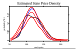
How Precise Are Price Distributions Predicted by Implied Binomial Trees?

Wolfgang Härdle

Institut für Statistik und Ökonometrie
CASE-Center for Applied Statistics and Economics
Humboldt-Universität zu Berlin

Jun Zheng

School of Mathematics Science, Peking University, China
Daimler-Benz-Stiftung Ph.D students program



Binomial Trees

- Option pricing model, recovers the state price density(SPD) from option prices

- State price density(SPD), density function assigning probabilities to the various possible values of the underlying at the option's expiration, $p(S_t, S_T, r, \tau)$

- Geometric Brownian Motion (GBm) model assumption

$$\frac{dS_t}{S_t} = r dt + \sigma dZ_t$$

yields (Black and Scholes - BS):

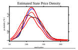
$$p(S_t, S_T, r, \tau) = \frac{1}{S_T\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{\left[\ln\left(\frac{S_T}{S_t}\right) - (r - \frac{\sigma^2}{2})\tau\right]^2}{2\sigma^2\tau}\right]$$

where $\tau = T - t$,

S_t is the stock price at time t,

r is the interest rate,

σ is the constant volatility.



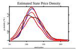
Let K be the exercise price at time T.

- Option prices

$$
\begin{aligned}
C(K,T) &= e^{-r\tau}\int_0^{+\infty}(S_T-K)^+ p(S_t,S_T,r,\tau)dS_T,\\
P(K,T) &= e^{-r\tau}\int_0^{+\infty}(K-S_T)^+ p(S_t,S_T,r,\tau)dS_T.
\end{aligned}
$$

- Implied local volatility surface

$$
\begin{aligned}
\sigma^2_{imp}(s,\tau) &= \mathrm{var}(\log S_T|S_t=s)\\
&= \int(\log S_T - E\log S_T)^2 p(s,S_T,r,\tau)\,dS_T.
\end{aligned}
$$

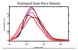


- CRR Binomial Tree (Cox, Ross,& Rubinstein (1979))

  Example

  $S = 100, T = 2$ years, $\triangle t = 1$ year, $\sigma = 10\%, r = 0.03, \tau = T$

		122.15
	110.52	
100.00		100.00
	90.48	
		81.88



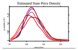
Why Implied Binomial Trees (IBT)?

Problem of the GBm model:

Volatility smile: the Black-Scholes implied volatility of market option prices decreases with the stock price; increases with the time.

Purpose of the IBT

- construction adapted to the volatility smile
- possibility to price derivative securities
- calculation of the state price density (SPD)
- calculation of the implied local volatility surfaces



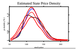
Diffusion process

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(S_t, t) dZ_t \tag{1}$$

$\sigma(S_t, t)$, local instantaneous volatility function.

Questions

- precision of the SPD estimations obtained from the IBT?
- relation between the local instantaneous volatility function and the Black Scholes implied volatility surface?



Overview

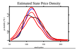
√ 1. Introduction and Motivation

2. Algorithm

3. Simulation and Comparison

4. Expectation on the elements of DAX

5. Conclusion



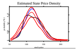
Construction Algorithm

Notation and Basic Requirements

- $s_{n,i}$, the stock price of the ith node at the nth level
- Forward prices $F_{n,i} = s_{n,i} \times e^{\triangle t}$ and transition probabilities $p_{n,i}$ satisfy the preference-free condition:

$$F_{n,i} = p_{n,i} s_{n+1,i+1} + (1 - p_{n,i}) s_{n+1,i}$$

- Arrow-Debreu prices $\lambda_{n,i}$ (discounted risk-neutral probability) the price of an option that pays 1 in one and only one state i at nth level, and otherwise pays 0.



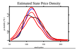
$$\begin{cases} \lambda_{n+1,1} = & e^{-r\triangle t}\left\{(1-p_{n,1})\lambda_{n,1}\right\} \\ \lambda_{n+1,i+1} = & e^{-r\triangle t}\left\{\lambda_{n,i}p_{n,i} + \lambda_{n,i+1}(1-p_{n,i+1})\right\}, \quad 2 \le i \le n \\ \lambda_{n+1,n+1} = & e^{-r\triangle t}\left\{\lambda_{n,n}p_{n,n}\right\} \end{cases}$$

$\triangle t$, the length of the time level

- Call option price $C(K, n\triangle t)$ satisfies

$$C(K, n\triangle t) = \sum_{i=1}^{n+1} \lambda_{n+1,i} \max(s_{n+1,i} - K, 0)$$

- $F_{n,i} < s_{n+1,i+1} < F_{n,i+1}$, in order to avoid arbitrage.



Algorithm: Derman and Kani IBT

Step 1: Central nodes

- Define $s_{n+1,i} = s_{1,1} = S$, $i = n/2 + 1$, for n even
- Start from $s_{n+1,i}$, $s_{n+1,i+1}$, $i = (n+1)/2$, suppose $s_{n+1,i} = s_{n,i}^2 / s_{n+1,i+1} = S^2 / s_{n+1,i+1}$, for n odd

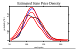
$$s_{n+1,i+1} = \frac{S\{e^{r\triangle t} C(S, n\triangle t) + \lambda_{n,i} S - \rho_u\}}{\lambda_{n,i} F_{n,i} - e^{r\triangle t} C(S, n\triangle t) + \rho_u} \quad \text{for} \quad i = (n+1)/2$$

Step 2: Upward

$$s_{n+1,i+1} = \frac{s_{n,i}\{e^{r\triangle t}C(s_{n,i}, n\triangle t) - \rho_u\} - \lambda_{n,i}s_{n,i}(F_{n,i} - s_{n+1,i})}{\{e^{r\triangle t}C(s_{n,i}, n\triangle t) - \rho_u\} - \lambda_{n,i}(F_{n,i} - s_{n+1,i})}$$

Step 3: Downward

$$s_{n+1,i} = \frac{s_{n,i+1}\{e^{r\triangle t}P(s_{n,i}, n\triangle t) - \rho_l\} - \lambda_{n,i}s_{n,i}(F_{n,i} - s_{n+1,i+1})}{\{e^{r\triangle t}P(s_{n,i}, n\triangle t) - \rho_l\} + \lambda_{n,i}(F_{n,i} - s_{n+1,i+1})}$$

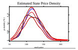


where

$$
\begin{aligned}
\rho_u &= \sum_{j=i+1}^{n} \lambda_{n,j}(F_{n,j} - s_{n,i}) \\
\rho_l &= \sum_{j=1}^{i-1} \lambda_{n,j}(s_{n,i} - F_{n,j})
\end{aligned}
$$

Technical Summary for Derman and Kani construction:

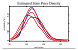
- prices options by CRR method
- satisfies the basic requirements above
- starts from the central nodes, define the current value as the stock price of the central node at the odd level



Algorithm: Barle and Cakici IBT

Major modifications

- align the center nodes of the tree with the forward price rather than with the current stock price
- use the forward price of the previous node to calculate the new option of the nodes at the next level
- Use Black-Scholes formula instead of CRR binomial tree method to calculate the interpolated option prices



Simulation and Comparison

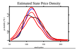
State space density estimation

- Estimation using the IBT

$$P(S_{n\triangle t} = s_{n+1,i}) = \lambda_{n+1,i} \times e^{rn\triangle t}$$

$\triangle t$, the length of the time level

- IBT is constructed from BS implied volatility surface, which is a direct assumption, or calculated from (implied by) market option prices.



- IBT is constructed from option prices interpolation directly, option price can be obtained from the market (difficult), or from the Monte-Carlo simulation samples of the diffusion process (1) by its definition

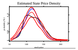
- Monte-Carlo Simulation of the diffusion process, Milstein scheme

  - From diffusion process model (1), get the random samples of S_T and estimate their density.

Implied local volatility surface

- Estimation using the IBT

$$
\begin{aligned}
EX &= \sum_{j=1}^{m} p_j \log(s_{n+m,j}) \\
\sigma_{imp}(s_{n,i}, m\triangle t) &= \sqrt{\sum_{j=1}^{m} p_j \left(\log(s_{n+m,j}) - EX\right)^2}
\end{aligned}
$$

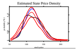


Example

- $S = \mathbf{100}$, $r = \mathbf{3}\%$, the annual BS implied volatility of a call is $\sigma = \mathbf{10}\%$, the implied volatility increases (decreases) linearly by $\mathbf{0.5}$ percentage points with every $\mathbf{10}$ point drop (rise) in the strike. (assumption on the BS implied volatility function)

- $$\frac{dS_t}{S_t} = \mu_t dt + \sigma(S_t, t) dZ$$

  where $\sigma(S_t, t) = 0.15 - 0.0005\, S_t$, drift function $\mu_t = r = 0.03$. (assumption on the local instantaneous volatility function)



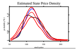
Output four-step four-year IBT of stock prices, transition probabilities, Arrow-Debreu prices respectively:

XFGimplt01

Output a plot of SPD, and a implied local volatility surface for five-year D & K IBT: XFGimplt02

Output a plot of SPD estimations for DAX index data at Jan. 4, 1999 using IBT method, tau=0.5 year:

XFGimplt05



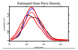
Derman and Kani one year(three step) IBT

stock price

			117.404
		111.616	
	105.944		105.964
100.000		100.000	
	94.389		94.372
		88.344	
			82.980

transition probability

		0.592
	0.603	
0.573		0.572
	0.600	
		0.549



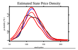
Arrow-Debreu price

			0.199
		0.339	
	0.567		
			0.405
1.000		0.474	
	0.423		
			0.292
		0.168	
			0.075

Barle and Cakici one year IBT

stock price

			120.017
		114.327	
	105.727		108.500
100.000		102.010	
	96.484		97.836
		89.160	
			86.044



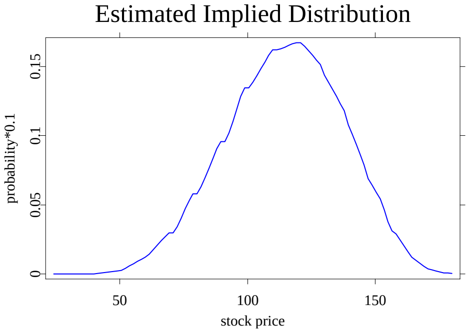
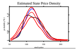


Figure 1: SPD estimation by the D & K IBT, level=20, calculate from Monte-Carlo simulated option prices(blue), $T = 5$ year, $\triangle t = 0.25$ year



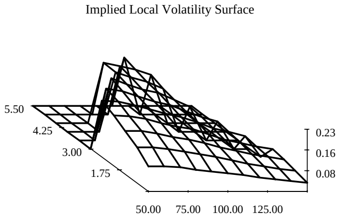
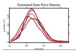


Figure 2: the implied local volatility surface estimation by the five year Derman and Kani IBT



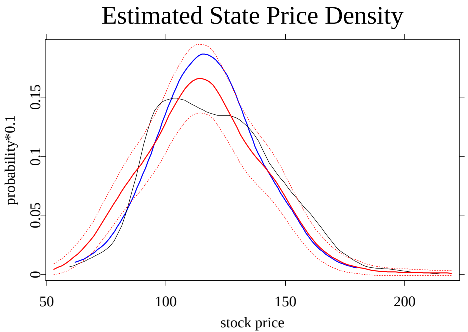


Figure 3: SPD estimation by Monte-Carlo simulation(red), and its confidence band (dashed), form the B & C IBT (blue), from the D & K IBT (black, thin), level =20, $T = 5$ year, $\triangle t = 0.25$ year

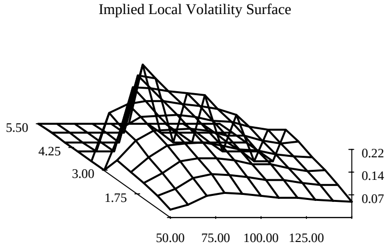


Figure 4: Implied local volatility surface estimation by D & K IBT, from Monte-Carlo simulated option prices

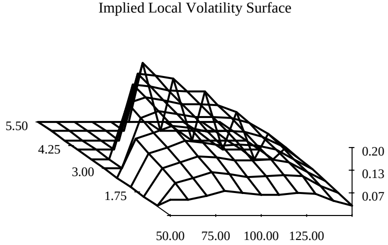


Figure 5: Implied local volatility surface estimation by B & C IBT, from Monte-Carlo simulated option prices

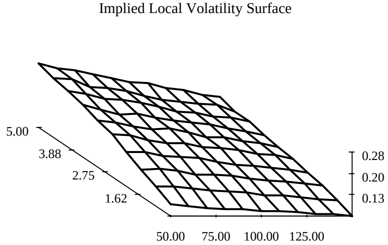


Figure 6: Implied local volatility surface estimation by Monte-Carlo simulation

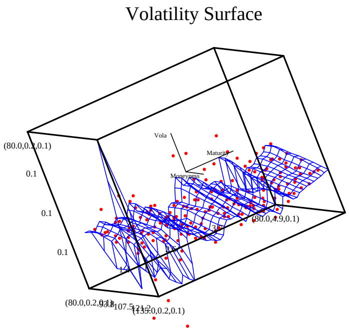


Figure 6: BS implied volatility surface estimation by Monte-Carlo simulation

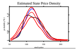
DAX data Example

German DAX index data included in MD*BASE

- German DAX option prices data at January 4, 1999
- DAX daily prices between January 1, 1997, and January 4. 1999

State price density estimation

- from the two IBTs (Derman and Kani, Barle and Cakici)
- historical time series density estimation (Aït-Sahalia, Wang & Yared (2000))



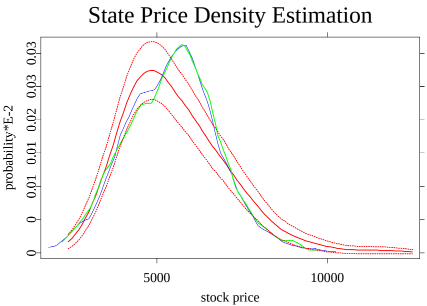
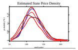
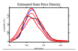


Figure 7 :SPD estimation of the DAX data, by historical time series density estimation and its confidence band (red), by the B & C IBT (blue), and by the D & K IBT (green), $\tau = 0.5$ year



Trading Rules to exploit SPD differences			
Skewness			sell OTM put,
	$(S1)$	skew$(f)>$ skew(g)	buy OTM call
Trade	$(S2)$	skew$(f) <$ skew(g)	buy OTM put,
			sell OTM call
Kurtosis			sell far OTM and ATM ,
	$(K1)$	kurt$(f)>$ kurt(g)	buy near OTM options
Trade	$(K2)$	kurt$(f) <$ kurt(g)	buy far OTM and ATM,
			sell near OTM options

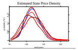
normal SPD is f and time series SPD is g. A far OTM call (put) is defined as one whose exercise price is 10% higher (lower) than the future price. A near OTM call (put) is defined as one whose exercise price is 5% higher (lower) but 10% lower(higher)than the future price.



Conclusion

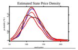
the IBT

- the IBT helps in assessing expectation about the future stock prices
- produces arbitrage-free binomial trees
- describes diffusion processes with variable volatility



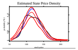
Limitation of the IBT

- negative probabilities are sometimes encountered
- redefinition causes losses of the information about the smile
- continuous diffusion is approximated by a binomial process



Precision of the SPD estimation

- SPD estimations from the two IBT methods coincide with the simulated SPD well, their precision depend on the precision of the implied volatility surface
- Difference between the SPD estimations from the two kinds of IBT construction
  - Running speed: Barle and Cakici method is faster
  - Precision: have no obvious difference
  - Special situation: when interest rate is high, the B & C IBT behaves better (Figure 8)
- Difference between volatility functions



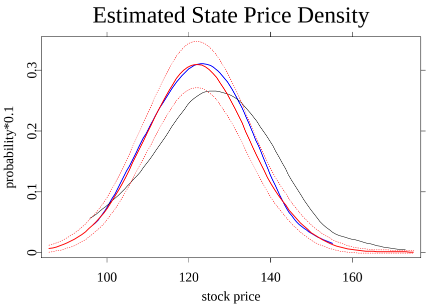
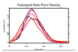


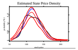
Figure 8: SPD estimation by Monte-Carlo simulation (red), by the B & C IBT (blue),and by the D & K IBT (black), where $r = 20\%, T = 1$ year



References

Aït-Sahalia, Y. & Lo, A. (1998). Nonparametric Estimation of State-Price Densities Implicit in Financial Asset Prices, *Journal of Finance*, **53**: 499–547.

Aït-Sahalia, Y. , Wang, Y.& Yared, F.(2000). Do Option Markets Correctly Price the Probabilities of Movement of the Underlying Asset? Forthcoming in the *Journal of Econometrics*



Barle, S & Cakici, N. (1998). How to Grow a Smiling Tree *The Journal of Financial Engineering*, **7**: 127–146.

Bingham, N.H.& Kiesel, R. (1998). *Risk-neutral Valuation: Pricing and Hedging of Financial Derivatives*, Springer Verlag, London.

Cox, J., Ross, S.& Rubinstein, M. (1979). Option Pricing: A simplified Approach, *Journal of Financial Economics* **7**: 229–263.

Derman, E.& Kani, I. (1994). The Volatility Smile and Its Implied Tree *http://www.gs.com/qs/*

Derman, E. & Kani, I. (1998). Stochastic Implied Trees: Arbitrage Pricing with Stochastic Term and Strike Structure of Volatility, *International Journal of Theoretical and Applied Finance* **1**: 7–22.

Dupire, B. (1994). Pricing with a Smile, *Risk* **7**: 18–20.

Fengler, M. R., Härdle, W. & Villa, Chr. (2001). *The Dynamics of Implied Volatilities: A Common Principal Components Approach*, SFB 373 Discussion Paper, HU Berlin. *http://sfb.wiwi.hu-berlin.de*

Härdle, W., Hlávka, Z. & Klinke, S. (2000). *XploRe Application Guide*, Springer Verlag, Heidelberg.

Hull, J. & White, A. (1987). The Pricing of Options on Assets with Stochastic Volatility, *Journal of Finance* **42**: 281–300.

Jackwerth, J. (1999). Optional-Implied Risk-Neutral Distributions and Implied Binomial Trees: A Literature Review, *Journal of Finance* **51**: 1611–1631.

Jackwerth, J. & Rubinstein, M. (1996). Recovering Probability Distributions from Option Prices, *Journal of Finance* **51**: 1611–

1631.

Kloeden, P., Platen, E. & Schurz, H. (1994). *Numerical Solution of SDE Through Computer Experiments*, Springer Verlag, Heidelberg.

Merton, R. (1976). Option Pricing When Underlying Stock Returns are Discontinuous, *Journal of Financial Economics* **January-March**: 125–144.

Rubinstein, M. (1994). Implied Binomial Trees. *Journal of Finance* **49**: 771–818.