simulation

Empirical rainfall I

Test the order of Markov chain using BIC (Katz, 1981):

Order/BIC	Changde	Enshi	Yichang
0	70.83	60.02	19.86
1	53.21	43.21	4.531
2	53.47	44.69	9.032
3	65.64	59.72	33.38

Table 1: BIC criterion for different orders of Markov chain for rainfall occurrences justifies 1st order Markov model ▸ BIC criterion.

Empirical rainfall II

Parameter	Changde	Enshi	Yichang
$\hat{p}_{01,\cdot,t\in\text{May}}$	0.38	0.27	0.17
$\hat{p}_{11,\cdot,t\in\text{May}}$	0.60	0.53	0.65

Table 2: Transitional probabilities to wet states for rainfall occurrences in May.

Empirical rainfall III

The estimated correlations of wet day occurrences in May ($X_{\text{min}} =$ 0.1 mm) $\widehat{\text{Corr}}(X_{s,t}, X_{s',t})$ (black) and $\text{Corr}(w_{\cdot,t}, w_{s',t})$ (red)

what is $w_{s',t}$:

	Changde	Enshi	Yichang
Changde	-	0.42 0.65	-0.01 0
Enshi	-	-	-0.04 0
Yichang	-	-	-

Table 3: Parameters for the generation of the rainfall occurrences in May.

Empirical rainfall IV

multi-site rainfall amount

Parameter	Changde	Enshi	Yichang
$\gamma_{\cdot,t\in\text{May}}$	0.73	0.60	0.67
$\beta_{1,\cdot,t\in\text{May}}$	16.02	13.84	8.99
$\beta_{2,\cdot,t\in\text{May}}$	0.73	0.85	0.90

Table 4: Estimated parameters of the mixture of exponential distributions.

Empirical rainfall V

The estimated rainfall amount correlations $\widehat{\text{Corr}}(R_{s,t}, R_{s',t})$ (black) and $\text{Corr}(v_{\cdot,t}, v_{s',t})$ (red) what is $v_{s',t}$:

	Changde	Enshi	Yichang
Changde	-	0.26 0.31	-0.01 0
Enshi	-	-	-0.02 0
Yichang	-	-	-

Table 5: Parameters for the generation of the rainfall amounts in May.

Simulated spatial rain patterns

Income-Rainfall Relationship

Indices: cumulative rainfall (RX) and wet day index (WX).

- $RX_{\tau_1,\tau_2,s} = \sum_{t=\tau_1}^{\tau_2} R_{ts}$ total rainfall in $[\tau_1, \tau_2]$.
  - important for planting and nutrition season
  - positive correlation with crop volumes
  - → price RX futures for May
- $WX_{\tau_1,\tau_2,s} = \sum_{t=\tau_1}^{\tau_2} X_{ts}$ number of wet days over $[\tau_1, \tau_2]$
  - important for harvesting, excess rainfall damage
  - crop volume distribution is better if $WX_{\tau_1,\tau_2,s\in\mathcal{S}_j} < WX_{crit}$
  - → price call options on WX futures for August with WX_{crit}=5 mm and K= 5 days.

Income-Rainfall Relationship

- WX: $\forall j \in \mathcal{J}$ ▸ go to simulation

$$I_j = \begin{cases} \mathrm{N}(\mu^+, \sigma^+), & \text{if } \forall\ s\ WX_{\tau_1,\tau_2,s\in\mathcal{S}_j} < WX_{crit}, \\ \mathrm{N}(\mu^0, \sigma^0), & \text{if } \exists\ s\ WX_{\tau_1,\tau_2,s\in\mathcal{S}_j} < WX_{crit}, \\ \mathrm{N}(\mu^-, \sigma^-), & \text{otherwise,} \end{cases}$$

- RX: insurers income $I_j \sim \mathrm{N}(\mu^+, \sigma^+)\ \forall j \in \mathcal{J}$

	Changde	Enshi	Yichang
I_1	$\rho_{11} = 0.5$	$\rho_{12} = 0.5$	$\rho_{13} = 0.0$
I_2	$\rho_{21} = 0.5$	$\rho_{22} = 0.0$	$\rho_{23} = 0.5$

Table 6: ρ-values used for simulation.

- set $\mu^+ = 500$, $\mu^0 = 100$, $\mu^- = 50$ and $\sigma^+ = \sigma^0 = \sigma^- = 100$.

Stylized Economy

- two crop insurance companies (farmers), one investor
- three traded stations in China go to map
- $r_t = r = 5\%$ p.a.,
- profit $\Pi(W_T, P_T)$, with P_T constant. go to table

Single Period: Investor's Supply and Insurers' Demand

Occurrences of wet days in Changde and Enshi are positive correlated

- payoffs of WX calls are positive associated,
- investor's supply $\downarrow$
- buyer's demand $\uparrow$ show Proposition

Single Period: Investor's Supply and Insurers' Demand

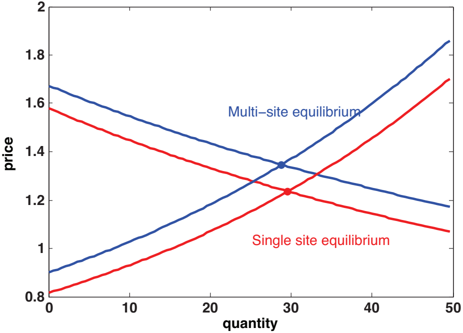


Figure 7: Supply/demand for WX call on Changde, $K=5$. Prices are given in index units.

Single Period WX call trading: Prices

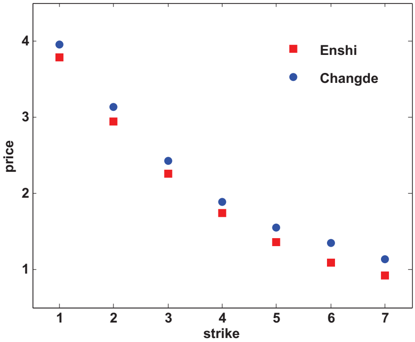


Figure 8: Prices of call options for different strikes K in a single-period WX call trading. Prices are given in index units.

Two-Period vs Single Period RX future trading: Equilibrium Prices

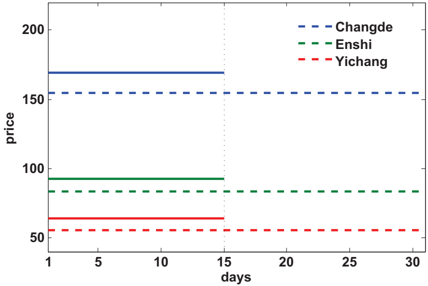


Figure 9: Single period (dashed) and two-period (solid) equilibrium prices for RX futures in May. A "flexibility" premium is paid by buyer for the possibility to rebalance the portfolio.

Two-Period RX future trading: Income

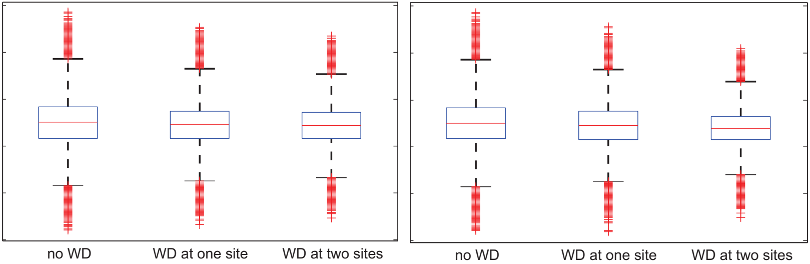


Figure 10: Income distribution of insurer 1 (left) and insurer 2 (right) at single and multiple sites two-period RX futures trading. Note: improvement of insurer 2 is better since payoffs of her RX futures (Changde and Yichang) are uncorrelated, for insurer 1 (Changde and Enshi) they are positive correlated. ▸ back

Summary

- pricing baskets of WDs in a multi-site, multi-period setting,
- agents trade with multiple sites simultaneously,
- insurer can hedge weather exposure in an optimal way,
- extension to more stations and agents possible, but computationally intensive.

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Appendix: Investor's Inverse Supply

$$W_{ts'} = \frac{1}{a_m \alpha_{mt+1s'} R^{T-t}}$$
$$\log \frac{\mathrm{E}_t \left\{ \exp\left(a_m \sum_{s \in \mathcal{S}} \alpha_{mt+1s} W_{t+1s} R^{T-(t+1)} \right) \Theta_{m,t+1} \right\}}{\mathrm{E}_t \left\{ \exp\left(a_m \sum_{s \neq s' s \in \mathcal{S}} \alpha_{mt+1s} W_{t+1s} R^{T-(t+1)} \right) \Theta_{m,t+1} \right\}},$$

$$\Theta_{mt} = \exp(-a_m R^{T-t} \sum_{s \in \mathcal{S}} \alpha_{mt+1s} W_{ts})$$
$$\mathrm{E}_t \{ \exp(a_m R^{T-(t+1)} \sum_{s \in \mathcal{S}} \alpha_{mt+1s} W_{t+1s}) \Theta_{mt+1} \},$$

with $R = 1 + r, \quad 0 \le t < T - 1, \quad \Theta_{mT} = 1.$

▸ back

Appendix: Buyer's Inverse Demand

$$
\begin{aligned}
W_{ts'} &= \frac{1}{a_j \alpha_{jt+1s'} R^{T-t}} \\
&\quad \log \frac{\mathrm{E}_t\{\exp(-a_j \sum_{s\in\mathcal{S}_j, s\neq s'} \alpha_{jt+1s} W_{t+1s} R^{T-(t+1)})\Theta_{jt+1}\}}{\mathrm{E}_t\{\exp(-a_j \sum_{s\in\mathcal{S}_j} \alpha_{jt+1s} W_{t+1s} R^{T-(t+1)})\Theta_{jt+1}\}}, \\
\Theta_{jt} &= \exp(a_j R^{T-t} \sum_{s\in\mathcal{S}_j} \alpha_{jt+1s} W_{ts}) \\
&\quad \mathrm{E}_t\{\exp(-a_j R^{T-(t+1)} \sum_{s\in\mathcal{S}_j} \alpha_{jt+1s} W_{t+1s})\Theta_{jt+1}\}, \\
\text{with } R &= 1+r, \quad 0 \leq t < T-1, \quad \Theta_{jT} = \exp(-a_j I_j).
\end{aligned}
$$

back

Appendix: single site vs multi-site in single period

Condition for Buyer j: If

$$\mathrm{Cov}[U_j(\alpha_{j,T,s'}W_{T,s'}), U_j\{(W_{T,s}\alpha_{j,T,s})_{s\in\mathcal{S}_j\setminus\{s'\}}\}] \geq (\leq)$$
$$\frac{\mathrm{Cov}\{U_j(I_j), U_j(\alpha_{j,T,s'}W_{T,s'})\}\mathrm{Cov}[U_j(I_j), U_j\{(W_{T,s}\alpha_{j,T,s})_{s\in\mathcal{S}_j\setminus\{s'\}}\}]}{\mathrm{E}\{U_j(I_j)\}^2}$$
$$- \frac{\mathrm{E}[\bar{U}_j(I_j)\bar{U}_j(\alpha_{j,T,s'}W_{T,s'})\bar{U}_j\{(W_{T,s}\alpha_{j,T,s})_{s\in\mathcal{S}_j\setminus\{s'\}}\}]}{\mathrm{E}\{U_j(I_j)\}} \quad (4)$$

then for $a_j > 0, j \in J$ and $(\alpha_{j,T,s})_{s\in\mathcal{S}_j\setminus\{s'\}}$ of the same sign buyers demand for WD in s' shifts downwards (upwards) compared to the single-site case. back

Appendix: BIC (Katz, 1981)

Estimator of Markov chain order $\hat{k}_{BIC}$:

$$\hat{k}_{BIC} = \operatorname*{argmin}_{0\leq k\leq m} \mathrm{BIC}(k),$$
$$\mathrm{BIC}(k) = -2\log\lambda_{k,m} - (s^m - s^k)(s-1)\log n,$$
$$\lambda_{k,m} = \frac{M_k(Y_1,\ldots,Y_n)}{M_m(Y_1,\ldots,Y_n)},$$
$$M_k(Y_1,\ldots,Y_n) = \prod_{i_1,\ldots,i_{k+1}} \frac{n_{i_1,i_{k+1}}}{\sum_{i_{k+1}} n_{i_1,\ldots,i_{k+1}}}$$

and $n_{i_1,i_{k+1}} > 0$ is number of transitions from states $i_1 \to i_2 \to \ldots \to i_{k+1}$ where $i_k = 1,\ldots,s$ are the state labels.

▸ back

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