The Stochastic Fluctuation of the Quantile Regression Curve

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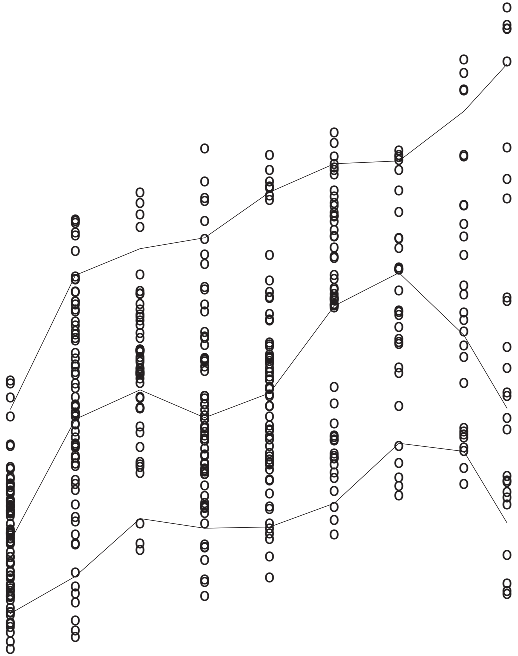
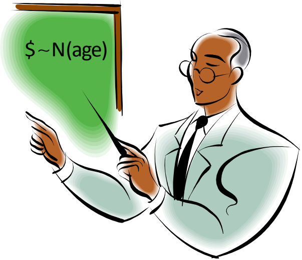




- log(Salary) $\sim$ Years
- "The rich got richer and the poor got poorer!"
- Yu et al. (2003)

Conditional Mean Regression

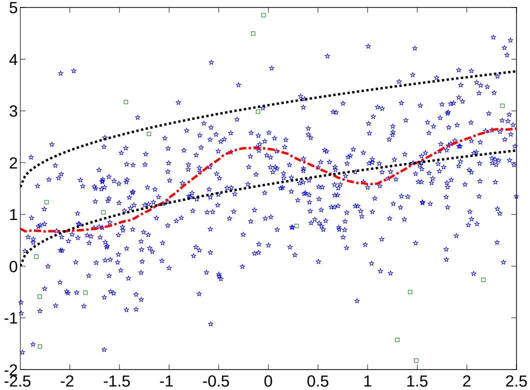
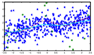


Figure 1: The conditional mean curve, the Nadaraya-Watson estimator and the 0.9-quantile curve. QR105



Conditional Quantile Regression

- Median regression = mean regression (symmetric)
- Describes the conditional behavior of a response Y in a broader spectrum
- Stochastic fluctuation effect (open!)
- Parametric model specification (open!)
- ...

Example

- Financial Market & Econometrics
  - VaR (Value at Risk) tool
  - Detect conditional heteroskedasticity
  - ...

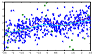
Example

- Labor Market
  - Analyze income of football players w.r.t. different ages, years, and countries, etc.
  - To detect discrimination effects, need split other effects at first.

$$\begin{aligned}\log(\text{Income}) &= A(\text{year, age, education, etc}) \\ &\quad + \beta\, B(\text{gender, nationality, union status, etc}) + \varepsilon\end{aligned}$$

  - ...

- Parametric e.g. polynomial model
  - Interior point method, Koenker and Bassett (1978), Koenker and Park (1996), Portnoy and Koenker (1996).
- Nonparametric model
  - Check function approach, Fan et al. (1994), Yu and Jones (1997, 1998).
  - Double-kernel technique, Fan et al. (1996).
  - Weighted NW estimation, Hall et al. (1999), Cai (2002).
  - Causality test, Jeong and Härdle (2008).

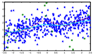


Outline

1. Motivation ✓
2. Basic Setup
3. Strong Uniform Consistency Rate
4. Asymptotic Uniform Confidence Band
5. Monte Carlo Simulation
6. Application
7. Further work

Basic Setup

- $\{(X_i, Y_i)\}_{i=1}^n$ i.i.d. rv's, pdf $f(x, y)$, $f(y|x)$, $f(x|y)$, cdf $F(x, y)$, $F(y|x)$, $F(x|y)$, and marginal f_X, f_Y, $x \in J = [0, 1] \subseteq \mathbb{R}^d$, $y \in \mathbb{R}$
- $l(x) = F_{Y|x}^{-1}(p)$ p-quantile regression curve
- $l_n(x)$ quantile-smoother



Check Function

$$\rho_p(u) = pu\mathbf{1}\{u \in (0,\infty)\} - (1-p)u\mathbf{1}\{u \in (-\infty,0)\}$$

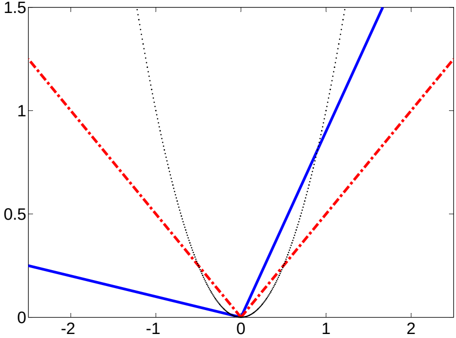
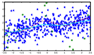


Figure 2: Check function for p=0.9, p=0.5 and weight function in conditional mean regression



- $l(x)$ minimizes (w.r.t. $\theta \in I = \mathbb{R}$)

$$L(\theta) = \mathsf{E}\{\rho_p(Y - \theta)|X = x\}$$

- $l_n(x)$ minimizes (w.r.t. θ)

$$L_n(\theta) = n^{-1} \sum_{i=1}^{n} \rho_p(Y_i - \theta) K_h(x - X_i) \tag{1}$$

- $K_h(u) = h^{-1} K(u/h)$ is a kernel with bandwidth h, and has the compact support $[-A, A]$.

- Iteratively reweighted least squares procedure, Lejeune and Sarda (1988), Yu et al. (2003).

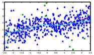
$$L_n(\theta) = n^{-1} \sum_{i=1}^{n} (Y_i - \theta)^2 \left\{ \frac{\rho_p(Y_i - \theta)}{(Y_i - \theta)^2} \right\} K_h(x - X_i)$$

- Define $w_p(Y_i; \theta)$ as $\rho_p(Y_i - \theta)/(Y_i - \theta)^2$

$$L_n(\theta) = n^{-1} \sum_{i=1}^{n} (Y_i - \theta)^2 w_p(Y_i; \theta) K_h(x - X_i)$$

- Integrate w_p into K_h, and rewrite $L_n(\theta)$ as a reweighted sum of squares:

$$L_n(\theta) = n^{-1} \sum_{i=1}^{n} (Y_i - \theta)^2 K_p(x; X_i; Y_i; \theta)$$



Define sequence $l_{n,k}$ (for any initial value $l_{n,1}$) through:

$$
\begin{aligned}
l_{n,k+1} &= \operatorname*{argmin}_{\theta} n^{-1} \sum_{i=1}^{n} (Y_i - \theta)^2 K_p(x; X_i; Y_i; l_{n,k}) \\
&= \frac{\sum_{i=1}^{n} K_p(x; X_i; Y_i; l_{n,k}) Y_i}{\sum_{i=1}^{n} K_p(x; X_i; Y_i; l_{n,k})} \qquad (2)
\end{aligned}
$$

Empirically we set $l_{n,1}$ as the global p-quantile.

Theorem

$\exists$ *some* k_0, *s.t.* $l_{n,k}(x) = l_n(x), \forall k \geqslant k_0$.

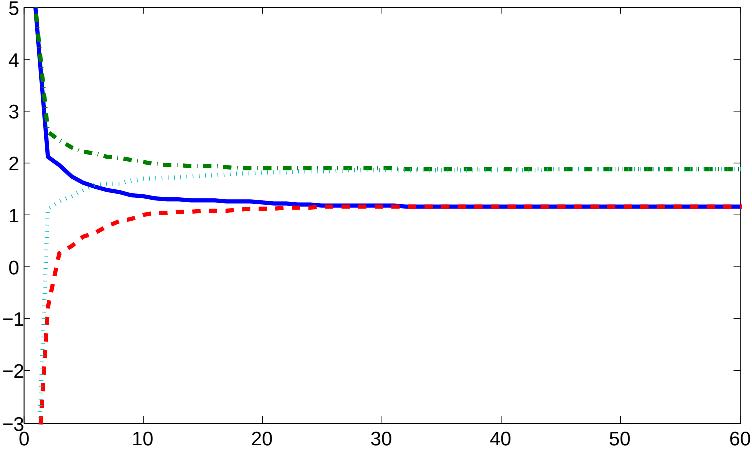


Figure 3: Convergence of $l_{n,k}(x)$ to 0.5-quantile smoother $l_n(x)$ with starting value 5, -5 and x value -1, 1. QR105

Weight Function

$$\begin{aligned}\psi(u) &= \psi_p(u) = p\mathbf{1}\{u \in (0,\infty)\} - (1-p)\mathbf{1}\{u \in (-\infty,0)\} \\ &= p - \mathbf{1}\{u \in (-\infty,0)\}\end{aligned}$$

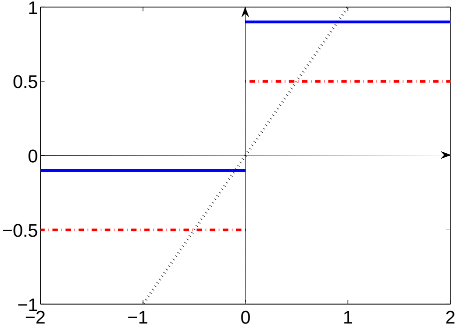


Figure 4: Weight function for p=0.9, p=0.5 and conditional mean regression

The Estimation Idea

$l_n(x)$ and $l(x)$ can also be treated as a zero (w.r.t. θ) of:

$$\widetilde{H}_n(\theta, x) \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^{n} K_h(x - X_i)\psi(Y_i - \theta) \tag{3}$$

$$\widetilde{H}(\theta, x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} f(x, y)\psi(y - \theta)dy \tag{4}$$

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Uniform Strong Consistency Rate

Lemma (Härdle et al. (1988))

For some constant A^* not depending on n, we have a.s. as $n \to \infty$

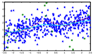
$$\sup_{\theta \in I} \sup_{x \in J} |\widetilde{H}_n(\theta, x) - \widetilde{H}(\theta, x)| \quad \leq \quad A^* \max\{(nh/\log n)^{-1/2}, h^{\tilde{\alpha}}\}$$

with $f(\cdot|y)$ is ulL-$\tilde{\alpha}$ $(0 < \tilde{\alpha} \leqslant 1)$ on J, uniformly in y.

Require additionally as in Härdle and Luckhaus (1984)

$$\inf_{x \in J} \Big| \int \psi\{y - l(x) + \varepsilon\} dF(y|x) \Big| \geqslant \tilde{q}|\varepsilon|, \quad \text{for } |\varepsilon| \leqslant \delta_1, \qquad (5)$$

with some positive constants $\tilde{q}$ and δ_1.



Theorem

With the additional assumption (5), we have a.s. as $n \to \infty$

$$\sup_{x \in J} |l_n(x) - l(x)| \quad \leq \quad B^* \max\{(nh/\log n)^{-1/2}, h^{\tilde{\alpha}}\} \tag{6}$$

with $B^* = A^*/m_1 \tilde{q}$ *not depending on* n *and* m_1 *a lower bound of* $f_X(t)$.

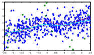
Remark

- Franke and Mwita (2003): without assuming K has compact support, we have a.s. as $n \to \infty$

$$\sup_{x \in J} |l_n(x) - l(x)| \leqslant B^{**}\{(nh/(s_n \log n))^{-1/2} + h^2\}$$

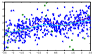
where B^{**} is some constant and $s_n, n \geqslant 1$ is an increasing positive integers sequence, $1 \leqslant s_n \leqslant \frac{n}{2}$ and some other criteria.

- $\{nh/(\log n)\}^{-1/2} \leqslant \{nh/(s_n \log n)\}^{-1/2}$.



Define

$$
\begin{aligned}
H_n(t) &= (nh)^{-1}\sum_{i=1}^{n} K\{(t-X_i)/h\}\psi\{Y_i - l(t)\} \\
D_n(t) &= \partial(nh)^{-1}\sum_{i=1}^{n} K\{(t-X_i)/h\}\psi\{Y_i - \theta\}/\partial\theta\ \{l(t)\} \\
\sigma^2(t) &= \mathsf{E}[\psi^2\{Y - l(t)\}|t] = p(1-p) \\
g'(t) &= \sigma^2(t) f_X(t) = p(1-p) f_X(t) \\
q(t) &= \partial\,\mathsf{E}\{\psi(Y-\theta)|t\}/\partial\theta\ \{l(t)\}\cdot f_X(t) \\
&= f\{l(t)|t\} f_X(t)
\end{aligned}
$$

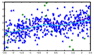


Asymptotic Uniform Confidence Band

Theorem

Let $h = n^{-\delta}$, $\frac{1}{5} < \delta < \frac{1}{3}$ *and* $\lambda(K) = \int_{-A}^{A} K^2(u)du$ *and*

$$\begin{aligned} d_n &= (2\delta \log n)^{1/2} + (2\delta \log n)^{-1/2}[\log\{c_1(K)/\pi^{1/2}\} \\ &\quad + \frac{1}{2}\{\log \delta + \log \log n\}], \\ &\quad \textit{if } c_1(K) = \{K^2(A) + K^2(-A)\}/\{2\lambda(K)\} > 0 \\ d_n &= (2\delta \log n)^{1/2} + (2\delta \log n)^{-1/2} \log\{c_2(K)/2\pi\} \\ &\quad \textit{otherwise with } c_2(K) = \int_{-A}^{A} \{K'(u)\}^2 du/\{2\lambda(K)\}. \end{aligned}$$

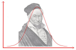


Then

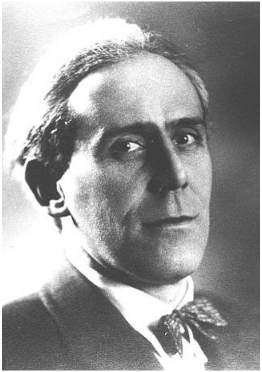
$$\mathrm{P}\left((2\delta \log n)^{1/2} \left[\sup_{t \in J} r(t) |\{ l_n(t) - l(t) \}| / \lambda(K)^{1/2} - d_n \right] < z \right)$$
$$\longrightarrow \exp\{-2\exp(-z)\}, \quad \text{as } n \to \infty. \tag{7}$$

with

$$r(t) = (nh)^{1/2} f\{l(t)|t\}\{f_X(t)/p(1-p)\}^{1/2}.$$

Emil Julius Gumbel on BBI: 

Emil Julius Gumbel



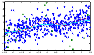
- Born on 18910718 in München
- Professor of Mathematical Statistics at Heidelberg
- Application of extreme value theory, particularly to climate and hydrology
- Gumbel distribution
- Died on 19660910 in New York

Corollary

An approximate $(1-\alpha) \times 100\%$ *confidence band over* $[0,1]$ *is*

$$\begin{aligned} l_n(t) \quad \pm \quad & (nh)^{-1/2}\{p(1-p)/\hat{f}_X(t)\}^{1/2}\hat{f}^{-1}\{l(t)|t\} \\ & \times\{d_n + c(\alpha)(2\delta\log n)^{-1/2}\}\cdot\{\lambda(K)\}^{1/2}, \end{aligned} \tag{8}$$

where $c(\alpha) = \log 2 - \log|\log(1-\alpha)|$ *and* $\hat{f}_X(t)$, $\hat{f}\{l(t)|t\}$ *are consistent estimates for* $f_X(t)$, $f\{l(t)|t\}$.



Remark

Asymptotic normality at interior points (local constant, local linear, reweighted NW methods, etc.):

$$l_n(t) - l(t) \sim \mathrm{N}\big(0, \tau^2(t)\big)$$

with $\tau^2(t) = \lambda(K)p(1-p)/[f_X(t)f^2\{l(t)|t\}]$.

Write (8) as:

$$l_n(t) \pm \{d_n + c(\alpha)(2\delta \log n)^{-1/2}\}\hat{\tau}(t)$$

Optimal Bandwidth Selection

- Minimize the approximation of AMSE (asymptotic mean square error)
- Rule-of-thumb for h_p from Yu and Jones (1998):
  1. Select optimal bandwidth h_{mean} from conditional mean regression
  2. $h_p = [p(1-p)/\varphi^2\{\Phi^{-1}(p)\}]^{1/5} \cdot h_{\text{mean}}$
     with φ, Φ as the pdf and cdf of a standard normal distribution

$$h_p = [p(1-p)/\varphi^2\{\Phi^{-1}(p)\}]^{1/5} \cdot h_{\text{mean}}$$

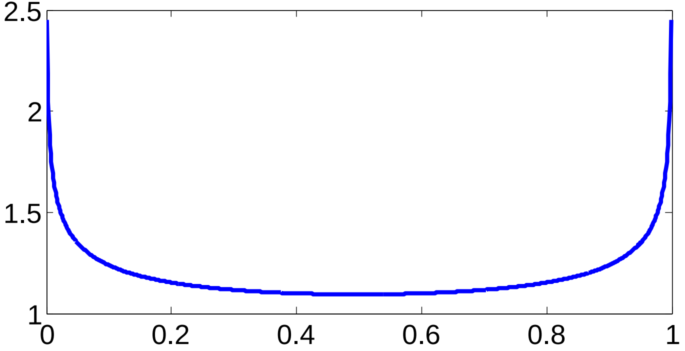


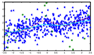
Figure 5: The relationship between h_p and p

Passus Argumentum

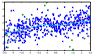
1. $l_n(t)$ as a zero (w.r.t. θ) of $\widetilde{H}_n(\theta, t)$, then:

$$\begin{aligned} 0 &= H_n(t) + \{l_n(t) - l(t)\} D_n(t) + \dots \\ l_n(t) - l(t) &= -\{H_n(t) - \operatorname{E} H_n(t)\}/q(t) - R_n(t) \qquad (9) \end{aligned}$$

2. Approximate the leading linear term by a weighted Wiener process similar to Johnston (1982), Bickel and Rosenblatt (1973).



- $\|R_n\| = \sup_{t \in J} |R_n(t)| = \mathcal{O}_p\{(nh \log n)^{-1/2}\}$
- Rosenblatt transformation: $T(x, y) = \{F_{X|y}(x|y), F_Y(y)\}$
- Empirical process: $Z_n(x, y) = n^{1/2}\{F_n(x, y) - F(x, y)\}$
- Brownian bridges: $B_n(x, y) = W_n(x, y) - xyW_n(1, 1)$



Brownian Bridge

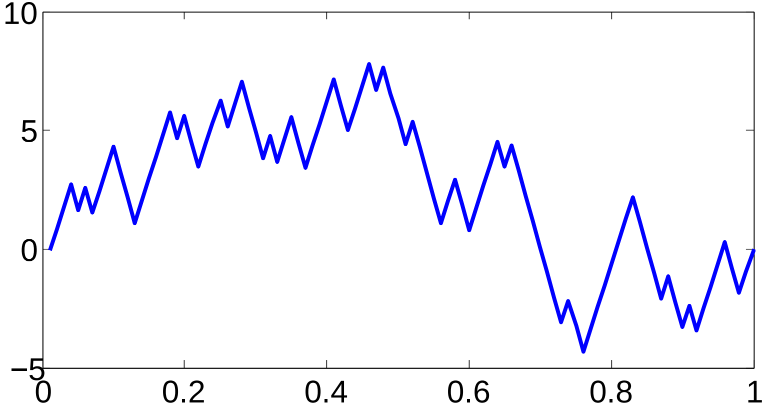


Figure 6: Brownian bridge with $d = 1$ QRbb

Lemma (Tusnady (1977))

On a suitable probability space there exists a sequence of Brownian bridges B_n such that

$$\sup_{x,y} |Z_n(x,y) - B_n\{T(x,y)\}| = \mathcal{O}\{n^{-1/2}(\log n)^2\} \quad a.s.,$$

where $Z_n(x,y) = n^{1/2}\{F_n(x,y) - F(x,y)\}$ denotes the empirical process of $\{(X_i, Y_i)\}_{i=1}^n$.

For $d = 2$, the optimal approximation rate is given in Rio (1996).
For $d > 2$, open problem!

Approximate the Linear Part

- $Y_n(t) = (nh)^{1/2}\{g'(t)\}^{-1/2}\{H_n(t) - \mathsf{E}\, H_n(t)\}$
- Stochastic integral w.r.t. $Z_n(x, y)$:

$$Y_n(t) = \{hg'(t)\}^{-1/2} \iint K\{(t-x)/h\}\psi\{y - l(t)\} dZ_n(x, y)$$

$$Y_{0,n}(t) = \{hg(t)\}^{-1/2} \iint_{\Gamma_n} K\{(t-x)/h\}\psi\{y - l(t)\} dZ_n(x, y)$$

$$\Gamma_n = \{|y| \leqslant a_n\}$$

$$g(t) = \mathsf{E}[\psi^2\{y - l(t)\} \cdot \mathbf{1}(|y| \leqslant a_n)|X = t] \cdot f_X(t)$$

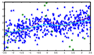
$$Y_{1,n}(t) = \{hg(t)\}^{-1/2} \iint_{\Gamma_n} K\{(t-x)/h\}\psi\{y - l(t)\} dB_n\{T(x,y)\}$$

$$Y_{2,n}(t) = \{hg(t)\}^{-1/2} \iint_{\Gamma_n} K\{(t-x)/h\}\psi\{y - l(t)\} dW_n\{T(x,y)\}$$

$$Y_{3,n}(t) = \{hg(t)\}^{-1/2} \iint_{\Gamma_n} K\{(t-x)/h\}\psi\{y - l(x)\} dW_n\{T(x,y)\}$$

$$Y_{4,n}(t) = \{hg(t)\}^{-1/2} \int g(x)^{1/2} K\{(t-x)/h\} dW(x)$$

$$Y_{5,n}(t) = h^{-1/2} \int K\{(t-x)/h\} dW(x)$$



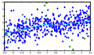
Lemma (Bickel and Rosenblatt (1973))

Let

$$Y_{5,n}(t) = h^{-1/2} \int K\{(t-x)/h\} dW(x).$$

Then, as $n \to \infty$, the supremum of $Y_{5,n}(t)$ has a Gumbel distribution.

$$\mathrm{P}\left\{(2\delta \log n)^{1/2}\left[\sup_{t \in J} |Y_{5,n}(t)|/\{\lambda(K)\}^{1/2} - d_n\right] < z\right\}$$
$$\longrightarrow \exp\{-2\exp(-z)\}.$$



Outline

Monte Carlo Simulation

Bivariate data $\{(X_i, Y_i)\}_{i=1}^{n}$, $n = 500$ with joint pdf

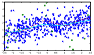
$$
\begin{aligned}
f(x,y) &= g(y - \sqrt{x+2.5})\mathbf{1}(x \in [-2.5, 2.5]) \qquad (10)\\
g(u) &= \frac{9}{10}\varphi(u) + \frac{1}{90}\varphi(u/9).
\end{aligned}
$$

$l(x)$ as a zero (w.r.t. θ) of:

$$9\Phi(\theta) + \Phi(\theta/9) = 10p,$$

0.5-quantile curve $l(x) = \sqrt{x+2.5}$, and 0.9-quantile curve $l(x) = 1.5296 + \sqrt{x+2.5}$. We used the quartic kernel

$$
\begin{aligned}
K(u) &= \frac{15}{16}(1-u^2)^2, \quad |u| \leqslant 1,\\
&= 0, \quad |u| > 1.
\end{aligned}
$$



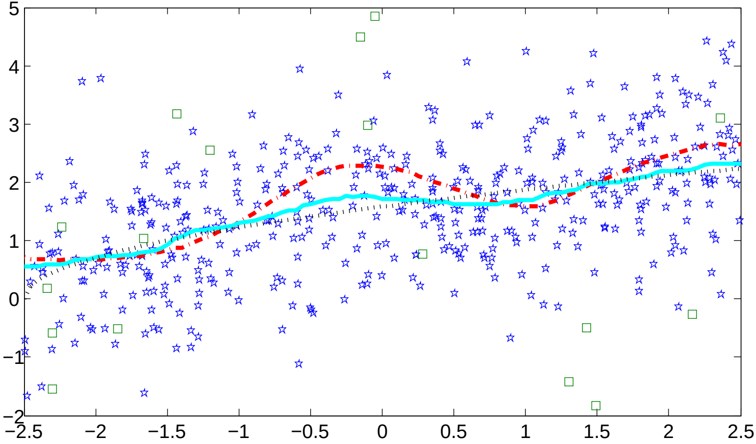


Figure 7: The 0.5-quantile curve, the Nadaraya-Watson estimator $m_n^*(x)$, and the 0.5-quantile smoother $l_n(x)$ with $h_{0.5} = 1.10$. QR105

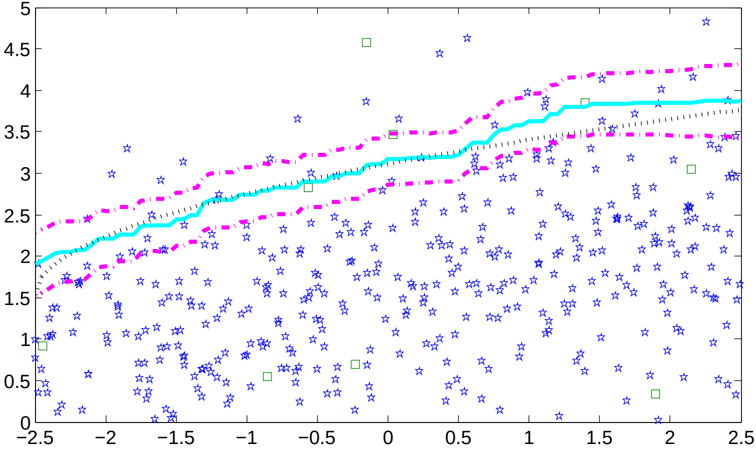


Figure 8: The 0.9-quantile curve, the 0.9-quantile smoother with $h_{0.9} = 1.25$ and 95% confidence bands. QR1

Labor Market Application

- For fixed $B(\cdot)$, study nondiscrimination effects in $A(\cdot)$
- Relation: $\log(\text{Wage}) \sim \text{Age}$
- Data: Current Population Survey (CPS) in 2005
- Male, 25 - 59, full-time, college graduate containing $16,731$ observations

- Conditional mean approach reveals a quadratic relation, Murphy and Welch (1990)
- Conditional quantile approach
- Ages reported as integer values
- Quartic kernel, $h = 2$

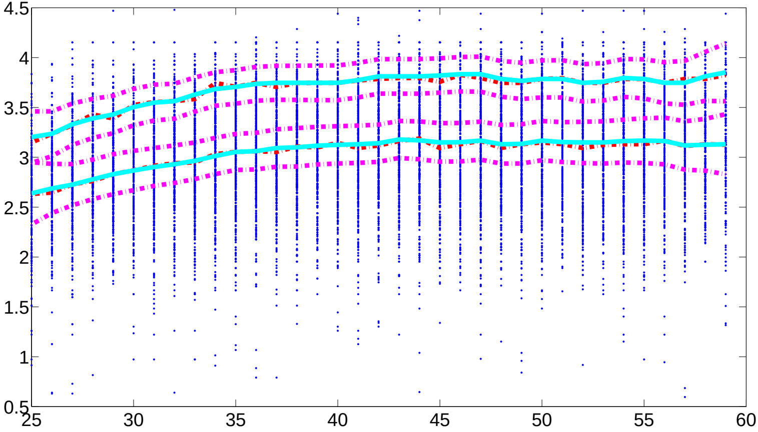


Figure 9: The original observations, local quantiles, 0.5, 0.9-quantile smoothers and corresponding 95% confidence bands. QRCPS

Parametric model specification test

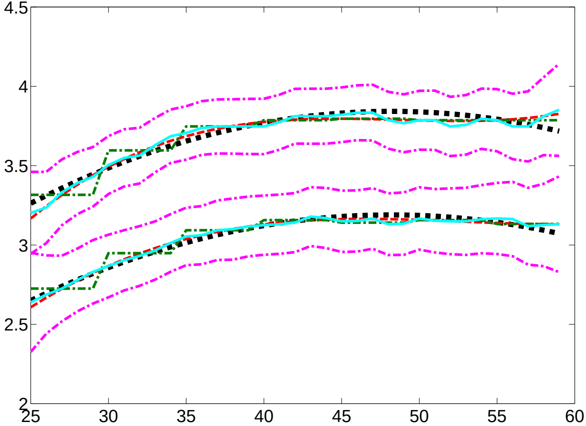


Figure 10: Quadratic, quartic, set of dummies (for age groups) estimates, quantile smoothers and corresponding 95% confidence bands. QRCPS

Conclusion

- At the 5% significance level
  - quadratic $\surd$, quartic $\surd$
  - set of dummies (for age groups) $\surd$
- At the 10% significance level
  - quadratic $\surd$, quartic $\surd$
  - set of dummies (for age groups) $\times$
- Suggest quadratic model measure wage-earning relation for simplicity.

Further work

- Panel data/Partial linear quantile regression
- Semiparametric quantile regression adequacy checking $\beta = 0$?
  - Semiparametric quantile regression from Yu
  - Semiparametric mean regression adequacy checking of Zhu, Zhu and Song (2008)
- Nonparametric local adaptive quantile regression
  - Local adaptive mean regression from Spokoiny (2008)
- Strengthen errors-in-variables quantile regression from Fan
  - Linear and partial linear errors-in-variables quantile regression of He and Liang (2000)
- ...

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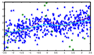

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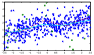

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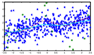

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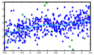

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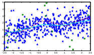

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