

Kalman Filter

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High Dimensional Nonstationary Time Series

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Not all variables (features) are observable!

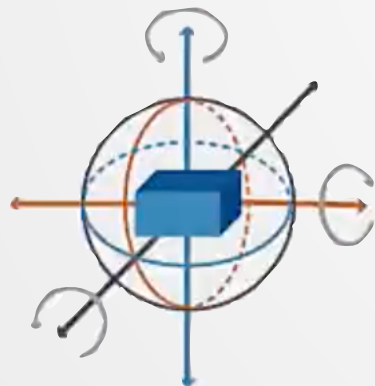
- Variable of interest measured indirectly

- ▶ Position

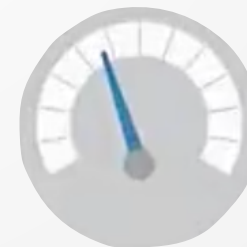


- Measurements might be subjected to noise

- ▶ Acceleration
- ▶ Angular velocity
- ▶ Relative position



Inertial Measurement Unit (IMU)



Odometer



Applications

Navigation

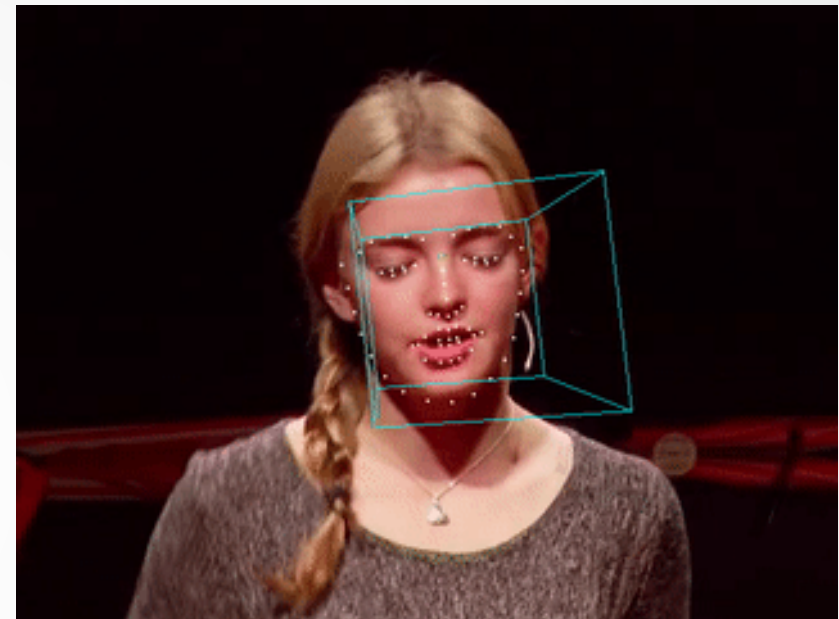


Satellite receiver information

γ

Location

Image Identification



Past photo information

γ

Facial centre in the next frame



APOLLO and Kalman



“[...] not because *[it is]* easy, but because *[it is]* hard; because that goal will serve to organise and measure the best of our energies and skills, because that change is one that we are willing to accept, one we are unwilling to postpone, and one we intend to win [...].”



How about the Temperature inside APOLLO?

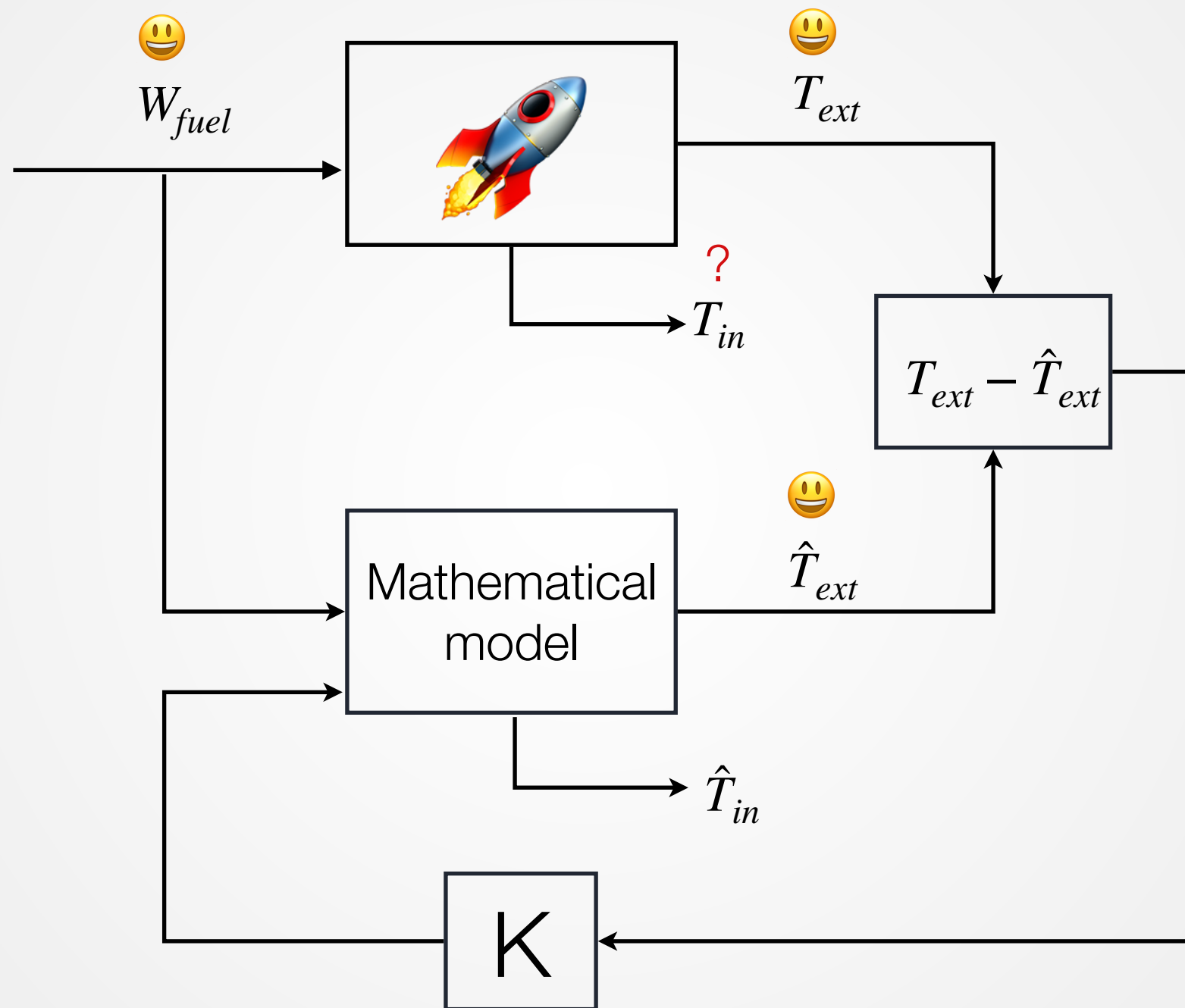
- ▣ Approximate what we not observe
 - ▶ Temperature inside the spaceship: T_{in}
 - ▶ Temperature in extreme situation
- ▣ Observable features 😊
 - ▶ Amount of fuel: W_{fuel}
 - ▶ Temperature outside the spaceship: T_{ext}
 - ▶ Mathematical model:

$$\hat{T}_{in} = B W_{fuel}$$

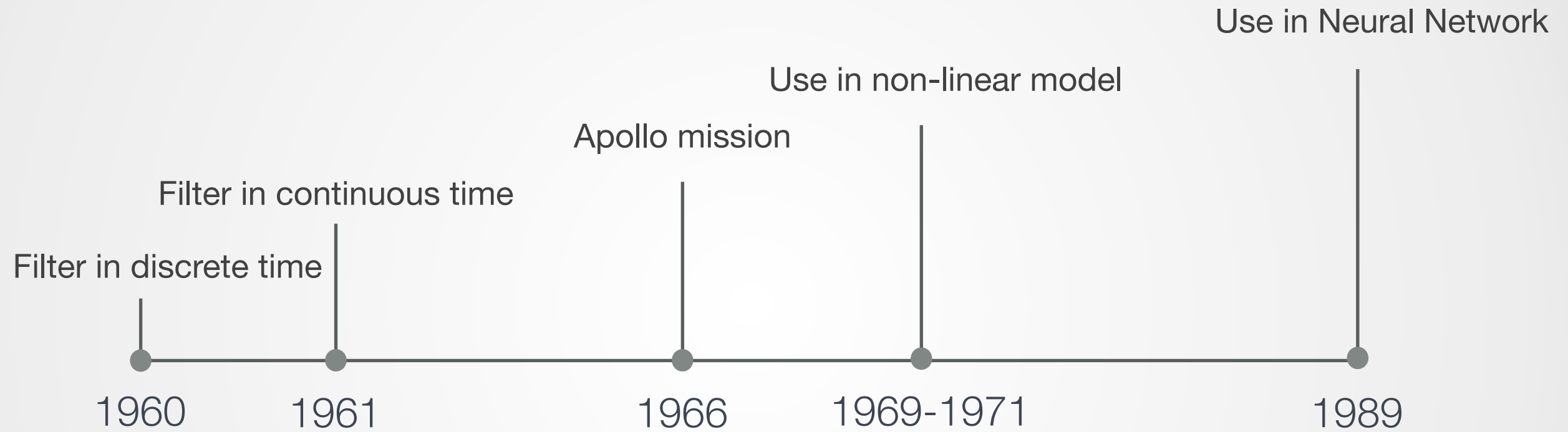
$$\hat{T}_{ext} = C \hat{T}_{in}$$



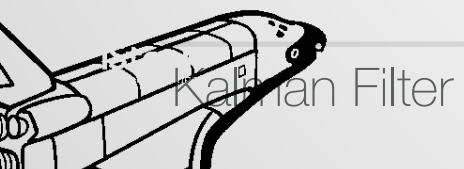
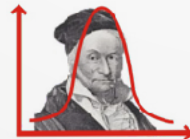
How to Measure the Temperature inside the Spaceship?



Kalman Filter Evolution



Rudolf Emil Kálmán on BBI

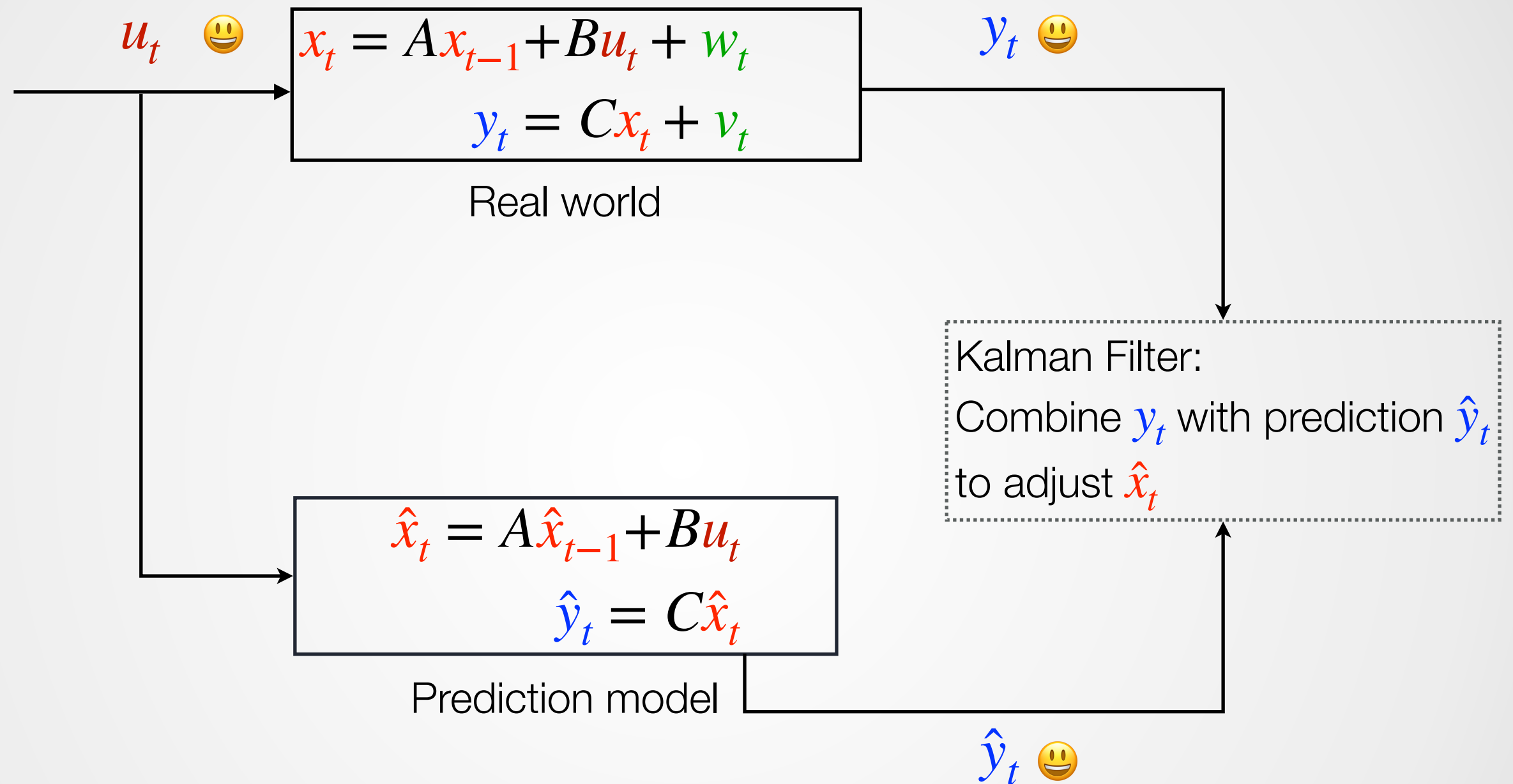


Outline

1. Motivation ✓
2. Algorithm Introduction
3. Application in Finance
4. References



Function of Kalman Filter



Mechanisms of Model, eg for the spaceship: temperature inside, temperature outside, errors, fuel, coefficient matrix



Estimation Process

□ Function of Kalman gain K_t

$$x_t = Ax_{t-1} + Bu_t + w_t$$

$$\hat{x}_t = A\hat{x}_{t-1} + Bu_t$$

$$= K_t v_t$$

$$= K_t (y_{t-1} - \hat{y}_{t-1})$$

$$= K_t C(x_{t-1} - \hat{x}_{t-1})$$

$$\underbrace{x_t - \hat{x}_t}_{e_t^{obj}} = A \underbrace{(x_{t-1} - \hat{x}_{t-1})}_{e_{t-1}^{obj}} + \underbrace{Bu_t - Bu_t}_0 + K_t C \underbrace{(x_{t-1} - \hat{x}_{t-1})}_{e_{t-1}^{obj}}$$

$$e_t^{obj} = (A + K_t C) e_{t-1}^{obj} \quad \Rightarrow \quad e_t^{obj} = e^{(A + K_t C)} e_0^{obj}$$

If $(A + K_t C) < 0$, then $\lim_{t \rightarrow \infty} e_t^{obj} = 0$ and $\lim_{t \rightarrow \infty} x_t = \hat{x}_t$



Estimation Process

- Prediction based on math model:

$$\begin{aligned} \hat{x}_t^- &= A\hat{x}_{t-1} + Bu_t \\ \hat{P}_t^- &= A\hat{P}_{t-1}A^\top + Q \\ \hat{y}_t^- &= C\hat{x}_t^- \end{aligned}$$

Without adjustment of y_t and Kalman gain K_t

Covariance matrix

$\hat{P}_t^-$: $E(x_t - \hat{x}_t^-)(x_t - \hat{x}_t^-)^\top$, covariance of estimation bias presenting uncertainty in prediction

Q : $w_t \sim N(0, Q)$, uncertainty of external environment



Estimation Process

- Update process:

$$K_t = \frac{\hat{P}_t^- C^T}{C \hat{P}_t^- C^T + R}$$

$$\hat{x}_t = \hat{x}_t^- + K_t v_t = \hat{x}_t^- + K_t (y_t - C \hat{x}_t^-)$$

$$\hat{P}_t = (I - K_t C) \hat{P}_t^-$$

$\hat{P}_t$: $E(x_t - \hat{x}_t)(x_t - \hat{x}_t)^T$ covariance of estimation bias of $\hat{x}_t$ after adjustment

R : $v_t \sim N(0, R)$ Uncertainty in measurement

K_t : Parameter for minimising $\hat{P}_t$

- Back to prediction process to estimate the state at $t+1$:

$$\hat{x}_{t+1}^- = A \hat{x}_t + B u_t$$

$$\hat{P}_{t+1}^- = A \hat{P}_t A^T + Q$$

$$\hat{y}_{t+1}^- = C \hat{x}_{t+1}^-$$

It is an application of Bayes

<http://ais.informatik.uni-freiburg.de/teaching/ss10/robotics/slides/10-kalman-filter.pdf>



Extreme Cases of Kalman Filter

- No measurement error

$$\lim_{R \rightarrow 0} K_t = \lim_{R \rightarrow 0} \frac{P_t^- C^\top}{C P_t^- C^\top + R} = \lim_{R \rightarrow 0} \frac{P_t^- C^\top}{C P_t^- C^\top + 0} = \frac{1}{C}$$

$$\hat{x}_t = \frac{1}{C} y_t$$

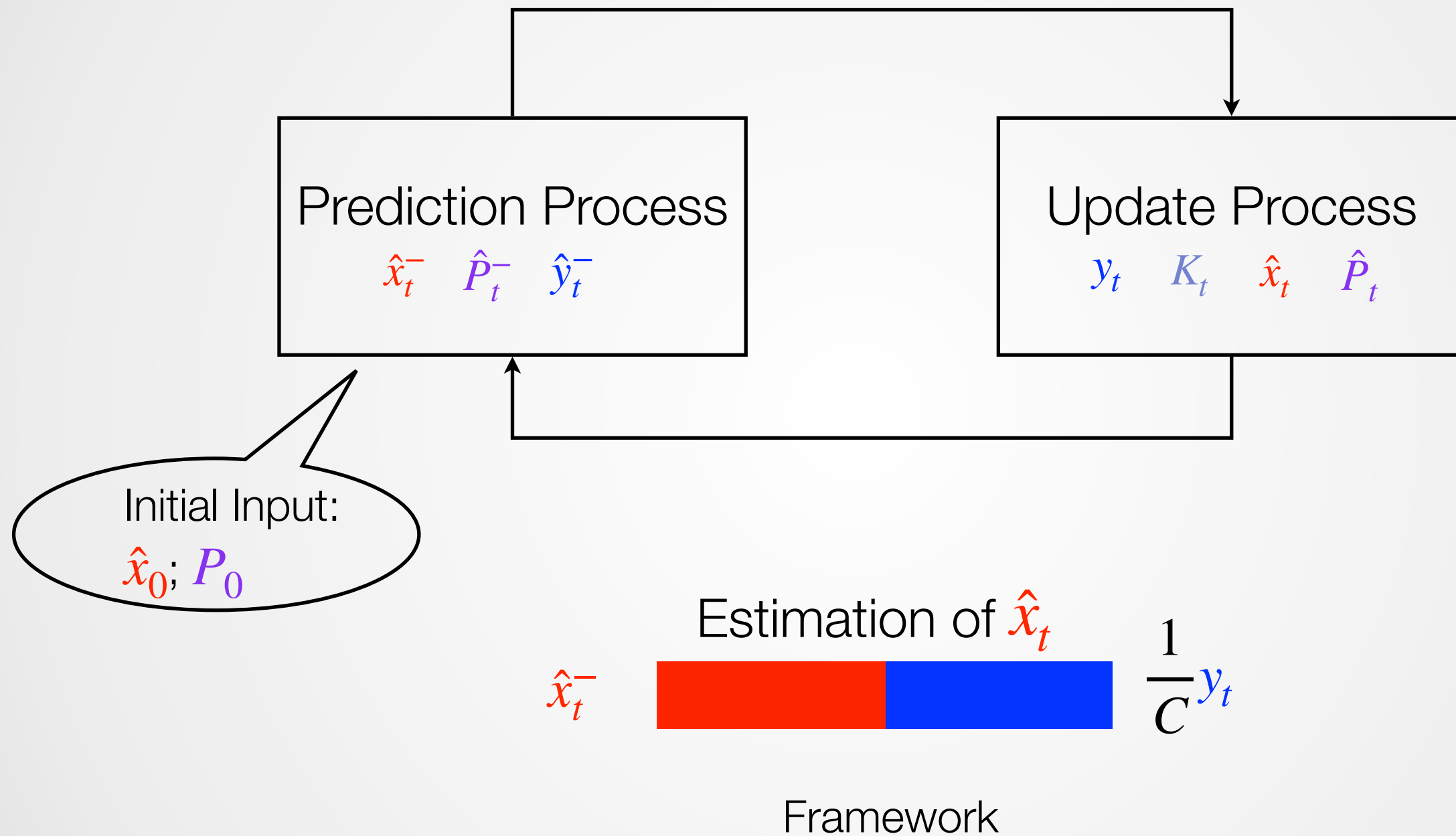
- No estimation error

$$\lim_{P_t^- \rightarrow 0} K_t = \lim_{P_t^- \rightarrow 0} \frac{P_t^- C^\top}{C P_t^- C^\top + R} = 0$$

$$\hat{x}_t = \hat{x}_t^-$$

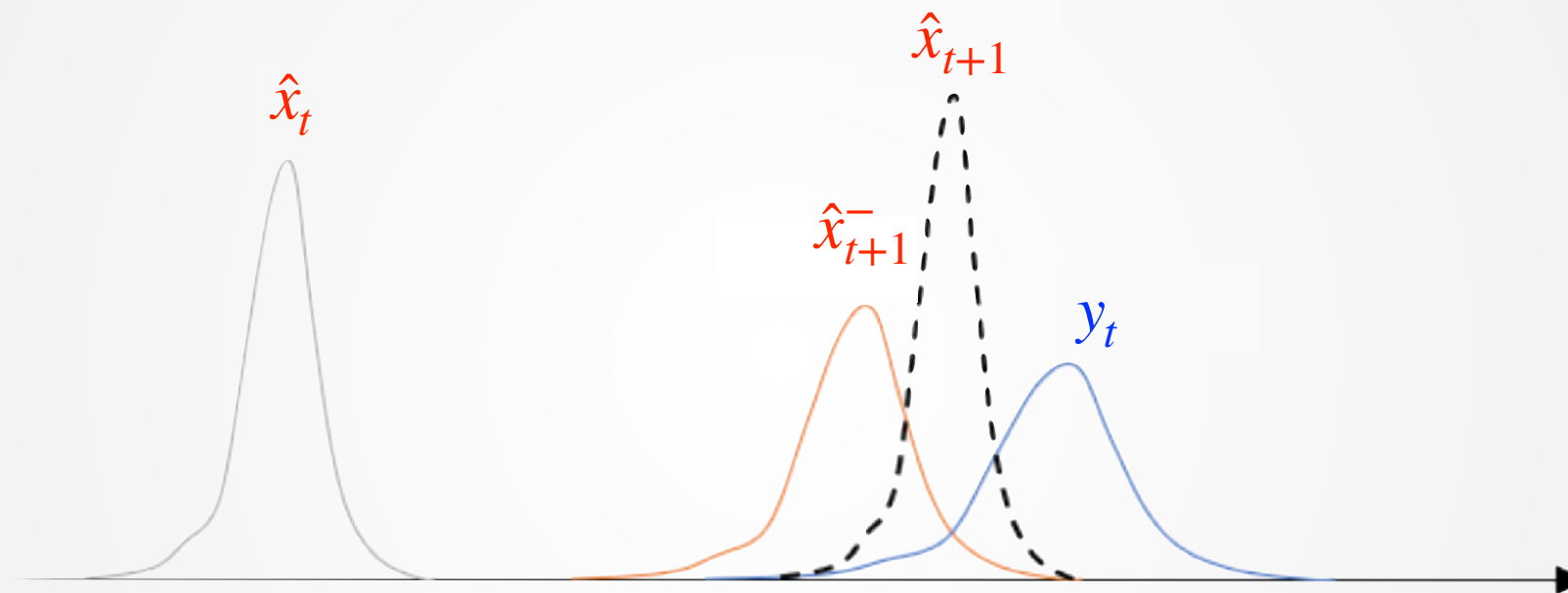


Dynamic Calculation Process



Estimation Process

- Combine error distributions to get precise estimations

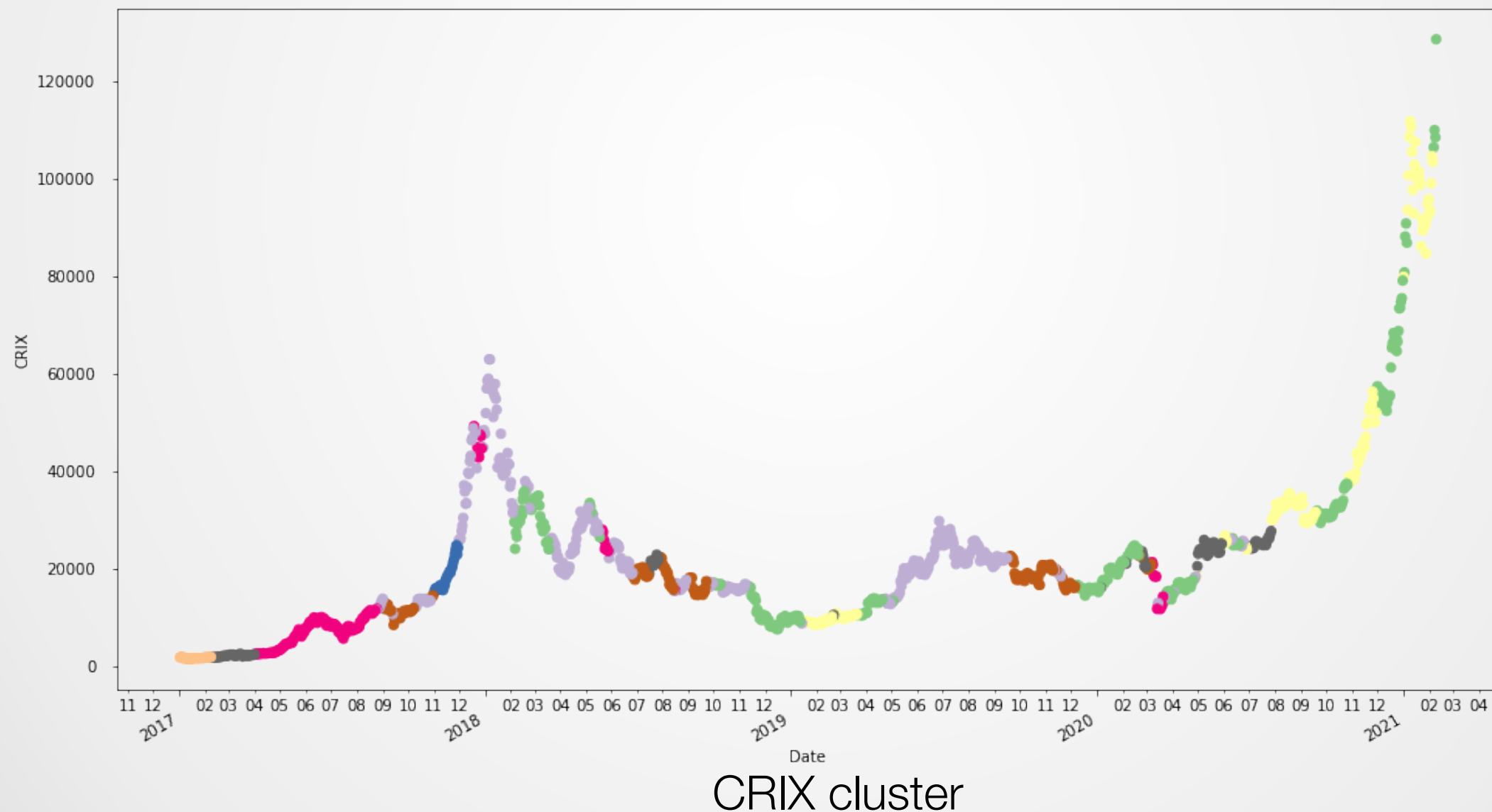


Dynamic Changes of Error Distribution



CRIX Return Prediction KF_modelres

- Kalman filter $\succ$ Return prediction
 - ▶ CRIX: A Laspeyre weighted index of market capitalisation.
 - ▶ Span: 2017-01-02 to 2021-02-09



Estimation Process

- ▣ Prediction based on math model:

$$\hat{r}_{x,t}^- = A \hat{r}_{x,t-1}$$

$$\hat{P}_t^- = A \hat{P}_{t-1} A^\top + Q$$

$$\hat{r}_{y,t}^- = C \hat{r}_{x,t}^-$$

$\hat{r}_{x,t}^-$: Predicted return of next day without adjustment

$\hat{r}_{y,t}^-$: return of today calculated by model

$Q : w_t \sim N(0, 0.03)$

$A = 1 \quad C = 1$

$\hat{r}_{x,0}$: The observable return on 2017-01-02

$\hat{P}_0 = 1$



Estimation Process

□ Update Process:

$$K_t = \frac{\hat{P}_t^-}{\hat{P}_t^- + R}$$

$$\hat{r}_{x,t} = \hat{r}_{x,t}^- + K_t(r_{y,t} - \hat{r}_{x,t}^-)$$

$$\hat{P}_t = (I - K_t)\hat{P}_t^-$$

$\hat{r}_{x,t}$: Predicted return after adjustment

$\hat{r}_{y,t}$: return on day t that we observe

$R : v_t \sim N(0, 0.03)$,

□ Back to prediction process to estimate the state at $t+1$:

$$\hat{r}_{x,t+1}^- = \hat{r}_{x,t}$$

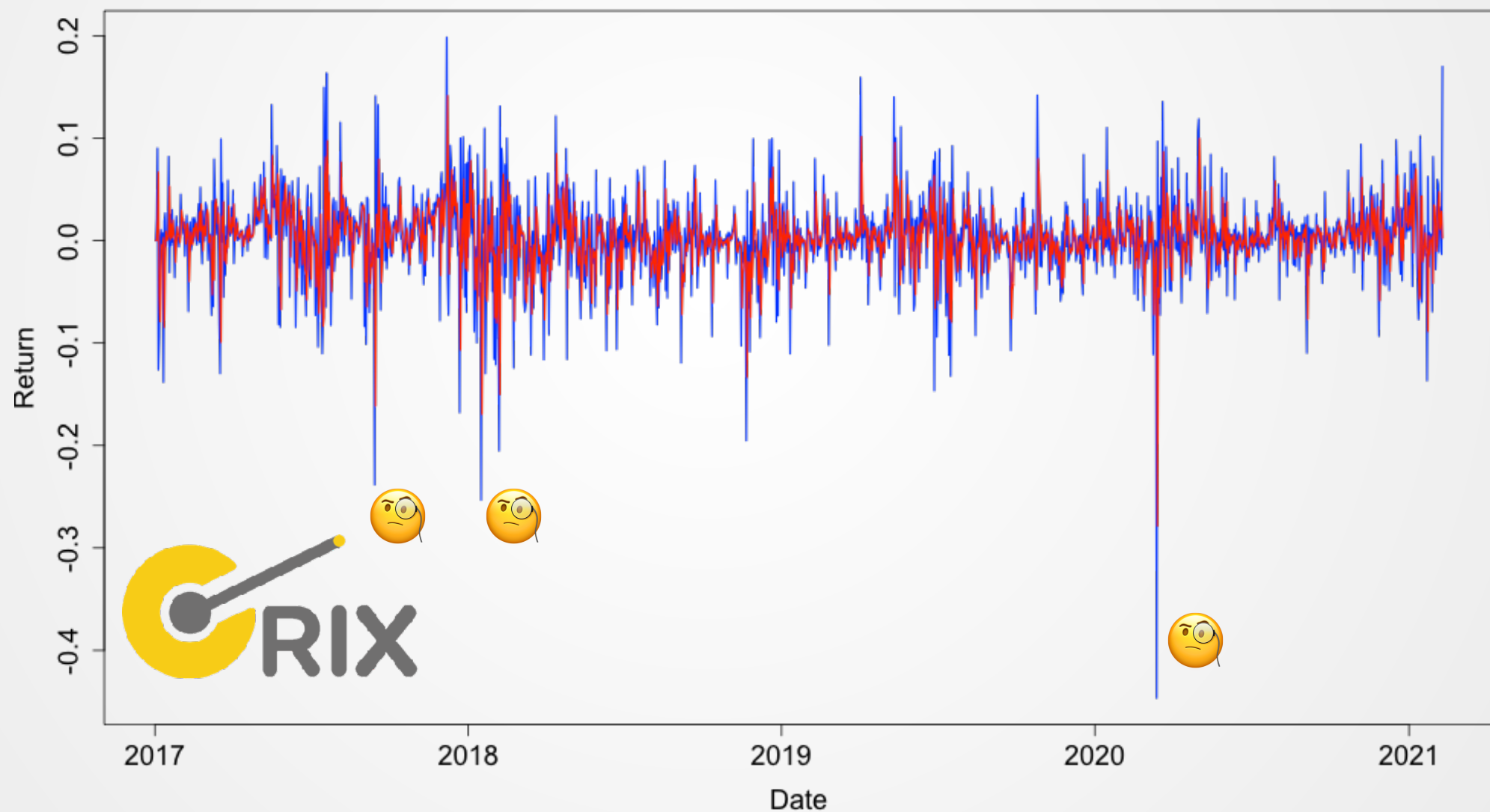
$$\hat{P}_{t+1}^- = \hat{P}_t + Q$$

$$\hat{r}_{y,t+1}^- = \hat{r}_{x,t+1}^-$$



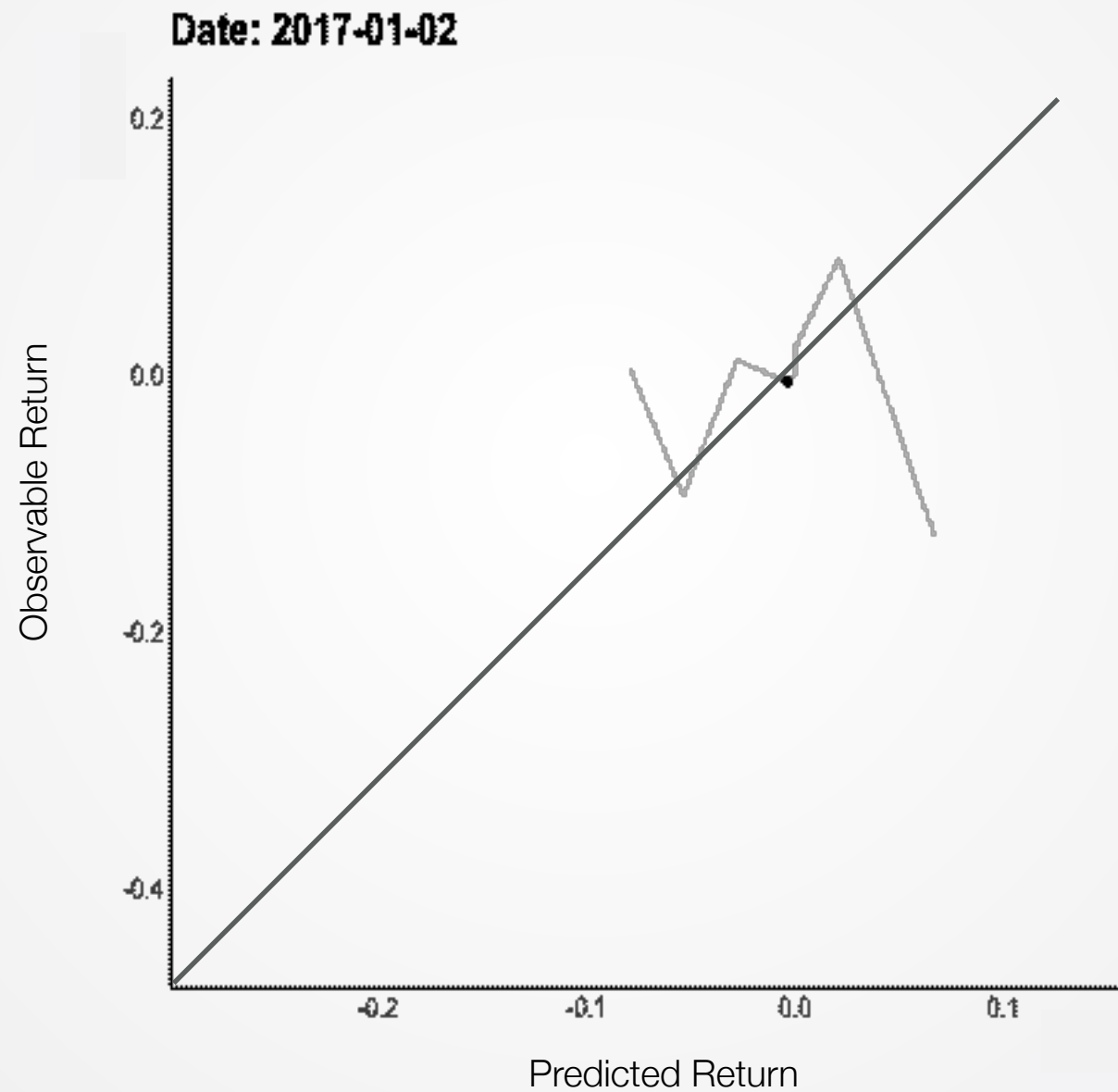
Estimation Process KF_modelres

▣ $Q = 0.03$ $R = 0.03$



Estimation Process KF_modelres

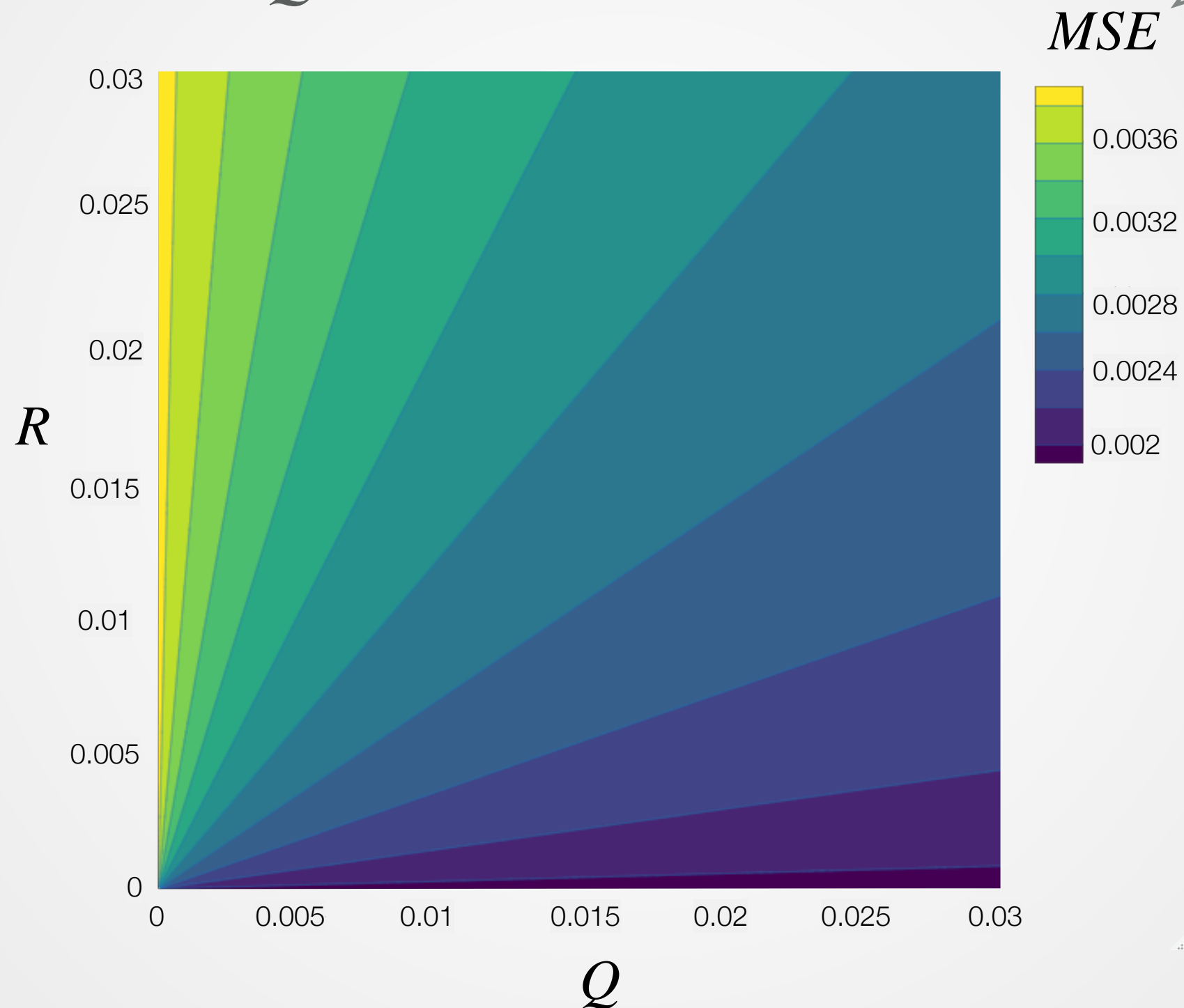
▣ $Q = 0.03 \quad R = 0.03$



Estimation Process

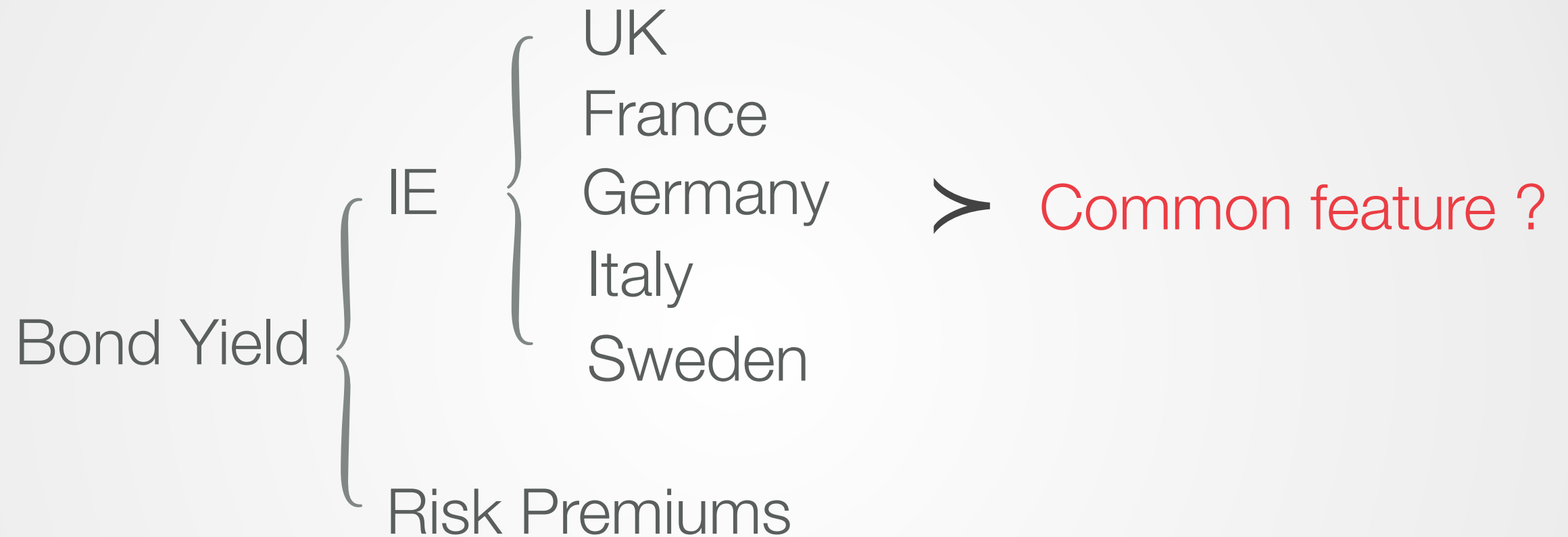
▣ *MSE* of Different Q and R

$$\frac{1}{N} \sum_{t=1}^N (r_{y,t} - \hat{r}_{x,t}^-)^2$$



Other applications

- ▣ The common feature of inflation expectations (IE)



Other applications

- ▣ The common factor of inflation expectations (IE)  MTS

$$\Pi_{t,\tau} = q_{\tau}\Pi_{t-1,\tau} + p_{\tau} + v_{t,\tau}$$

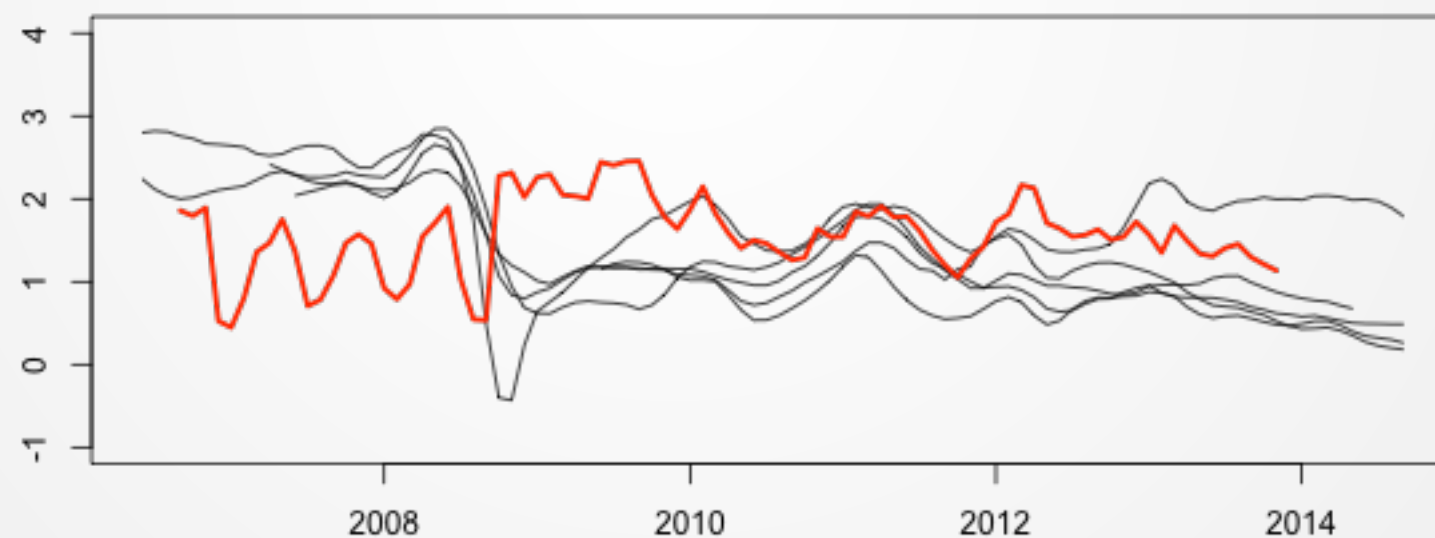
$$\hat{\pi}_{it,\tau} = n_{i,\tau}\Pi_{t,\tau} + m_{i,\tau} + u_{it,\tau}$$

$\Pi_{t,\tau}$: Common feature of τ year IE at time t

$\hat{\pi}_{it,\tau}$: i country's τ year IE at time t

$m_{i,\tau}$, $n_{i,\tau}$, p_{τ} , and q_{τ} are unknown

$v_{it,\tau}$ and $u_{it,\tau}$ are i.i.d



Result of Extraction

3 year IE of five European countries, Common feature (Kalman Filter)



Reference

- ▣ Kalman RE (1960)

A New Approach to Linear Filtering and Prediction Problems

Journal of Basic Engineering, 82: 34–45.

- ▣ Kalman RE, Bucy RS (1961)

New Results in Linear Filtering and Prediction Theory

Journal of Basic Engineering, 83: 95–108.

- ▣ Nkomo R, Kabundi A (2013)

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Working paper 394, Economic Research Southern Africa



Reference

- Chen S, Härdle WK, Wang W (2021) The common and specific components of inflation expectations across European countries. Empir Econ (2021). <https://doi.org/10.1007/s00181-021-02027-1>
- <https://www.zhihu.com/question/23971601>



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1960	Filter in discrete time	Kalman, R. E.(1960). A new approach to linear filtering and prediction problems. <i>Journal of Basic Engineering Transactions</i> , 82, 35-45.
1961	Filter in continuous time	Kalman, R. E., & Bucy, R. S. (1961). New Results in Linear Filtering and Prediction Theory. <i>Trans. ASME, Ser. D, J. Basic Eng</i> (Vol.83, pp.109).
1960s	Use in air and space navigation (Schmidt- Kalman filter)	Schmidt, S. (1966). "Applications of State-space Methods to Navigation Problems". In Leondes, C. (ed.). <i>Advances in Control Systems</i> . 3 . New York, NY: Academic Press. pp. 293–340.
1960s	Use in non-linear field(Extended Kalman Filter, EKF)	Sunahara, Y. (1969). "An approximate method of state estimation for nonlinear dynamical systems," in Proceedings of the Joint Automatic Control Conference, University of Colorado, Boulder, Colo, USA, Bucy, R. S. and Senne, K. D. (1971) "Digital synthesis of non-linear filters," <i>Automatica</i> , vol. 7, no. 3, pp. 287–298, 1971
1989	Use in Neural Network	Singhal, S., & Wu, L. (1989). Training Multilayer Perceptrons with the Extended Kalman Algorithm. <i>neural information processing systems</i> .

