Statistics II: Exercise Session 5

1 Theory of estimation

1.1 Confidence intervals

Confidence interval for expected value μ
Assumption: X_i is normally distributed or the probability distribution of the population is unknown, but $n \geq 30$.

Variance of population σ^2 **is known**	
Confidence interval	$\mathrm{P}\left(\overline{X} - z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} \leq \mu \leq \overline{X} + z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$ $\left[\overline{X} - z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}; \overline{X} + z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right]$
Estimated interval	$\left[\overline{x} - z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}; \overline{x} + z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right]$
Length l	$l = 2 \cdot e = 2 \cdot z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$, where e = error
Sample size	$n \geq \dfrac{\sigma^2 \cdot z^2_{1-\frac{\alpha}{2}}}{e^2}$
Variance of population σ^2 **is unknown**	
Confidence interval	$\mathrm{P}\left(\overline{X} - t_{1-\frac{\alpha}{2};f} \cdot \frac{S}{\sqrt{n}} \leq \mu \leq \overline{X} + t_{1-\frac{\alpha}{2};f} \cdot \frac{S}{\sqrt{n}}\right) = 1-\alpha$ $\left[\overline{X} - t_{1-\frac{\alpha}{2};f} \cdot \frac{S}{\sqrt{n}}; \overline{X} + t_{1-\frac{\alpha}{2};f} \cdot \frac{S}{\sqrt{n}}\right]$
Estimated interval	$\left[\overline{x} - t_{1-\frac{\alpha}{2};f} \cdot \frac{s}{\sqrt{n}}; \overline{x} + t_{1-\frac{\alpha}{2};f} \cdot \frac{s}{\sqrt{n}}\right]$
Length l	$l = 2 \cdot e = 2 \cdot t_{1-\frac{\alpha}{2};f} \cdot \frac{S}{\sqrt{n}}$
Approximative CI for $n > 30$	$\mathrm{P}\left(\overline{X} - z_{1-\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}} \leq \mu \leq \overline{X} + z_{1-\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}}\right) \approx 1-\alpha$ $\left[\overline{X} - z_{1-\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}}; \overline{X} + z_{1-\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}}\right]$

Confidence interval for proportion π for normal approximation
Assumption: $X \sim B(n;p)$ and $\widehat{\Pi}$ is approximately normally distributed.

Approximative CI	$\mathrm{P}\left(\frac{X}{n} - z_{1-\frac{\alpha}{2}} \cdot \sigma_{\widehat{\Pi}} \leq \pi \leq \frac{X}{n} + z_{1-\frac{\alpha}{2}} \cdot \sigma_{\widehat{\Pi}}\right) = 1-\alpha$ $\left[\frac{X}{n} - z_{1-\frac{\alpha}{2}} \cdot \sqrt{\frac{\frac{X}{n}\left(1-\frac{X}{n}\right)}{n}}; \frac{X}{n} + z_{1-\frac{\alpha}{2}} \cdot \sqrt{\frac{\frac{X}{n}\left(1-\frac{X}{n}\right)}{n}}\right]$
Estimated interval	$\left[\frac{x}{n} - z_{1-\frac{\alpha}{2}} \cdot \sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}}; \frac{x}{n} + z_{1-\frac{\alpha}{2}} \cdot \sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}}\right]$
Sample size	$n \geq \dfrac{z^2_{1-\frac{\alpha}{2}}}{4 \cdot e^2}$

2 Hypothesis testing

2.1 Basic concepts

Statistical hypothesis - A statement about the parameters describing a population (not a sample).

Null hypothesis (H_0) - A simple hypothesis associated with a contradiction to a theory one would like to prove.

Statistic - A sample function $V = V(X_1, \ldots, X_n)$, often to summarize the sample for comparison purposes.

Region of rejection / Critical region - The set of values of the test statistic for which the null hypothesis is rejected.

Critical value - The threshold value delimiting the regions of acceptance and rejection for the test statistic.

Power of a test $(1-\beta)$ - The test's probability of correctly rejecting the null hypothesis. The complement of the false negative rate, $\beta = \mathrm{P}("H_0"|H_1)$.

Size / Significance level of a test (α) - For simple hypotheses, this is the test's probability of incorrectly rejecting the null hypothesis, $\mathrm{P}("H_1"|H_0)$.

Type I. Error - Rejecting H_0 when it is true.

Type II. Error - Not rejecting H_0 when it is wrong.

Test		**Null hypothesis** H_0	**Alternative hypothesis** H_1
Both-sided		$\vartheta = \vartheta_0$	$\vartheta \neq \vartheta_0$
One-sided	(left-sided)	$\vartheta \leq \vartheta_0$	$\vartheta > \vartheta_0$

2.2 Test for expected value μ

Variance σ^2		known	unknown
Test statistic V		$\dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$	$\dfrac{\overline{X} - \mu_0}{S/\sqrt{n}}$
$X_i \sim N(\mu;\sigma)$	$n \leq 30$	$N(0{,}1)$	t_{n-1}
	$n > 30$	$N(0{,}1)$	
Any distr.	$n > 30$	$\approx N(0{,}1)$	

2.3 Power function

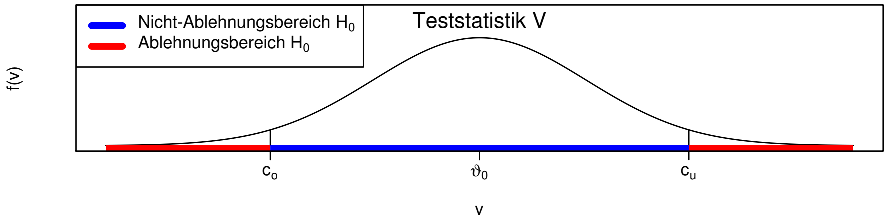
Function of a parameter, giving probability of rejection (based on the settings of hypothesis).

$$G(\vartheta) = P(\text{“}H_1\text{”}|\vartheta) \text{ with } \begin{cases} G(\vartheta) \leq \alpha & \text{for all } \vartheta \in \Theta_0 \\ G(\vartheta) = 1 - \beta(\vartheta) & \text{for all } \vartheta \in \Theta_1 \end{cases}$$

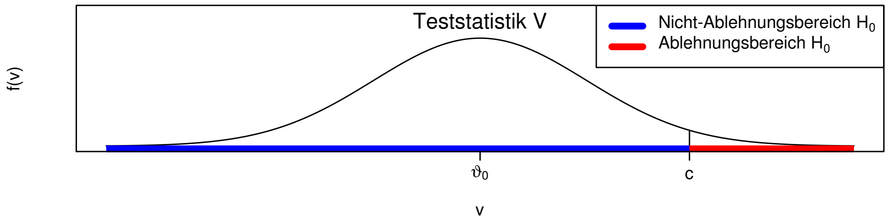
Power function for test of μ

$G(\mu)$ for both-sided test	
$1 - \left[P\left(V \leq z_{1-\frac{\alpha}{2}} - \dfrac{\mu-\mu_0}{\sigma/\sqrt{n}}\right) - P\left(V < -z_{1-\frac{\alpha}{2}} - \dfrac{\mu-\mu_0}{\sigma/\sqrt{n}}\right)\right]$	
$G(\mu)$ for left-sided Test	$G(\mu)$ for right-sided Test
$P\left(V < -z_{1-\alpha} - \dfrac{\mu-\mu_0}{\sigma/\sqrt{n}}\right)$	$1 - P\left(V \leq z_{1-\alpha} - \dfrac{\mu-\mu_0}{\sigma/\sqrt{n}}\right)$

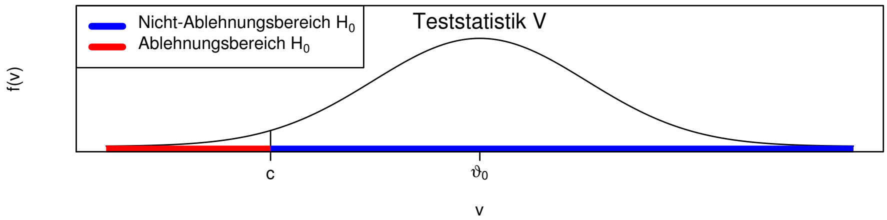
Zweiseitiger Test: $H_0: \vartheta = \vartheta_0$ vs. $H_1: \vartheta \neq \vartheta_0$



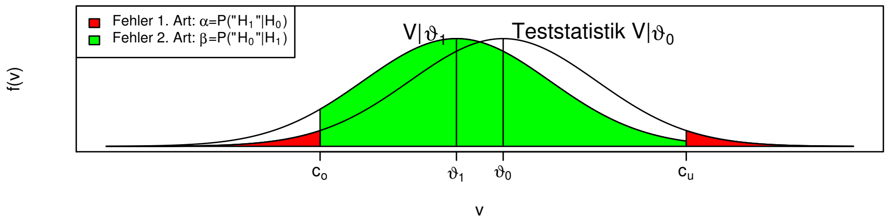
Rechtsseitiger Test: $H_0: \vartheta \leq \vartheta_0$ vs. $H_1: \vartheta > \vartheta_0$



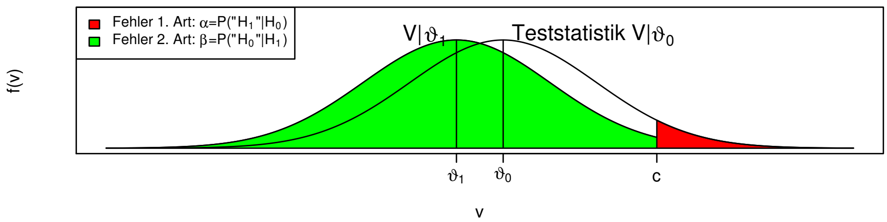
Linksseitiger Test: $H_0: \vartheta \geq \vartheta_0$ vs. $H_1: \vartheta < \vartheta_0$



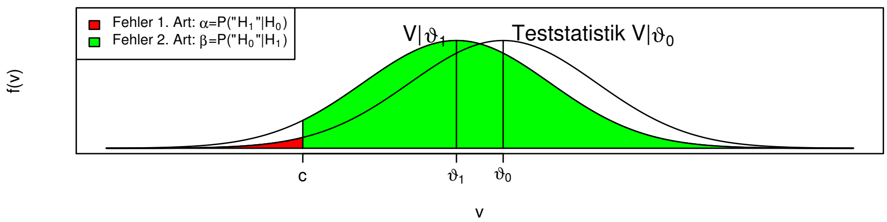
Zweiseitiger Test: $H_0: \vartheta = \vartheta_0$ vs. $H_1: \vartheta \neq \vartheta_0$



Rechtsseitiger Test: $H_0: \vartheta \leq \vartheta_0$ vs. $H_1: \vartheta > \vartheta_0$



Linksseitiger Test: $H_0: \vartheta \geq \vartheta_0$ vs. $H_1: \vartheta < \vartheta_0$



Exercises

7-26: *Dioxin emissions*

It is believed that the dioxin emissions of a cosmetic factory per minute are normally distributed with the mean 5 kg and the deviation 1 kg.

a) What is the probability that the average of a sample of size $n = 9$ is between 4 and 6 kg?

b) What is the area, where the average value lies with the probability of 95 %?

c) How large a sample has to be, so that the average dioxin emissions are exactly estimated with the probability of 95% and an estimation error $e = 0.5$ kg/min?

d) Compute the confidence interval for the average dioxin emissions at the confidence level $1 - \alpha$.

The dioxin emissions were 9 times randomly measured (kg/min):

$$7.0,\ 4.0,\ 5.0,\ 10.0,\ 9.0,\ 6.0,\ 8.0,\ 6.5,\ 7.5.$$

e) Calculate the estimation interval at a confidence level $1 - \alpha = 0.98$.

7-45: *Kilometrage*

A For a test 49 randomly drawn cars of the same type were fuelled with the same amount of fuel. With this amount of fuel the cars went 50 km on average. Assume that the standard deviation of the population is known to be 7km.

  a) Give an explicit confidence interval $[V_L, V_U]$ for average kilometrage μ for this type of cars at the confidence level $1 - \alpha$.

  b) Determine the interval for μ when $1 - \alpha = 95\%$.

  c) What sample size n is needed, if the estimated interval for μ at the same level shall have the width of 2 km?

B Some visitors of this test event were randomly chosen by journalist and asked about their membership in the ADAC (German automobile club). Among 200 people 40 were ADAC members. Determine the interval for π when $1 - \alpha = 99$ %.

C A coffee machine was installed at the tribune for this event. It fills 0.2l cup with coffee. Assume that the quantity is normally distributed. Random sample of size $n = 5$ has following values: 0.18, 0.25, 0.12, 0.20, 0.25.

  a) Give an explicit confidence interval $[V_L, V_U]$ for average quantity μ for this machine at the confidence level $1 - \alpha$.

  b) Determine the interval for μ when $1 - \alpha = 95\%$.

8-1: *Special refrigerators*

A company produces special refrigerators to conserve certain goods. The wished temperature for that type of refrigerators is -25°C. When goods are insufficiently cooled, they go bad very easily and since the client base of the company is not big, defective products would cause the worst case - the ruin of the company. That is why the cooling performance of 100 randomly chosen produced refrigerators shall be tested on a significance level of 2.275 % in order to decide whether the production can be carried on or if constructional changes on the refrigerators need to be done. Experience shows that the cooling temperature is normally distributed with standard deviation of 2°C.

a) What are the hypotheses for this test? Justify.

b) Formulate the underlying sample function formally and verbally and give its distribution under H_0.

c) What is the testing function and what is its distribution under H_0?

d) Determine the region of rejection.

e) Determine the value of power function if the true mean of cooling temperature is:

(i) -24.8°C;

(ii) -25.8°C;

(iii) -29.0°C.

f) Sketch the power function.

g) The random sampling yielded a mean of cooling temperature of -26°C and standard deviation of 1.5°C.

(i) What is the test decision?

(ii) Interpret the test result in an exact way statistically as well as from the context point of view.

h) Random sampling yielded a mean of cooling temperature of -25.3°C.

(i) What is the test decision?

(ii) Interpret the test result in an exact way statistically as well as from the context point of view.

(iii) Which mistake can be made by taking this test decision?

(iv) What is the probability that this mistake has really been made?

(v) How big is the probability to make this mistake when using this test procedure and the real μ is -29 degrees?

i) Why is it sufficient for one-sided test to consider under the null hypothesis only the case $\mu = \mu_0$?