Statistics of Risk Aversion

Enzo Giacomini and Wolfgang Härdle

CASE Center for Applied Statistics and Economics
Humboldt-Universität zu Berlin

Pricing Kernels & Risk Aversion

1. S_t - asset value at time t in a complete market
2. u - risk averse utility function from representative investor
3. marginal rate of substitution or pricing kernel, $\tau = T - t$

$$M(S_T) = \frac{u'(S_T)}{u'(s_t)}$$

under risk aversion u concave: M decreasing

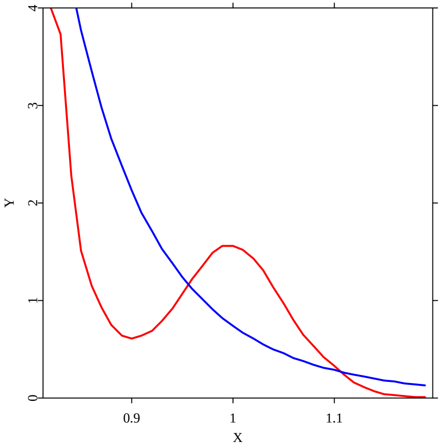


Figure 1: Theoretical (blue) and empirical (red) pricing kernels, estimated from DAX on 19990502 for $\tau = 10$ days, expressed in moneyness $\kappa = S_T/S_t e^{r\tau}$

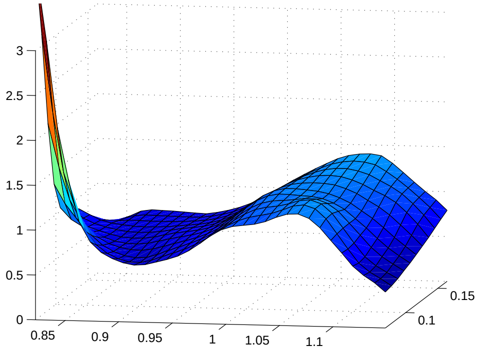


Figure 2: Estimated PK across moneyness κ and maturity τ, DAX on 20010710

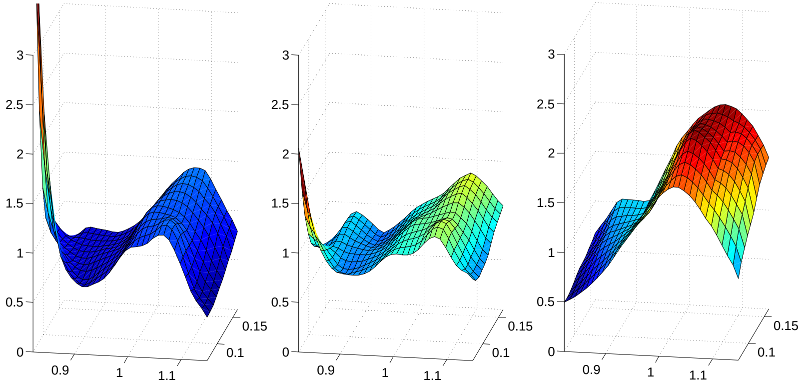
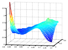


Figure 3: Empirical PK across κ and τ, estimated form DAX on 20010710, 20010904 and 20011130



Empirical pricing kernels

1. do not reflect risk aversion across all strikes
2. vary across time to maturity τ and time t

$$M(x) = M_{t,\tau}(x)$$

How to explain pricing kernel and risk aversion dynamics ?

Outline

1. Motivation ✓
2. Pricing Kernels
3. DSFM and Pricing Kernel Estimation
4. Empirical Results
5. References

Pricing Kernels

Asset price follows diffusion process

$$\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t)dB_t$$

where $0 \le t \le s \le T$ and B_t is standard Brownian motion under measure P. The risk neutral measure Q is obtained by $\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = \zeta_t$. For a payoff $\Psi(S_s)$ with maturity $\tau = s - t$,

$$e^{-r\tau} E^Q \left[\Psi(S_s) \middle| \mathcal{F}_t \right] = E^P \left[\Psi(S_s) e^{-r\tau} \frac{\zeta_s}{\zeta_t} \middle| \mathcal{F}_t \right]$$

The pricing kernel is defined as

$$M_{t,\tau} = e^{-r\tau} \frac{\zeta_s}{\zeta_t}$$

Merton Optimization Problem

Market completness, representative investor with concave utility function u

1. wealth $\{W_s\}$ and consumption processes $\{C_s\}$, $C_s = 0$
2. all wealth consumed at T, $C_T = W_T$
3. amount $\{\xi_s\}$ invested in S_s chosen by

$$\max_{\{\xi_s, t \leq s \leq T\}} E[u(W_T)|\mathcal{F}_t]$$

subjected to

$$\begin{aligned} W_s &\geq 0 \\ dW_s &= \{rW_s + \xi_s(\mu - r)\}ds + \xi_s \sigma dB_s \end{aligned}$$

Merton Equilibrium

In Merton equilibrium the pricing kernel (PK) is path independent and equals the marginal rate of substitution

$$e^{-r\tau}\frac{q_t(S_T)}{p_t(S_T)} = M_{t,\tau}(S_T) = \frac{u'(S_T)}{u'(s_t)}$$

where

1. q_t is conditional density of S_T under the risk neutral measure Q - state price density (SPD)
2. p_t is conditional density of S_T under the objective measure P - objective density

Merton Equilibrium

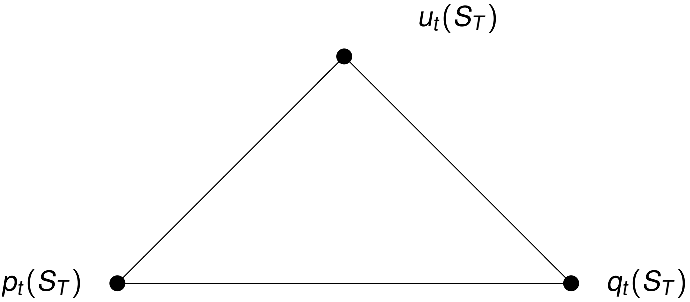


Figure 4: Utility function, risk neutral (SPD) and objective densities

Pricing Kernel Estimation

Ait-Sahalia and Lo (2000) estimate PK as the ratio between estimated SPD and estimated objective density

$$\widehat{M}_{t,\tau}(S_T) = e^{-r\tau}\frac{\widehat{q}_t(S_T)}{\widehat{p}_t(S_T)}$$

q is estimated from option and p from underlying prices

SPD Estimation

1. Breeden and Litzenberger (1978) obtain SPD from option prices
2. Ait-Sahalia and Lo (1998) used the estimate

$$\widehat{q}_t(S_T) = e^{r\tau} \left. \frac{\partial^2 C_{t,BS}\{S_t, K, \tau, r_t, \widehat{\sigma}_t(\kappa,\tau)\}}{\partial K^2} \right|_{K=S_T} \qquad (1)$$

3. $C_{t,BS}$ is the Black-Scholes price at time t
4. $\widehat{\sigma}_t(\kappa,\tau)$ is a nonparametric estimator for the implied volatility (Implied Volatility Surface - IVS)

Implied Volatility

1. at day i there are J_i options traded
2. each trade $j = 1, \ldots, J_i$ at day $i = 1, \ldots, I$ corresponds to an implied volatility $\sigma_{i,j}$ and a pair of moneyness and maturity $X_{i,j} = (\kappa_{i,j}, \tau_{i,j})^\top$
3. $\kappa_{i,j} = \frac{K}{F(t_{i,j})}$ is moneyness
4. K strike
5. $F(t_{i,j}) = S_{t_{i,j}} \exp(r_{\tau_{i,j}} \tau_{i,j})$ futures prices

IV - Degenerated Design

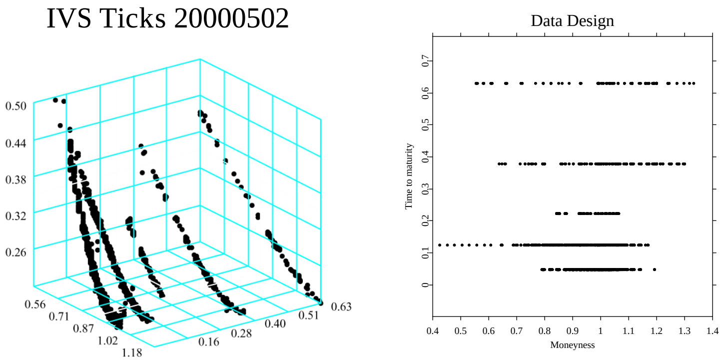


Figure 5: Left panel: call and put implied volatilities observed on 20000502. Right panel: data design on 20000502; ODAX, difference-dividend correction according to Hafner and Wallmeier (2001) applied.

Dynamic Semiparametric Factor Models (DSFM)

regress log implied volatilities $Y_{i,j} = \log \sigma_{i,j}$ on $X_{i,j}$

$$Y_{i,j} = \sum_{l=0}^{L} z_{i,l} m_l(X_{i,j}) + \varepsilon_{i,j}$$

1. $m_l(\cdot)$ are smooth basis functions, $l = 0, \ldots, L$
2. $z_{i,l}$ are time dependent factors
3. $\varepsilon_{i,j}$ is noise

The basis functions expanded using a series estimator, Borak et al. (2007)

$$m_l(X_{i,j}) = \sum_{k=1}^{K} \gamma_{l,k}\psi_k(X_{i,j})$$

for functions $\psi_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, K$ and coefficients $\gamma_{l,k} \in \mathbb{R}$. Defining $Z = (z_{i,l})$, $\Gamma = (\gamma_{l,k})$ the least square estimators are

$$(\widehat{\Gamma}, \widehat{Z}) = \arg \min_{\Gamma \in \mathcal{G}, Z \in \mathcal{Z}} \sum_{i=1}^{I} \sum_{j=1}^{J} \left\{ Y_{i,j} - z_i^\top \Gamma \psi(X_{i,j}) \right\}^2$$

where

1. $z_i = (z_{i,0}, \ldots, z_{i,L})^\top$, $\psi = (\psi_1, \ldots, \psi_K)^\top$
2. $\mathcal{G} = \mathcal{M}(L+1, K)$, $\mathcal{Z} = \{Z \in \mathcal{M}(I, L+1) : z_{i,0} \equiv 1\}$, $\mathcal{M}(a, b)$ is the set of $(a \times b)$ matrices

IVS and DSFM

The implied volatility surface at day i is estimated as

$$\widehat{\sigma}_i(\kappa, \tau) = exp\left\{\widehat{z}_i^\top \widehat{m}(\kappa, \tau)\right\} \tag{2}$$

where

1. $\widehat{m} = (\widehat{m}_0, \ldots, \widehat{m}_L)^\top$
2. $\widehat{m}_l = \widehat{\gamma}_l^\top \psi$
3. $\gamma_l = (\gamma_{l,1}, \ldots, \gamma_{l,K})^\top$

Implied SPD and DSFM

Using (1) the implied SPD may be approximated by

$$\widehat{q}_t(\kappa, \tau, \widehat{z}_t, \widehat{m}) \quad = \quad \varphi(d_2)\left\{\frac{1}{K\widehat{\sigma}_t\sqrt{\tau}} + \frac{2d_1}{\widehat{\sigma}_t}\frac{\partial\widehat{\sigma}_t}{\partial K} + \frac{K\sqrt{\tau}d_1 d_2}{\widehat{\sigma}_t}\left(\frac{\partial\widehat{\sigma}_t}{\partial K}\right)^2 + K\sqrt{\tau}\frac{\partial^2\widehat{\sigma}_t}{\partial K^2}\right\}\Bigg|_{K=S_T}$$

where $\varphi(x)$ is the standard normal pdf, $d_1 = \frac{\log\left(\frac{S_t}{K}\right)+(r+\frac{1}{2}\widehat{\sigma}_t^2)\tau}{\widehat{\sigma}_t\sqrt{\tau}}$ and $d_2 = d_1 - \widehat{\sigma}_t\sqrt{\tau}$

PK and DSFM

As in Ait-Sahalia and Lo (2000) we define an estimate $\widehat{M}_t(\kappa, \tau)$ of the PK as the ratio between the estimated SPD and the estimated p:

$$\widehat{M}_t(\kappa, \tau, \widehat{z}_t, \widehat{m}) = e^{-r_t \tau} \frac{\widehat{q}_t(\kappa, \tau, \widehat{z}_t, \widehat{m})}{\widehat{p}_t(\kappa, \tau)}$$

Here $\widehat{p}_t$ is estimated by a GARCH(1,1) model.

It is our interest to examine the dynamic structure of $\widehat{M}_t$

Empirical Results

Intraday DAX index and option data

1. from 20010101 to 20020101
2. 253 trading days
3. $L = 3$
4. $\widehat{q}_t$ estimated with DSFM
5. $\widehat{p}_t$ estimated from last 240 days with GARCH(1,1)

Figure 6: Loading factors $\widehat{z}_{tl}$, $l = 1, 2, 3$ from the top

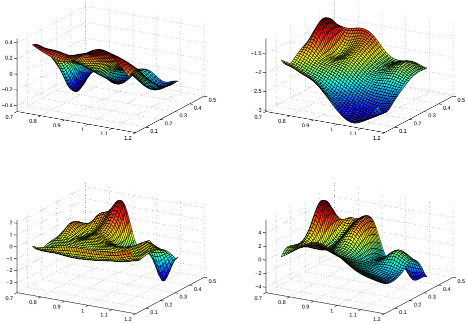


Figure 7: Basis functions $\widehat{m}_l$, $l = 0, \ldots, 3$

	$\widehat{z_{t1}}$	$\widehat{z_{t2}}$	$\widehat{z_{t3}}$
min	0.36	-0.37	-0.07
max	0.75	0.49	0.05
median	0.66	0.01	0.00
mean	0.63	0.00	0.00
std.dev.	0.09	0.05	0.02
u	1.13	0.73	0.07
d	0.18	-0.57	-0.10

Table 1: Descriptive statistics of loading factors.

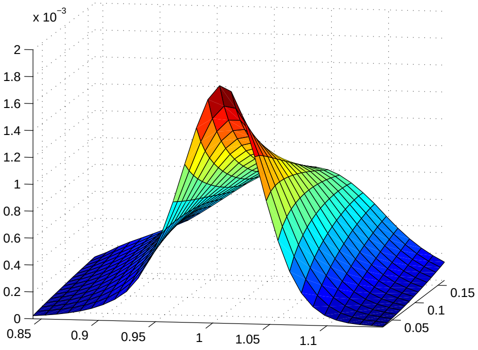


Figure 8: Estimated SPD across κ and τ at $t = 20010710$

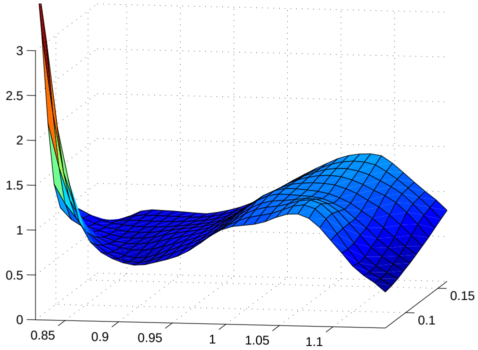
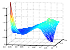


Figure 9: Estimated PK across κ and τ at $t = 20010710$



IV, SPD and PK dynamics

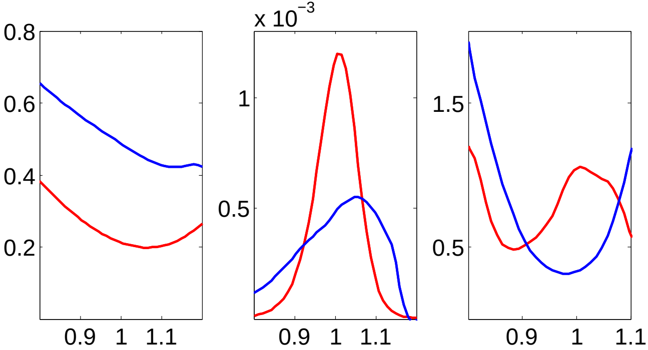


Figure 10: IV (left), SPD (middle) and PK (right), $\tau = 20$ days. Red: $t = 20010824$, $\widehat{z}_{t1} = 0.68$, blue: $t = 20010921$, $\widehat{z}_{t1} = 0.36$

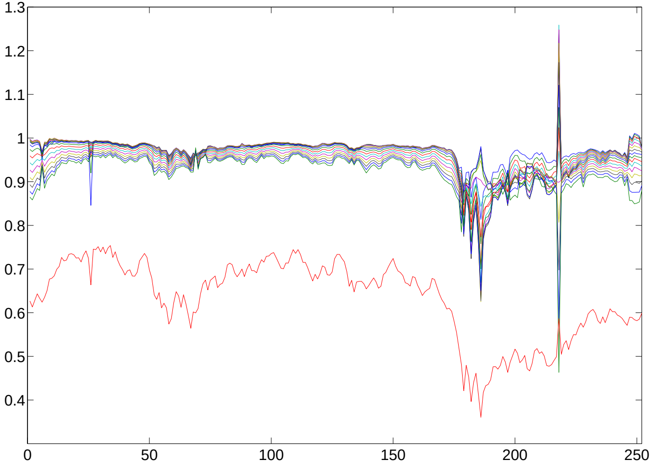


Figure 11: Mean of SPD, $\tau = 18, \ldots, 55$ days, $\widehat{z}_1$ (below)

	ρ		
	$\widehat{z}_1$	$\widehat{z}_2$	$\widehat{z}_3$
Mean	0.66	-0.32	-0.52
Var	-0.53	-0.42	0.11
Skew	-0.86	0.19	0.40

Table 2: Correlation between SPD mean, variance and skewness and loading factors, $\tau = 20$ days

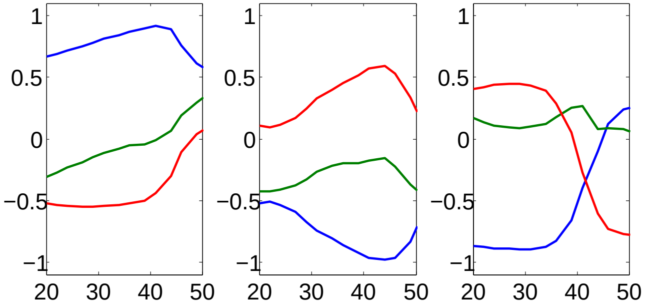
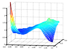


Figure 12: Correlation between $\widehat{z}_l$ and SPD mean (left), variance (middle) and skewness (right), $l = 1$ (blue), 2 (green) and 3 (red). Horizontal axis: τ



Scenario loadings W^l

1. linear increase in N steps on loading of factor l from levels $d_l = \min \widehat{z}_{t,l} - 0.5|\min \widehat{z}_{t,l}|$ to $u_l = \max \widehat{z}_{t,l} + 0.5|\max \widehat{z}_{t,l}|$
2. remaining loading factors constant at median of estimated values
3. scenario loadings to factor l in matrices $W^l = (w^l_{n,j})$, $l, j = 0, \ldots, 3$, and $n = 1, \ldots, N$ with

$$w^l_{n,j} = \left\{ d_j + \frac{n-1}{N-1}(u_j - d_j) \right\} \mathbf{1}(j = l) + med(\widehat{z}_{t,j}) \mathbf{1}(j \neq l)$$

Scenario l

1. IV, SPD and PK estimated with loadings W^l: influence of variations in factor l with remaining factors constant at median
2. observed changes in mean, variance and skewness: typical effect of variation in factor l

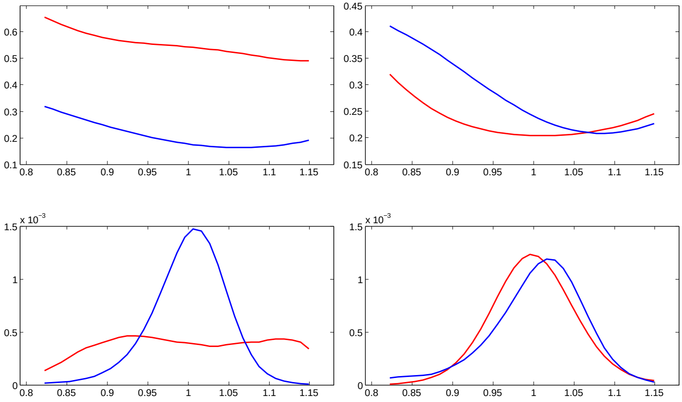


Figure 13: IV (above), SPD (below), for variation in loading factor 1 (left) and 3 (right), $\tau = 20$ days

Scenario W^1: SPD

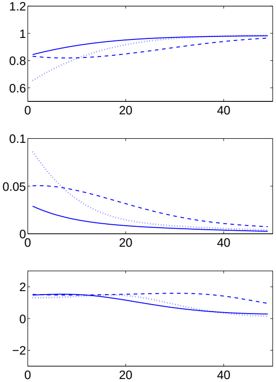


Figure 14: Mean, variance and skewness from SPD (from the top) plotted against n. For W^1, $\tau = 25$ (full), 40 (dotted) and 75 (dashed) days, $N = 50$

Scenario W^2: SPD

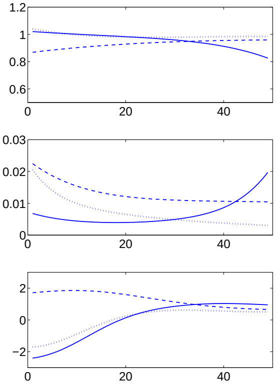


Figure 15: Mean, variance and skewness from SPD (from the top) plotted against n. For W^2, $\tau = 25$ (full), 40 (dotted) and 75 (dashed) days, $N = 50$

Scenario W^3: SPD

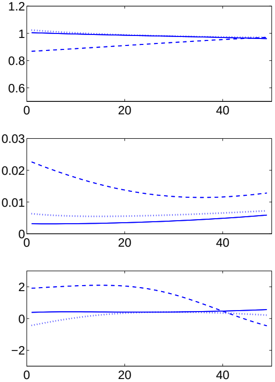


Figure 16: Mean, variance and skewness from SPD (from the top) plotted against n. For W^3, $\tau = 25$ (full), 40 (dotted) and 75 (dashed) days, $N = 50$

Scenario W^1: PK

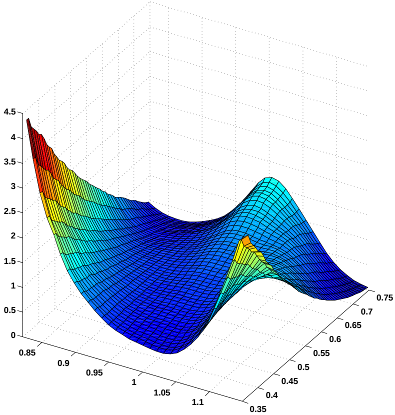


Figure 17: PK(κ) plotted against w_1^1, $\tau = 20$ days

Scenario W^2: PK

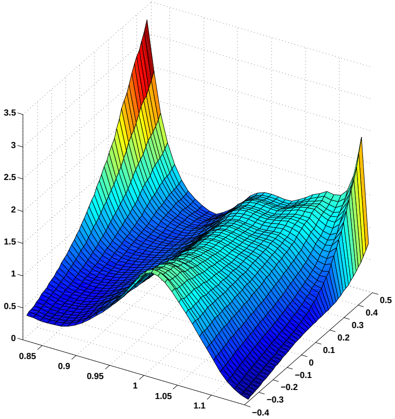
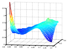


Figure 18: PK(κ) plotted against w_2^2, $\tau = 20$ days



Scenario W^3: PK

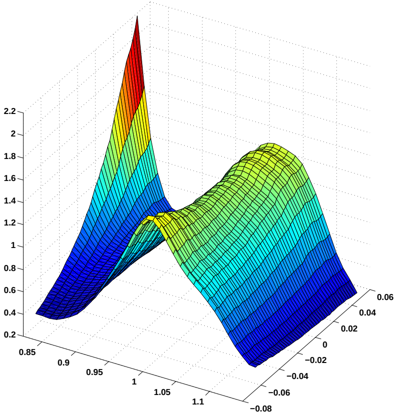


Figure 19: PK(κ) plotted against w_3^3, $\tau = 20$ days

Scenario W^1

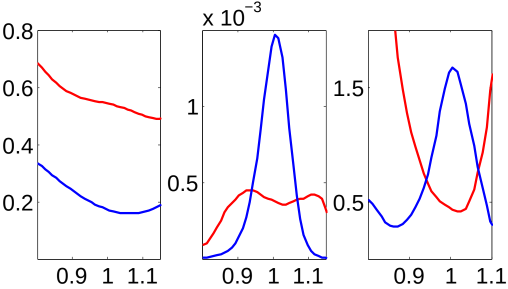
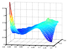


Figure 20: IV (left), SPD (middle) and PK (right), for $w_1^1 = d_1$ (red) and u_1 (blue) $\tau = 20$ days



Scenario W^2

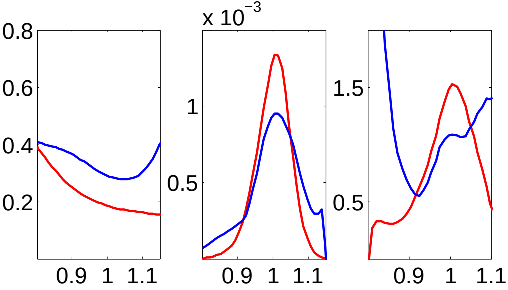


Figure 21: IV (left), SPD (middle) and PK (right), for $w_2^2 = d_2$ (red) and u_2 (blue) $\tau = 20$ days

Scenario W^3

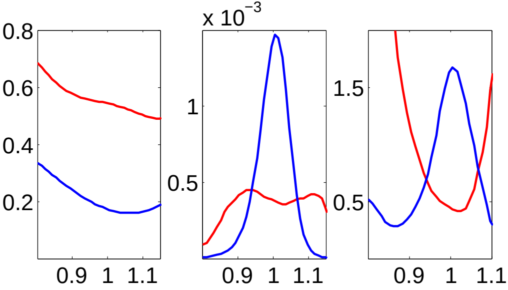
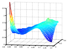


Figure 22: IV (left), SPD (middle) and PK (right), for $w_3^3 = d_3$ (red) and u_3 (blue) $\tau = 20$ days



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