IRTG 1792 Discussion Paper 2018-037

Default probabilities and default correlations under stress

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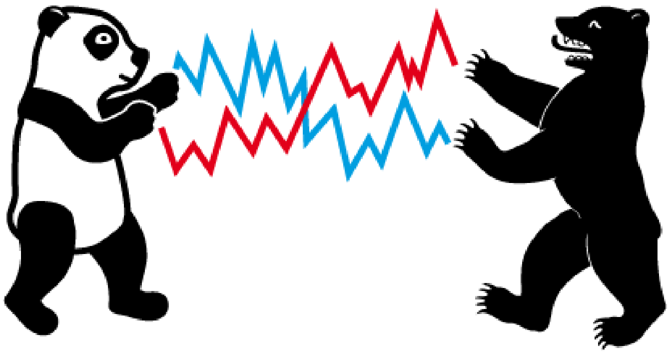

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This research was supported by the Deutsche Forschungsgemeinschaft through the International Research Training Group 1792 "High Dimensional Nonstationary Time Series".

http://irtg1792.hu-berlin.de
ISSN 2568-5619

International Research Training Group 1792

Default probabilities and default correlations under stress

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April 1, 2014

Abstract

We investigate default probabilities and default correlations of Merton-type credit portfolio models in stress scenarios where a common risk factor is truncated. The analysis is performed in the class of elliptical distributions, a family of light-tailed to heavy-tailed distributions encompassing many distributions commonly found in financial modelling. It turns out that the asymptotic limit of default probabilities and default correlations depend on the max-domain of the elliptical distribution's mixing variable. In case the mixing variable is regularly varying, default probabilities are strictly smaller than 1 and default correlations are in $(0, 1)$. Both can be expressed in terms of the Student t-distribution function. In the rapidly varying case, default probabilities are 1 and default correlations are 0. We compare our results to the tail dependence function and discuss implications for credit portfolio modelling.

Keywords: financial risk management, credit portfolio modelling, stress testing, elliptic distribution, max-domain

MSC classification: 60G70, 91G40

1 Introduction

In the aftermath of the subprime crisis and the European sovereign debt crisis, stress testing of bank portfolios has become an integral part of financial risk management and banking supervision (Turner, 2009; Larosière and others, 2009; Brunnermeier *et al.*, 2009; BIS, 2009). Stress tests for credit portfolios are of particular importance, since in a typical bank risk capital for credit risk far outweighs capital requirements for any other risk class.

In this paper, we analyse the behaviour of credit portfolio models under stress depending on the joint distribution of the stochastic variables of the model. Although widely questioned, the industry standard is still to employ multivariate normally distributed random variables. In order to cover a wide range of light-tailed to heavy-tailed distributions we use the family of elliptical distributions, which contains the normal distribution as a special case. More formally, let $Z = (Z_0, \ldots, Z_d)^T$ be a random vector on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$. We assume that Z follows an *elliptical distribution* with representation

$$Z \stackrel{\mathcal{L}}{=} GAU, \tag{1}$$

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The views expressed in this paper are those of the authors and do not necessarily reflect the position of Deutsche Bank AG.

where $G > 0$ is a scalar random variable, the so-called *mixing variable*, A is a deterministic $(d+1) \times (d+1)$ matrix with $AA^T := \Sigma$, which in turn is a $(d+1) \times (d+1)$ nonnegative definite symmetric matrix of rank $d+1$, and U is a $(d+1)$-dimensional random vector uniformly distributed on the unit sphere $\mathcal{S}_{d+1} := \{z \in \mathbb{R}^{d+1} : z^T z = 1\}$, and U is independent of G. Recent papers study the asymptotic properties of value-at-risk in a similar setup, e.g. Embrechts *et al.* (2009); Mainik and Embrechts (2013).

In the next section, we provide a short survey of structural credit portfolio models. In this setting, Z_0 will be interpreted as a risk factor of the model and $Z_1, \ldots, Z_d$ as asset return variables of d firms. The default of the i-th firm is represented by $\{Z_i \leq D_i\}$ for a given default threshold $D_i \in \mathbb{R}$ and the corresponding default probability (PD) is defined by $p_i := \mathbf{P}(Z_i \leq D_i) = \mathbb{E}(\mathbf{1}_{\{Z_i \leq D_i\}})$. The default correlations are defined as the correlations of the default indicators $\mathbf{1}_{\{Z_i \leq D_i\}}$ and $\mathbf{1}_{\{Z_j \leq D_j\}}$. To simplify the exposition, we assume throughout that the correlations of $Z_0, Z_1, \ldots, Z_d$ are in $(0,1)$.

The objective of this paper is to analyse the impact of stress on default probabilities and default correlations. Stress scenarios are specified by truncating the risk factor Z_0, i.e., by conditioning on $\{Z_0 \leq C\}$ with stress level $C \in \mathbb{R}$. Using techniques from Extreme Value Theory (EVT), we derive the limit of conditional default probabilities and default correlations as $C \to -\infty$. The limit depends on whether the mixing variable G is in the max-domain of the Fréchet or the Gumbel distribution, or more generally, on whether the tail distribution function $\mathbf{P}(G > \cdot)$ is regularly varying or rapidly varying. For stressed default probabilities, we show that for any $D_i \in \mathbb{R}$

$$\lim_{C \to -\infty} \mathbf{P}(Z_i \leq D_i | Z_0 \leq C) = 1,$$

if $\mathbf{P}(G > \cdot)$ is rapidly varying. In contrast, if $\mathbf{P}(G > \cdot)$ is regularly varying with tail index $-\alpha$, then

$$\lim_{C \to -\infty} \mathbf{P}(Z_i \leq D_i | Z_0 \leq C) = t_{\alpha+1}\left(\frac{\sqrt{\alpha+1}\,\rho}{\sqrt{1-\rho^2}}\right) \in [1/2, 1),$$

where t_ν denotes the Student t distribution function with parameter ν and ρ denotes the correlation of Z_0 and Z_i. These results imply that the limiting default probability under stress is strictly smaller in the heavy-tailed case than in the light-tailed case. Essentially, in the heavy-tailed case, extreme outcomes are driven by the joint mixing variable, implying a strictly positive probability for a conditional extreme positive outcome of Z_i.

It is interesting to note that this behaviour of limiting default probabilities is fundamentally different to tail dependence, which is positive for heavy-tailed G, and converges to 0 as the tail index of G tends to infinity, that is, to the light-tailed case, see (Schmidt, 2002; Klüppelberg *et al.*, 2008; Hult and Lindskog, 2002). Limiting default correlations, on the other hand, behave like tail dependence: we show that $\mathrm{Corr}(\mathbf{1}_{\{Z_i \leq D_i\}}, \mathbf{1}_{\{Z_j \leq D_j\}} | Z_0 \leq C)$ converges to 0 in the light-tailed case and to a positive number in the heavy-tailed case.

The paper is structured as follows: in Section 2, we define stress tests in structural credit portfolio models. The results on asymptotic stressed default probabilities are derived in section 3. Section 4 focuses on stressed default correlations. In section 5, implications for credit portfolio modelling are discussed.

2 Preliminaries

2.1 Structural credit portfolio models

Depending on their formulation, credit portfolio models can be divided into reduced-form models and structural (or firm-value) models. The progenitor of all structural models is the model of

Merton (Merton, 1974), which links the default of a firm to the relationship between its assets and the liabilities at the end of a given time period $[0, T]$. More precisely, in a structural credit portfolio model the i-th counterparty defaults if its asset return (or ability-to-pay) variable Z_i falls below a default threshold D_i: the default event at time T is defined as $\{Z_i \leq D_i\} \subseteq \Omega$, where Z_i is a real-valued random variable on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and $D_i \in \mathbb{R}$. The portfolio loss variable is defined by

$$L := \sum_{i=1}^{d} l_i \cdot \mathbf{1}_{\{Z_i \leq D_i\}}, \tag{2}$$

where d denotes the number of counterparties and l_i is the loss-at-default of the i-th counterparty. In order to reflect risk concentrations, each Z_i is decomposed into a sum of systematic factors $X_1, \ldots, X_m$, which are often identified with geographic regions or industries, and a firm-specific factor ε_i, that is,

$$Z_i = \sqrt{R_i^2} \sum_{j=1}^{m} w_{ij} X_j + \sqrt{1 - R_i^2} \varepsilon_i. \tag{3}$$

The impact of the risk factors on Z_i is determined by $R_i^2 \in [0, 1]$ and the factor weights $w_{ij} \in \mathbb{R}$.

In order to quantify portfolio risk, measures of risk are applied to the portfolio loss distribution (2). The expected loss of the credit portfolio is used for specifying credit reserves. It is defined as the mean of L:

$$\mathbb{E}(L) = \sum_{i=1}^{d} l_i \cdot p_i,$$

where $p_i = \mathbf{P}(Z_i \leq D_i) = \mathbb{E}(\mathbf{1}_{\{Z_i \leq D_i\}})$ denotes the default probability of the i-th counterparty. Capital requirements for covering unexpected losses are typically derived from the value-at-risk $\mathrm{VaR}_\alpha(L)$ for a predefined probability $\alpha \in (0, 1)$, where $\mathrm{VaR}_\alpha(L)$ is simply defined as the α-quantile of L. Obviously, the default probabilities and risk concentrations specified by the dependence structure of the default variables $\mathbf{1}_{\{Z_i \leq D_i\}}$ determine the value-at-risk of the credit portfolio. Default correlations

$$\mathrm{Corr}(\mathbf{1}_{\{Z_i \leq D_i\}}, \mathbf{1}_{\{Z_j \leq D_j\}}) = \frac{\mathbf{P}(Z_i \leq D_i, Z_j \leq D_j) - p_i \, p_j}{\sqrt{p_i(1 - p_i) p_j (1 - p_j)}}$$

are used as a measure of dependence by portfolio management to identify risk concentrations on counterparty level.

2.2 Distribution of model variables

The standard approach in credit risk management is to model the risk factors and ability-to-pay variables through a joint multi-variate Gaussian distribution. Since the purpose of this paper is to analyze the impact of stress scenarios under different distribution assumptions we use a more general framework and consider elliptical distributions instead.

Elliptical distributions cover a variety of light-tailed to heavy-tailed distributions depending on the tail behaviour of the mixing variable G, i.e., whether G is rapidly varying or regularly varying. A special role is played by *normal variance mixture (NVM) distributions* (see, for instance, McNeil *et al.* (2005); Bingham and Kiesel (2002)). First, NVM distributions encompass a number of distributions commonly used in financial modelling, most prominently normal distributions, t-distributions and symmetric generalised hyperbolic distributions. Second, all elliptical distributions of interest in the credit portfolio context can be represented as NVM distributions: any elliptical distribution whose so-called characteristic generator does not depend on the dimension d can be represented as an NVM distribution, see Theorem 2.21 of Fang *et al.* (1990), or Theorem 3.25 of McNeil *et al.* (2005). For details on elliptical distributions, we refer to (Fang *et al.*, 1990; Cambanis *et al.*, 1981) and for their application in finance and risk management we refer to (McNeil *et al.*, 2005).

2.3 Stress testing in credit portfolio models

In a stress test, credit portfolios are typically evaluated under the assumption of adverse economic conditions. A natural way for implementing stress tests in portfolio models is to translate the stress scenario into constraints on risk factors. In our setup, the constraints are formalised by truncating risk factor variables $X_1, \ldots, X_m$, that is, by conditioning on the range of values that a risk factor may attain. This is a commonly used stress testing technique for credit risk management and capital management of financial institutions, see e.g. (Bonti *et al.*, 2006; Duellmann and Erdelmeier, 2009; Kalkbrener and Packham, 2013). More precisely, let us consider the situation when the risk factor $Z_0 \in \{X_1, \ldots, X_m\}$ is truncated by $C \in \mathbb{R}$, that is, $Z_0 \leq C$ and write

$$\mathbf{P}^C(A) = \mathbf{P}(A|Z_0 \leq C), \quad A \in \mathcal{A},$$

for the corresponding conditional distribution. In this setting, C is interpreted as the level of stress applied to the risk factor Z_0. The objective of this paper is to calculate the limit of default probabilities, joint default probabilities and corresponding default correlations under $\mathbf{P}^C$ as $C \to -\infty$.

3 Default probabilities under stress

Let h be a positive, Lebesgue-measurable function on $(0, \infty)$. We write $h \in RV_\alpha$ if h is regularly varying with index $\alpha \in \mathbb{R}$, i.e.,

$$\lim_{x\to\infty} \frac{h(tx)}{h(x)} = t^\alpha, \qquad t > 0,$$

and $h \in RV_{-\infty}$ if h is rapidly varying with index $-\infty$, i.e.,

$$\lim_{x\to\infty} \frac{h(tx)}{h(x)} = \begin{cases} 0, & t > 1, \\ \infty, & 0 < t < 1. \end{cases}$$

For details on regularly varying functions, we refer to Bingham *et al.* (1987).

Let $Z = GAU$ denote an elliptical random vector as in Equation (1). We assume that all variables are standardised so that $\Sigma = AA^T$ is the correlation matrix of $(Z_0, \ldots, Z_d)^T$. The correlation of Z_i and Z_j is denoted by ρ_{ij}, $i, j = 0, 1, \ldots, d$. We assume that the correlations with respect to the risk factor are positive, i.e., $\rho_{0i} > 0$. The case $\rho_{0i} \leq 0$ can be treated analogously.

In the following, denote by $A_{i\cdot}$ the i-th row of A and let F_U denote the uniform distribution on $\mathcal{S}_{d+1}$.

It is well-known that $\mathbf{P}(G > \cdot) \in RV_{-\alpha}$ implies $\mathbf{P}(Z_i > \cdot) \in RV_{-\alpha}$, $i = 0, \ldots, d$, see e.g. Theorem 7.35 of McNeil *et al.* (2005). For many distributions of interest in $RV_{-\infty}$, such as the normal distribution and the generalised hyperbolic distributions, we know that the mixing variable is in $RV_{-\infty}$, see e.g. Section 7.3 of McNeil *et al.* (2005).

Theorem 1. *(i) If $\mathbf{P}(G > \cdot) \in RV_{-\alpha}$, then*

$$\lim_{C\to-\infty} \mathbf{P}^C(Z_1 \leq D_1, \dots, Z_d \leq D_d)$$
$$= \int_{u\in\mathcal{S}_{d+1}, A_0.u>0,\dots,A_d.u>0} (A_0.u)^\alpha \, \mathrm{d}F_U(u) \left(\int_{u\in\mathcal{S}_{d+1}, A_0.u>0} (A_0.u)^\alpha \, \mathrm{d}F_U(u) \right)^{-1}.$$

(ii) If $\mathbf{P}(G > \cdot) \in RV_{-\infty}$, then

$$\lim_{C\to-\infty} \mathbf{P}^C(Z_1 \leq D_1, \dots, Z_d \leq D_d) = 1.$$

Proof. We first give a proof for the special case $D_i = 0$ for $i = 1, \dots, d$.

Since the elliptical random vector is symmetric and continuous we can write

$$\lim_{C\to-\infty} \mathbf{P}^C(Z_1 \leq 0, \dots, Z_d \leq 0) = \lim_{C\to\infty} \frac{\mathbf{P}(Z_0 > C, Z_1 > 0, \dots, Z_d > 0)}{\mathbf{P}(Z_0 > C)}. \tag{4}$$

For the numerator

$$\begin{aligned}\mathbf{P}(Z_0 > C, Z_1 > 0, \dots, Z_d > 0) &= \mathbf{P}(G > \frac{C}{A_0.U}, A_0.U > 0, \dots, A_d.U > 0) \\ &= \int_{u\in\mathcal{S}_{d+1}, A_i.u>0, i=0,\dots,d} \mathbf{P}\left(G > \frac{C}{A_0.u}\right) F_U(\mathrm{d}u). \end{aligned} \tag{5}$$

For *(i)*, it follows from $\mathbf{P}(G > \cdot) \in RV_{-\alpha}$ that

$$\lim_{C\to\infty} \frac{\mathbf{P}\left(G > C/(A_0.u)\right)}{\mathbf{P}(G > C)} = (A_0.u)^\alpha, \quad \text{for } A_0.u > 0.$$

Potter's bounds (de Haan and Ferreira, 2006, Proposition B.1.9) state that for arbitrary $\varepsilon > 0$ and $\delta > 0$ there exists C_0 such that for all $C \geq C_0$, $C/(A_0.u) \geq C_0$,

$$\frac{\mathbf{P}\left(G > C/(A_0.u)\right)}{\mathbf{P}(G > C)} < (1+\varepsilon)(A_0.u)^\alpha \max\left((A_0.u)^\delta, (A_0.u)^{-\delta}\right),$$

and since the right-hand side is integrable, we obtain by Dominated Convergence that

$$\lim_{C\to\infty} \int_{u\in\mathcal{S}_{d+1}, A_i.u>0, i=0,\dots,d} \frac{\mathbf{P}\left(G > C/(A_0.u)\right)}{\mathbf{P}(G > C)} F_U(\mathrm{d}u) = \int_{u\in\mathcal{S}_{d+1}, A_i.u>0, i=0,\dots,d} (A_0.u)^\alpha \, F_U(\mathrm{d}u).$$

Applying the same method to the denominator of Equation (4) completes the proof of *(i)*.

For *(ii)*, it suffices to consider the case $d = 1$, i.e., $\lim_{C\to\infty} \mathbf{P}(Z_1 > 0 | Z_0 > C) = 1$, since the general case follows from

$$\mathbf{P}(Z_1 > 0, \dots, Z_d > 0 | Z_0 > C) \geq 1 - \sum_{i=1}^{d} (1 - \mathbf{P}(Z_i > 0 | Z_0 > C)).$$

Equality (5) implies

$$\mathbf{P}(Z_1 > 0 | Z_0 > C) =$$
$$= \int_{u\in\mathcal{S}_2, A_i.u>0, i=0,1} \mathbf{P}\left(G > \frac{C}{A_0.u}\right) F_U(\mathrm{d}u) \left(\int_{u\in\mathcal{S}_2, A_0.u>0} \mathbf{P}\left(G > \frac{C}{A_0.u}\right) F_U(\mathrm{d}u) \right)^{-1}.$$

Write $u \in S_2$ in polar coordinates as $u = (\cos\theta, \sin\theta)$, $\theta \in [-\pi, \pi]$, and let A be the Cholesky decomposition of the correlation matrix, i.e., $A_{0\cdot} = (1,0)^T$, $A_{1\cdot} = (\rho, \sqrt{1-\rho^2})$ with $\rho := \rho_{01}$ the correlation of Z_0, Z_1. Hence, $A_{0\cdot}u = \cos\theta > 0$ if $\theta \in (-\pi/2, \pi/2)$ and $A_{1\cdot}u = \rho\cos\theta + \sqrt{1-\rho^2}\sin\theta = \sin(\theta + \arcsin\rho) > 0$ if $\theta \in (-\arcsin\rho, \pi - \arcsin\rho)$. It follows that

$$\int_{u\in\mathcal{S}_2, A_{i\cdot}u>0, i=0,1} \mathbf{P}\left(G > \frac{C}{A_{0\cdot}u}\right) F_U(\mathrm{d}u) = \int_{-\arcsin\rho}^{\pi/2} \mathbf{P}\left(G > \frac{C}{\cos\theta}\right) \mathrm{d}\theta$$
$$= \int_{-\pi/2}^{\pi/2} \mathbf{P}\left(G > \frac{C}{\cos\theta}\right) \mathrm{d}\theta - \int_{\arcsin\rho}^{\pi/2} \mathbf{P}\left(G > \frac{C}{\cos\theta}\right) \mathrm{d}\theta$$

and

$$\int_{u\in\mathcal{S}_2, A_{0\cdot}u>0} \mathbf{P}\left(G > \frac{C}{A_{0\cdot}u}\right) F_U(\mathrm{d}u) = \int_{-\pi/2}^{\pi/2} \mathbf{P}\left(G > \frac{C}{\cos\theta}\right) \mathrm{d}\theta.$$

Since $\rho > 0$, we have $\cos\theta < \cos(\arcsin\rho) = \sqrt{1-\rho^2}$ for $\theta \in (\arcsin\rho, \pi/2)$. Hence, by definition of rapidly varying functions and by Dominated Convergence,

$$\lim_{C\to\infty} \int_{\arcsin\rho}^{\pi/2} \frac{\mathbf{P}\left(G > \frac{C}{\sqrt{1-\rho^2}} \frac{\sqrt{1-\rho^2}}{\cos\theta}\right)}{\mathbf{P}\left(G > \frac{C}{\sqrt{1-\rho^2}}\right)} \mathrm{d}\theta = 0.$$

On the other hand, for $\theta \in (-\arcsin\rho, \arcsin\rho)$,

$$\lim_{C\to\infty} \frac{\mathbf{P}\left(G > \frac{C}{\sqrt{1-\rho^2}} \frac{\sqrt{1-\rho^2}}{\cos\theta}\right)}{\mathbf{P}\left(G > \frac{C}{\sqrt{1-\rho^2}}\right)} = \infty,$$

so that, putting everything together, we obtain

$$\lim_{C\to\infty} \mathbf{P}(Z_1 > 0 | Z_0 > C) = 1.$$

It remains to show that

$$\lim_{C\to-\infty} \mathbf{P}^C(Z_1 \le 0, \ldots, Z_d \le 0) = \lim_{C\to-\infty} \mathbf{P}^C(Z_1 \le D_1, \ldots, Z_d \le D_d), \tag{6}$$

for arbitrary $D_1, \ldots, D_d$. Let $i \in \{1, \ldots, d\}$ and $a > 0$. Note that for $C < -|D_i|/a$,

$$\mathbf{P}^C(Z_i - aZ_0 \le 0) \le \mathbf{P}^C(Z_i \le aC) < \mathbf{P}^C(Z_i \le D_i) < \mathbf{P}^C(Z_i \le -aC) \le \mathbf{P}^C(Z_i + aZ_0 \le 0).$$

Hence,

$$\lim_{C\to-\infty} \mathbf{P}^C(Z_1 - aZ_0 \le 0, Z_2 \le 0, \ldots, Z_d \le 0) \le \lim_{C\to-\infty} \mathbf{P}^C(Z_1 \le D_1, Z_2 \le 0, \ldots, Z_d \le 0)$$
$$\le \lim_{C\to-\infty} \mathbf{P}^C(Z_1 + aZ_0 \le 0, Z_2 \le 0, \ldots, Z_d \le 0).$$

Since Equation (5) is continuous in $A_{1\cdot}$, it follows that $\lim_{C\to-\infty} \mathbf{P}^C(Z_1 + aZ_0 \le 0, Z_2 \le 0, \ldots, Z_d \le 0)$ is a continuous function in $a \in \mathbb{R}$, hence

$$\lim_{C\to-\infty} \mathbf{P}^C(Z_1 \le D_1, Z_2 \le 0, \ldots, Z_d \le 0) = \lim_{C\to-\infty} \mathbf{P}^C(Z_1 \le 0, Z_2 \le 0, \ldots, Z_d \le 0).$$

and therefore (6) is obtained by reiterating this argument. $\square$

Remarks

(i) Default thresholds $D_1, \ldots, D_d$ determine the unconditional default probabilities $\mathbf{P}(Z_i \leq D_i)$. Note, however, that $\lim_{C\to-\infty} \mathbf{P}^C(Z_1 \leq D_1, \ldots, Z_d \leq D_d)$ does not depend on the D_i, i.e., in the limit, stressed default probabilities do not depend on the unconditional default probabilities, but only on the dependence structure of the Z_i.

(ii) Theorem 1 implies in particular that the limiting default probability under stress is strictly smaller than 1 in the heavy-tailed case, provided that the variables are not perfectly correlated. This result can be attributed to the special structure of elliptical distributions, where a stress event may be caused by a large mixing variable and a uniform random vector on the sphere $\mathcal{S}_{d+1}$ with components close to zero, whose signs may well differ, thus overall leading to potentially very large positive or negative realisations of the asset returns. In the light-tailed case, the tail behaviour of the mixing variable is too moderate to produce extreme overall behaviour of opposite signs.

(iii) Even without further making the limiting distribution of part (i) concrete, it can be numerically determined efficiently using Monte Carlo simulation. An efficient method to simulate uniform random variates on the unit sphere $\mathcal{S}_{d+1}$, is to draw $(d+1)$-dimensional independent normally distributed random variables $Y = (Y_0, \ldots, Y_d)^T$, and transform them according to $Y/\|Y\|$, which produces the desired random variates, see e.g. Corollary 3.23 of McNeil *et al.* (2005) or Section 3.4.1.E. of Knuth (1998). Simulation has proven to be significantly faster than the numerical calculation of integrals in Proposition 3 below.

In the following two propositions, we express the integral in Theorem 1(i) in terms of beta functions: the incomplete beta function $B(z; a, b)$ is defined by

$$B(z; a, b) := \int_0^z u^{a-1}(1-u)^{b-1}\,\mathrm{d}u = 2\int_0^{\arcsin(\sqrt{z})} (\sin\theta)^{2a-1}(\cos\theta)^{2b-1}\,\mathrm{d}t, \tag{7}$$

where the last equation follows from substituting $u = (\sin\theta)^2$. The regularized incomplete beta function is defined as

$$I_x(a, b) = \frac{B(x; a, b)}{B(a, b)}, \tag{8}$$

where $B(a, b) := B(1, a, b)$. Note that there exists the following relationship between an incomplete beta function and the distribution function t_ν of the Student-t distribution with parameter ν:

$$t_\nu(x) = \begin{cases} \frac{1}{2} I_{\nu/(x^2+\nu)}\left(\frac{\nu}{2}, \frac{1}{2}\right), & x \leq 0, \\ \frac{1}{2}\left[1 + I_{x^2/(x^2+\nu)}\left(\frac{1}{2}, \frac{\nu}{2}\right)\right], & x > 0. \end{cases} \tag{9}$$

Proposition 2 covers the case $d = 1$, which corresponds to stressed default probabilities, whereas Proposition 3 deals with stressed bivariate default probabilities.

Proposition 2. *Let* $\mathbf{P}(G > \cdot) \in RV_{-\alpha}$. *Then,*

$$\lim_{C\to-\infty} \mathbf{P}(Z_1 \leq D_1 | Z_0 \leq C) = \frac{1}{2} + \frac{1}{2} I_{\rho^2}\left(\frac{1}{2}, \frac{\alpha+1}{2}\right) = t_{\alpha+1}\left(\frac{\sqrt{\alpha+1}\,\rho}{\sqrt{1-\rho^2}}\right) \in [1/2, 1), \tag{10}$$

where $\rho := \rho_{01}$ *denotes the correlation of* Z_0 *and* Z_1.

Proof. By Theorem 1(i),

$$\lim_{C\to-\infty} \mathbf{P}^C(Z_1 \le D_1) = \int_{u\in\mathcal{S}_2, A_{0\cdot}u>0, A_{1\cdot}u>0} (A_{0\cdot}u)^\alpha \, \mathrm{d}F_U(u) \left(\int_{u\in\mathcal{S}_2, A_{0\cdot}u>0} (A_{0\cdot}u)^\alpha \, \mathrm{d}F_U(u) \right)^{-1}.$$

Write $u \in \mathcal{S}_2$ in polar coordinates as $u = (\cos\theta, \sin\theta)$, $\theta \in [-\pi, \pi]$. As in the proof of Theorem 1(ii) we obtain

$$\lim_{C\to-\infty} \mathbf{P}^C(Z_1 \le D_1) = \int_{-\arcsin\rho}^{\pi/2} (\cos\theta)^\alpha \, \mathrm{d}\theta \left(\int_{-\pi/2}^{\pi/2} (\cos\theta)^\alpha \, \mathrm{d}\theta \right)^{-1}.$$

Using Equalities (7) and (8) yields

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} (\cos\theta)^\alpha \, \mathrm{d}\theta &= B\left(\frac{1}{2}, \frac{\alpha+1}{2}\right), \\
\int_{-\arcsin\rho}^{\pi/2} (\cos\theta)^\alpha \, \mathrm{d}\theta &= \int_0^{\arcsin\rho} (\cos\theta)^\alpha \, \mathrm{d}\theta + \frac{1}{2} B\left(\frac{1}{2}, \frac{\alpha+1}{2}\right) \\
&= \frac{1}{2} B\left(\rho^2; \frac{1}{2}, \frac{\alpha+1}{2}\right) + \frac{1}{2} B\left(\frac{1}{2}, \frac{\alpha+1}{2}\right)
\end{aligned}$$

and therefore

$$\lim_{C\to-\infty} \mathbf{P}^C(Z_1 \le D_1) = \frac{1}{2} + \frac{1}{2} I_{\rho^2}\left(\frac{1}{2}, \frac{\alpha+1}{2}\right).$$

The claim follows by observing that this expression corresponds to the respective Student t-distribution function, see Equation (9). □

We shall also assume in the following proposition that $\rho_{12} \ge \rho_{01}\rho_{02}$, which expresses that the specific components of Z_1 and Z_2 are correlated in a non-negative way. The sole reason for this assumption is to avoid awkward case differentiations, and it can easily be lifted.

Proposition 3. *Let* $\mathbf{P}(G > \cdot) \in RV_{-\alpha}$*. Then,*

$$\begin{aligned}
\lim_{C\to-\infty} \mathbf{P}(Z_1 \le D_1, Z_2 \le D_2 | Z_0 \le C) &= \frac{1}{2} t_{\alpha+1}\left(\frac{\sqrt{(\alpha+1)}\, t}{\sqrt{1-t^2}} \right) \\
&+ \int_{-\arcsin t}^{\pi/2} \left[\frac{1}{2} - t_{\alpha+2}\left(-\left| \frac{\sqrt{\alpha+2}}{q_3(\varphi)} \right| \right) \right] (\cos\varphi)^\alpha \, \mathrm{d}\varphi \left(B\left(\frac{1}{2}, \frac{\alpha+1}{2}\right) \right)^{-1} \\
&+ \int_{-\arcsin\rho_{01}}^{-\arcsin t} t_{\alpha+2}\left(-\left| \frac{\sqrt{\alpha+2}}{q_3(\varphi)} \right| \right) (\cos\varphi)^\alpha \, \mathrm{d}\varphi \left(B\left(\frac{1}{2}, \frac{\alpha+1}{2}\right) \right)^{-1},
\end{aligned} \tag{11}$$

where $q_3(\varphi) = \dfrac{\sqrt{1-\rho_{02}^2 - q_1^2}}{\rho_{02}\cos\varphi + q_1 \sin\varphi}$ *and* $q_1 = \dfrac{\rho_{12} - \rho_{01}\rho_{02}}{\sqrt{1-\rho_{01}^2}}$ *and* $t := \rho_{01} \wedge \frac{\rho_{02}}{\sqrt{q_1^2+\rho_{02}^2}}$.

Proof. Write $u \in \mathcal{S}_3$ in polar coordinates as $u = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^T$, with $\theta \in [0, \pi]$, $\varphi \in [-\pi, \pi]$. Let A be given by $A_{0\cdot} = (1, 0, 0)^T$, $A_{1\cdot} = (\rho_{01}, \sqrt{1-\rho_{01}^2}, 0)^T$ and $A_{2\cdot} = (\rho_{02}, q_1, q_2)^T$ with $q_1 = \dfrac{\rho_{12} - \rho_{01}\rho_{02}}{\sqrt{1-\rho_{01}^2}}$ and $q_2 = \sqrt{1-\rho_{02}^2 - q_1^2}$. We have $A_{0\cdot}u = \sin\theta\cos\varphi > 0$ if $\varphi \in (-\pi/2, \pi/2)$ and $\theta \in (0, \pi)$. For $A_{1\cdot}u = \rho_{01}\sin\theta\cos\varphi + \sqrt{1-\rho_{01}^2}\sin\theta\sin\varphi = \rho_{01}\sin\theta\sin(\varphi + \arcsin\rho_{01}) > 0$ we obtain $\varphi \in (-\arcsin\rho_{01}, \pi - \arcsin\rho_{01})$ on $\theta \in (0, \pi)$. Finally, for $A_{2\cdot}u = \sin\theta(\rho_{02}\cos\varphi + q_1\sin\varphi) + q_2\cos\theta > 0$ we need to distinguish four cases: First, if $q_4(\varphi) :=$

$\rho_{02}\cos\varphi + q_1 \sin\varphi > 0$ and $\theta \in (0, \pi/2)$, then $A_2.u > 0$ if $\tan\theta > -q_2/q_4$, which is fulfilled for all $\theta \in (0,\pi)$. Second, if $q_4 > 0$ and $\theta \in (\pi/2, \pi)$, then $A_2.u > 0$ if $\tan\theta < -q_2/q_4$, which implies $\theta < \pi - \arctan(q_2/q_4) = \pi - \arcsin\left(\frac{q_3}{\sqrt{1+q_3^2}}\right)$, where $q_3(\theta) := q_2/q_4$. Third, for $q_4 < 0$ and $\theta \in (0,\pi/2)$, we have $A_2.u > 0$ if $\tan\theta < -q_2/q_4 = q_2/|q_4|$, so that $\theta < \arctan(q_2/|q_4|) = \arcsin(\frac{|q_3|}{\sqrt{1+q_3^2}})$. Fourth, if $q_4 < 0$ and $\theta \in (\pi/2,\pi)$, then $A_2.u > 0$ if $\tan\theta > q_2/|q_4|$, but this is not fulfilled for any $\theta \in (\pi/2,\pi)$, since $\tan\theta < 0$. Finally, we have $q_4 = \rho_{02}\cos\varphi + q_1\sin\varphi > 0$ for $\varphi \in (-\pi/2, \pi/2)$ if $\tan\varphi > -\rho_{02}/q_1$, resp. $\varphi > \arctan(-\rho_{02}/q_1) = \arcsin(-\rho_{02}/\sqrt{q_1^2+\rho_{02}^2})$. Putting everything together, we obtain

$$\lim_{C\to-\infty} \mathbf{P}(Z_1 \le D_1, Z_2 \le D_2 | Z_0 \le C) = \left(\int_{-\pi/2}^{\pi/2}\int_0^{\pi} (\sin\theta)^{\alpha+1}(\cos\varphi)^\alpha \,\mathrm{d}\theta\,\mathrm{d}\varphi\right)^{-1}$$
$$\Bigg[\int_t^{\pi/2}\int_0^{\pi/2}(\sin\theta)^{\alpha+1}(\cos\varphi)^\alpha\,\mathrm{d}\theta\,\mathrm{d}\varphi + \int_t^{\pi/2}\int_{\pi/2}^{\pi-\arcsin(q_3/\sqrt{1+q_3^2})}(\sin\theta)^{\alpha+1}(\cos\varphi)^\alpha\,\mathrm{d}\theta\,\mathrm{d}\varphi$$
$$+\int_{-\arcsin\rho_{01}}^{t}\int_0^{\arcsin(q_3/\sqrt{1+q_3^2})}(\sin\theta)^{\alpha+1}(\cos\varphi)^\alpha\,\mathrm{d}\theta\,\mathrm{d}\varphi\Bigg],$$

with $t := -\arcsin(\min(\rho_{01}, \rho_{02}/\sqrt{q_1^2+\rho_{02}^2}))$.

First,

$$\int_{-\pi/2}^{\pi/2}\int_0^{\pi}(\sin\theta)^{\alpha+1}(\cos\varphi)^\alpha\,\mathrm{d}\theta\,\mathrm{d}\varphi = B\left(\frac{\alpha+2}{2},\frac{1}{2}\right)B\left(\frac{1}{2},\frac{\alpha+1}{2}\right).$$

Second,

$$\int_t^{\pi/2}\int_0^{\pi/2}(\sin\theta)^{\alpha+1}(\cos\varphi)^\alpha\,\mathrm{d}\theta\,\mathrm{d}\varphi$$
$$= \frac{1}{4}B\left(\frac{\alpha+2}{2},\frac{1}{2}\right)\left[B\left(\frac{1}{2},\frac{\alpha+1}{2}\right) + B\left(\rho_{01}^2 \wedge \frac{\rho_{02}^2}{q_1^2+\rho_{02}^2};\frac{1}{2},\frac{\alpha+1}{2}\right)\right]$$

and

$$\int_t^{\pi/2}\int_{\pi/2}^{\pi-\arcsin(q_3(\varphi)/\sqrt{1+q_3(\varphi)^2})}(\sin\theta)^{\alpha+1}(\cos\varphi)^\alpha\,\mathrm{d}\theta\,\mathrm{d}\varphi$$
$$= \int_t^{\pi/2}\frac{1}{2}\left[B\left(\frac{\alpha+2}{2},\frac{1}{2}\right) - B\left(\frac{q_3(\varphi)^2}{1+q_3(\varphi)^2};\frac{\alpha+2}{2},\frac{1}{2}\right)\right](\cos\varphi)^\alpha\,\mathrm{d}\varphi$$

and

$$\int_{-\arcsin\rho_{01}}^{t}\int_0^{\arcsin(q_3(\varphi)/\sqrt{1+q_3(\varphi)^2})}(\sin\theta)^{\alpha+1}(\cos\varphi)^\alpha\,\mathrm{d}\theta\,\mathrm{d}\varphi$$
$$= \int_{-\arcsin\rho_{01}}^{t}\frac{1}{2}B\left(\frac{q_3(\varphi)^2}{1+q_3(\varphi)^2};\frac{\alpha+2}{2},\frac{1}{2}\right)(\cos\varphi)^\alpha\,\mathrm{d}\varphi.$$

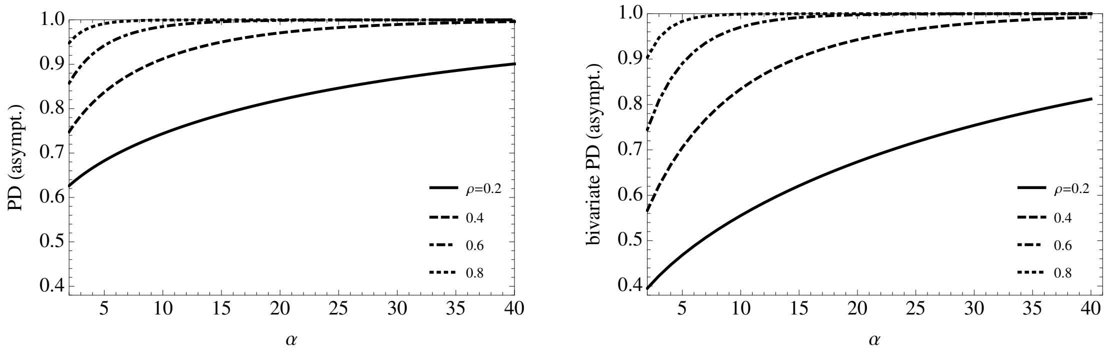

Figure 1: Asymptotic univariate PD's (left) and bivariate PD's (right) as a function of the tail index. Correlations are $\rho_{01} = \rho_{02} = \rho$ and $\rho_{12} = \rho^2$. The choice of ρ_{12} implies that the asset returns are correlated only via the risk factor.

Putting everything together yields

$$
\begin{aligned}
\lim_{C\to-\infty} \mathbf{P}(Z_1 \le D_1, Z_2 \le D_2 | Z_0 \le C) = \frac{1}{4}\left[1 + I_{\rho_{12}^2 \wedge \frac{\rho_{02}^2}{q_1^2+\rho_{02}^2}}\left(\frac{1}{2}, \frac{\alpha+1}{2}\right)\right] \\
+\frac{1}{2}\int_t^{\pi/2}\left[1 - I_{\frac{q_3(\varphi)^2}{1+q_3(\varphi)^2}}\left(\frac{\alpha+2}{2}, \frac{1}{2}\right)\right](\cos\varphi)^\alpha \,\mathrm{d}\varphi \left(B\left(\frac{1}{2}, \frac{\alpha+1}{2}\right)\right)^{-1} \\
+\frac{1}{2}\int_{-\arcsin\rho_{01}}^{t} I_{\frac{q_3(\varphi)^2}{1+q_3(\varphi)^2}}\left(\frac{\alpha+2}{2}, \frac{1}{2}\right)(\cos\varphi)^\alpha \,\mathrm{d}\varphi \left(B\left(\frac{1}{2}, \frac{\alpha+1}{2}\right)\right)^{-1},
\end{aligned}
$$

and replacing the incomplete beta functions by the Student-t distributions, cf. Equation (9), yields the claim. □

Figure 1 shows examples of asymptotic univariate and bivariate PD's for varying tail index α. This demonstrates how PD's depend on both the tail index and the correlations.

4 Default correlations under stress

In the case where $\mathbf{P}(G > \cdot) \in RV_{-\alpha}$, default correlations can be explicitly calculated using the results from Propositions 2 and 3. For the case where G is rapidly varying, we have the following result.

Proposition 4. *Let $\mathbf{P}(G > \cdot) \in RV_{-\infty}$. Then,*

$$\lim_{C\to-\infty} Corr^C(\mathbf{1}_{\{Z_1\le D_1\}}, \mathbf{1}_{\{Z_2\le D_2\}}) = 0,$$

where $Corr^C$ denotes the correlation under $\mathbf{P}^C$.

Proof. It is easily seen that for any probability measure $\mathbf{P}$,

$$\mathrm{Corr}(\mathbf{1}_{\{Z_1\le D_1\}}, \mathbf{1}_{\{Z_2\le D_2\}}) = \frac{\mathbf{P}(Z_1 > D_1, Z_2 > D_2) - (1-\mathbb{E}(\mathbf{1}_{\{Z_1\le D_1\}}))\,(1-\mathbb{E}(\mathbf{1}_{\{Z_2\le D_2\}}))}{\sqrt{\mathbb{E}(\mathbf{1}_{\{Z_1\le D_1\}})\,(1-\mathbb{E}(\mathbf{1}_{\{Z_1\le D_1\}}))}\sqrt{\mathbb{E}(\mathbf{1}_{\{Z_2\le D_2\}})\,(1-\mathbb{E}(\mathbf{1}_{\{Z_2\le D_2\}}))}}. \tag{12}$$

Using Theorem 1, this simplifies in the case of $\mathbf{P}(G > \cdot) \in RV_{-\infty}$ to

$$\lim_{C\to-\infty} \operatorname{Corr}^C(\mathbf{1}_{\{Z_1\le D_1\}}, \mathbf{1}_{\{Z_2\le D_2\}}) = \lim_{C\to-\infty} \frac{\mathbf{P}(Z_1 > D_1, Z_2 > D_2|Z_0 \le C)}{\sqrt{\mathbf{P}(Z_1 > D_1|Z_0 \le C)}\sqrt{\mathbf{P}(Z_2 > D_2|Z_0 \le C)}}$$
$$= \lim_{C\to\infty} \frac{\mathbf{P}(Z_1 \le D_1, Z_2 \le D_2, Z_0 > C)}{\sqrt{\mathbf{P}(Z_1 \le D_1, Z_0 > C)}\sqrt{\mathbf{P}(Z_2 \le D_2, Z_0 > C)}}, \tag{13}$$

where the second equality follows from the symmetry of the Z_i.

Suppose first that $\rho_{01} \ne \rho_{02}$; wlog. let $\rho_{02} > \rho_{01}$. As before, we first prove the claim for $D_1 = D_2 = 0$.

We write (13) in the form

$$\lim_{C\to-\infty} \operatorname{Corr}^C(\mathbf{1}_{\{Z_1\le 0\}}, \mathbf{1}_{\{Z_2\le 0\}}) = \lim_{C\to\infty} \frac{\mathbf{P}(Z_1 \le 0, Z_2 \le 0, Z_0 > C)}{\mathbf{P}(Z_2 \le 0, Z_0 > C)} \sqrt{\frac{\mathbf{P}(Z_2 \le 0, Z_0 > C)}{\mathbf{P}(Z_1 \le 0, Z_0 > C)}}. \tag{14}$$

The first term is bounded by 1. For $C > 0$, write the second term in the form

$$\begin{aligned}
\mathbf{P}(Z_i \le 0, Z_0 > C) &= \mathbf{P}\left(G > \frac{C}{A_0.U}, A_0.U > 0, A_i.U < 0\right) \\
&= \int_{u\in\mathcal{S}_2, A_0.u>0, A_i.u<0} \mathbf{P}\left(G > \frac{C}{A_0.u}\right) F_U(\mathrm{d}u) \\
&= \int_{-\pi/2}^{-\arcsin\rho_{0i}} \mathbf{P}\left(G > \frac{C}{\cos\theta}\right) \mathrm{d}\theta.
\end{aligned}$$

Since $\rho_{02} > \rho_{01}$, we have $-\arcsin\rho_{02} < -\arcsin\rho_{01}$, and

$$\sup_{\theta\in(-\pi/2, -\arcsin\rho_{02}]} \cos\theta = \cos(-\arcsin\rho_{02}) = \sqrt{1-\rho_{02}^2}.$$

Hence, using that G is rapidly varying and via Dominated Convergence,

$$\lim_{C\to\infty} \int_{-\pi/2}^{-\arcsin\rho_{02}} \frac{\mathbf{P}(G > C/\cos\theta)}{\mathbf{P}(G > C/\sqrt{1-\rho_{02}^2})} \mathrm{d}\theta = 0,$$

whereas

$$\lim_{C\to\infty} \frac{\mathbf{P}(G > C/\cos\theta)}{\mathbf{P}(G > C/\sqrt{1-\rho_{02}^2})} = \infty, \quad \text{for } \theta \in (-\arcsin\rho_{02}, -\arcsin\rho_{01}],$$

which implies that

$$\lim_{C\to\infty} \frac{\mathbf{P}(Z_2 \le 0, Z_0 > C)}{\mathbf{P}(Z_1 \le 0, Z_0 > C)} = 0 \tag{15}$$

and therefore Equation (14) is 0.

To complete the proof for $\rho_{02} > \rho_{01}$ it remains to show that

$$\lim_{C\to-\infty} \operatorname{Corr}^C(\mathbf{1}_{\{Z_1\le D_1\}}, \mathbf{1}_{\{Z_2\le D_2\}}) = \lim_{C\to-\infty} \operatorname{Corr}^C(\mathbf{1}_{\{Z_1\le 0\}}, \mathbf{1}_{\{Z_2\le 0\}}) = 0$$

for arbitrary $D_1, D_2 \in \mathbb{R}$. For $a > 0$ and $C > |D_2|/a$,

$$\mathbf{P}(Z_2 + aZ_0 \le 0, Z_0 > C) \le \mathbf{P}(Z_2 \le D_2, Z_0 > C) \le \mathbf{P}(Z_2 - aZ_0 \le 0, Z_0 > C).$$

For sufficiently small a we obtain from (15)

$$\begin{aligned} 0 = \lim_{C\to\infty} \frac{\mathbf{P}(Z_2 + aZ_0 \le 0, Z_0 > C)}{\mathbf{P}(Z_1 \le 0, Z_0 > C)} &\le \lim_{C\to\infty} \frac{\mathbf{P}(Z_2 \le D_2, Z_0 > C)}{\mathbf{P}(Z_1 \le 0, Z_0 > C)} \\ &\le \lim_{C\to\infty} \frac{\mathbf{P}(Z_2 - aZ_0 \le 0, Z_0 > C)}{\mathbf{P}(Z_1 \le 0, Z_0 > C)} = 0 \end{aligned}$$

and therefore

$$\lim_{C\to\infty} \frac{\mathbf{P}(Z_2 \le D_2, Z_0 > C)}{\mathbf{P}(Z_1 \le 0, Z_0 > C)} = 0.$$

Applying the same argument to the numerator yields

$$\lim_{C\to\infty} \frac{\mathbf{P}(Z_2 \le D_2, Z_0 > C)}{\mathbf{P}(Z_1 \le D_1, Z_0 > C)} = 0.$$

Obviously, $\dfrac{\mathbf{P}(Z_1 \le D_1, Z_2 \le D_2 | Z_0 > C)}{\mathbf{P}(Z_2 \le D_2 | Z_0 > C)}$ is bounded by 1. Hence, by Equality (14),

$$\lim_{C\to-\infty} \mathrm{Corr}^C(\mathbf{1}_{\{Z_1 \le D_1\}}, \mathbf{1}_{\{Z_2 \le D_2\}}) = 0.$$

It remains to show that the claim also holds for $\rho_{01} = \rho_{02}$. Wlog. assume that $D_1 \le D_2$. We now provide a proof for the case $D_2 \le 0$. The case $D_2 > 0$ is shown analogously.

It follows from (13) that

$$\lim_{C\to-\infty} \mathrm{Corr}^C(\mathbf{1}_{\{Z_1 \le D_1\}}, \mathbf{1}_{\{Z_2 \le D_2\}}) \le \lim_{C\to\infty} \frac{\mathbf{P}(Z_1 \le 0, Z_2 \le 0, Z_0 > C)}{\mathbf{P}(Z_1 \le D_1, Z_0 > C)}.$$

The numerator can be written in the form

$$\mathbf{P}(Z_1 \le 0, Z_2 \le 0, Z_0 > C) = \int_{A_0.u>0, A_i.u<0, i=1,2} \mathbf{P}\left(G > \frac{C}{A_0.u}\right) F_U(\mathrm{d}u).$$

Express $u \in \mathcal{S}_3$ in polar coordinates by $u = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^T$ with $\theta \in [0,\pi]$, $\varphi \in [-\pi,\pi]$. Observe that the conditions $A_0.u = \sin\theta\cos\varphi > 0$ and $A_1.u = \rho_{01}\sin\theta\,\sin(\varphi + \arcsin\rho_{01}) < 0$ imply $\theta \in (0,\pi)$ and $\varphi \in (-\pi/2, -\arcsin\rho_{01})$. Hence,

$$\begin{aligned} x &:= \sup_{A_0.u>0, A_1.u<0} A_0.u \\ &= \sup_{\theta\in(0,\pi), \varphi\in(-\pi/2, -\arcsin\rho_{01})} \sin\theta\cos\varphi \\ &= \sin(\pi/2)\,\cos(-\arcsin\rho_{01}) \\ &= \cos(-\arcsin\rho_{01}) = \sqrt{1-\rho_{01}^2}. \end{aligned}$$

At $\theta = \pi/2, \varphi = -\arcsin\rho_{01}$, the condition $A_2.u < 0$ is not satisfied: since $A_2.u = \sin\theta(\rho_{02}\cos\varphi + q_1\sin\varphi) + q_2\cos\theta$, where q_1 and q_2 are as in Proposition 3, resp. the proof of Proposition 3 it follows from $\rho_{02} = \rho_{01}$ that

$$A_2.u = \rho_{01}\underbrace{\cos(-\arcsin\rho_{01})}_{=\sqrt{1-\rho_{01}^2}} - \frac{\rho_{12} - \rho_{01}^2}{\sqrt{1-\rho_{01}^2}}\rho_{01} = \frac{\rho_{01}}{\sqrt{1-\rho_{01}^2}}(1-\rho_{12}) \ge 0.$$

This implies that

$$y := \sup_{A_0.u>0, A_1.u<0, A_2.u<0} A_0.u < x.$$

Hence, by the property that G is rapidly varying and via Dominated Convergence,

$$\lim_{C\to\infty} \int_{u\in\mathcal{S}_3, A_0.u>0, A_1.u<0, A_2.u<0} \frac{\mathbf{P}\left(G > C/(A_1.u)\right)}{\mathbf{P}(G > C/y)} F_U(\mathrm{d}u) = 0. \tag{16}$$

For the denominator we have

$$\begin{aligned}\mathbf{P}(Z_1 \le D_1, Z_0 > C) &= \int_{u\in\mathcal{S}_3, A_0.u>0, A_1.u<0} \mathbf{P}\left(G > \max\left(\frac{|D_1|}{|A_1.u|}, \frac{C}{A_0.u}\right)\right) F_U(\mathrm{d}u) \\ &= \int_0^{\pi}\int_{-\pi/2}^{-\arcsin\rho_{01}} \mathbf{P}\left(G > \max\left(f(\theta,\varphi), \frac{C}{\sin\theta\cos\varphi}\right)\right)\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta, \end{aligned} \tag{17}$$

where

$$f(\theta,\varphi) := \frac{|D_1|}{|\rho_{01}\sin\theta\sin(\varphi+\arcsin\rho_{01})|}.$$

Obviously, the maximum in the integrand of (17) is given by $f(\theta,\varphi)$ if

$$\frac{\sin\theta\cos\varphi}{|\rho_{01}\sin\theta\sin(\varphi+\arcsin\rho_{01})|} = \frac{\cos\varphi}{|\rho_{01}\sin(\varphi+\arcsin\rho_{01})|} \ge \frac{C}{|D_1|}.$$

Hence, $f(\theta,\varphi)$ is the maximum in a neighbourhood of $\varphi = -\arcsin\rho_{01}$. On the other hand, for every $\varphi \in (-\pi/2, -\arcsin\rho_{01})$ the maximum is given by $C/(A_0.u)$ for C sufficiently large. Hence, we choose C large enough so that

$$\max\left(\frac{|D_1|}{|\rho_{01}\sin\theta\sin(\varphi+\arcsin\rho_{01})|}, \frac{C}{A_0.u}\right) = \frac{C}{A_0.u} \qquad \text{and} \qquad A_0.u = \sin\theta\cos\varphi > y$$

for $(\theta,\varphi) \in M$, where $M \subset (0,\pi)\times(-\pi/2, -\arcsin\rho_{01})$ is a set of positive Lebesgue measure. Hence, by the property that G is rapidly varying and via Dominated Convergence, we obtain from (17) that

$$\lim_{C\to\infty} \frac{\mathbf{P}(Z_1 \le D_1, Z_0 > C)}{\mathbf{P}(G > C/y)} \ge \lim_{C\to\infty} \int_M \frac{\mathbf{P}\left(G > C/(\sin\theta\cos\varphi)\right)}{\mathbf{P}(G > C/y)} \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta = \infty.$$

Together with (16), this proves the claim. □

That default correlations under stress converge to zero in light-tailed models can be explained as follows: In regression analysis, correlation – expressed as R^2 – measures the degree of the linear relationship between two random variables. In the case of default correlations, there are only four possible scenarios: both variables are zero, both variables are one, and exactly one variable is one and the other is zero (and vice versa). In the light-tailed case, since, asymptotically, default is a sure event, only the event that both variables take value one remains. However, for any large stress level, the probability that both variables are zero vanishes most quickly and probability mass is pushed into the remaining three cases. No line will succeed in adequately describing the relationship of those variables and in particular will not capture the variance in the centralized variables.

5 Implications for credit portfolio modelling

The main results of this paper are formulae for asymptotic stressed default probabilities in credit portfolio models with elliptically distributed risk factors and asset variables $Z \overset{\mathcal{L}}{=} GAU$. We have shown that for any $D_i \in \mathbb{R}$

$$\lim_{C\to-\infty} \mathbf{P}(Z_i \le D_i | Z_0 \le C) = 1,$$

if $\mathbf{P}(G > \cdot)$ is rapidly varying and, if $\mathbf{P}(G > \cdot)$ is regularly varying,

$$\lim_{C\to-\infty} \mathbf{P}(Z_i \leq D_i | Z_0 \leq C) = t_{\alpha+1}\left(\frac{\sqrt{\alpha+1}\,\rho}{\sqrt{1-\rho^2}}\right) \in [1/2, 1).$$

This behaviour of limiting default probabilities is fundamentally different to tail dependence, which is a popular measure in finance to assess the ability of a bivariate distribution to generate joint extreme events: For two random variables Y_1 and Y_2 with distribution functions F_1 and F_2, the coefficient of (lower) tail dependence of Y_1 and Y_2 is

$$\lambda_l(Y_1, Y_2) := \lim_{q\to 0^+} \mathbf{P}(Y_2 \leq F_2^{\leftarrow}(q) | Y_1 \leq F_1^{\leftarrow}(q)), \tag{18}$$

where $F_i^{\leftarrow}$ denotes the inverse of the df F_i. The tail dependence coefficient depends only on the copula rather than the bivariate distribution function, see e.g. Joe (1997); Nelsen (1999); McNeil *et al.* (2005). For heavy-tailed elliptical distributions, i.e., the mixing variable G is in $RV_{-\alpha}$, the tail dependence is given by

$$\lambda_l(Y_1, Y_2) = \frac{\int_{\pi/2-\arcsin\rho}^{\pi/2} (\cos\theta)^\alpha \, d\theta}{\int_0^{\pi/2} (\cos\theta)^\alpha \, d\theta} = 2t_{\alpha+1}\left(-\sqrt{\frac{(\alpha+1)(1-\rho)}{1+\rho}}\right),$$

where $\rho := \mathrm{Corr}(Y_1, Y_2)$, see Hult and Lindskog (2002); Schmidt (2002); McNeil *et al.* (2005). Hence, $\lambda_l(Y_1, Y_2) > 0$ for $\rho > -1$. In contrast, the tail dependence is zero for a normal distribution, which is the most frequently used distribution in structural credit portfolio models, e.g. normal distributions are used in Moody's KMV model (Crosbie and Bohn (2002)). Due to zero tail dependence, normally distributed models are usually considered less sensitive to extreme stress than heavy-tailed models. The results in this paper show that this is not necessarily the case: in the limit, the impact of stress on default probabilities is higher in light-tailed models than in heavy-tailed models.

To analyze the precise difference between tail dependence and asymptotic stressed PD's, we write the tail dependence of Z_0 and Z_1 in the form

$$\lambda_l(Z_0, Z_1) = \lim_{C\to-\infty} \mathbf{P}(Z_1 \leq C | Z_0 \leq C), \tag{19}$$

which is equivalent to definition (18) since Z_0 and Z_1 are identically distributed. Hence, for calculating tail dependence the conditional probability $\mathbf{P}^C(Z_1 \leq C)$ has to be evaluated whereas the stressed default probability $\mathbf{P}^C(Z_1 \leq D)$ is evaluated at a constant D. For light-tailed elliptical distributions, the variable Z_1 only attains limit values in the range

$$\lim_{C\to-\infty} \mathbf{P}^C(C \leq Z_1 \leq D) = 1$$

for any $D \in \mathbb{R}$, whereas extreme events outside this range have positive probability in the heavy-tailed case:

$$\lim_{C\to-\infty} \mathbf{P}^C(Z_1 \leq C) > 0, \quad \lim_{C\to-\infty} \mathbf{P}^C(Z_1 > D) > 0.$$

The resulting difference between tails dependence and asymptotic stressed PD's as a function of the tail index α is shown in Figure 2.

Turning now to stressed default correlations, we observe a behaviour similar to tail dependence: we have shown that stressed default correlations converge to 0 in the light-tailed case and to a positive number in the heavy-tailed case. Hence, in light-tailed models extreme stress

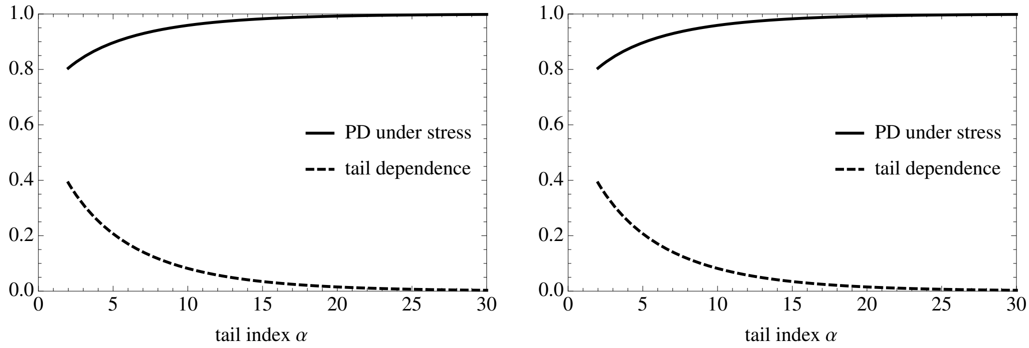

Figure 2: Tail dependence coefficient and asymptotic PD under stress as a function of the tail index α. Left: correlation parameter 0.4; right: correlation parameter 0.7.

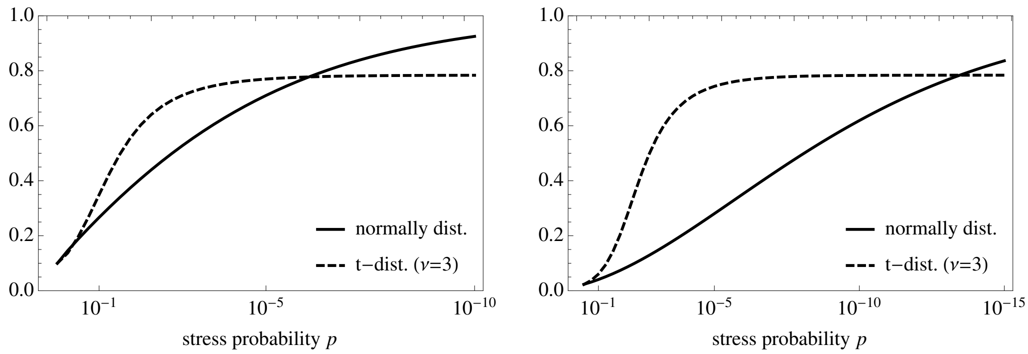

Figure 3: PD's under stress as a function of the stress probability $p := F(C)$, with F the distribution function of the respective model and C the stress level. Models considered are the normal distribution and the t-distribution with parameter 3. Correlations are 0.4. Left: unconditional PD is 0.1 (i.e., $D = F^{(-1)}(0.1)$); right: unconditional PD is 0.01.

scenarios tend to heavily increase the expected loss whereas tail risk measures, which are driven by the dependence of default events, are less affected.

It is important to note, however, that the asymptotic behaviour analysed in this paper is not necessarily representative for typical stress scenarios in credit risk management. To gain further insight and provide a heuristic answer, we consider PD's under stress for various stress levels and compare them in light- and heavy-tailed models. Figure 3 shows PD's under stress for both normally distributed and t-distributed ($\nu = 3$) models as a function of the stress probability $p := F(C)$, where F is the distribution function of the respective model and C is the stress level. The correlation is chosen to be 0.4. Despite converging to a value smaller than 1, PD's under stress in the t-distributed model dominate the normally distributed case unless the stress probability is very small: If the unconditional PD is 10%, then for stress probabilities greater than 10^{-6}, the PD under stress in the t-distributed model is greater than the respective PD in the normal model. If the unconditional PD is 1%, then the threshold lies beyond 10^{-13}.

Hence, aside from providing useful information for stress testing, our results indicate that gauging the suitability of a distribution family for credit portfolio modelling solely on asymptotic behaviour may be misleading.

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This research was supported by the Deutsche Forschungsgemeinschaft through the IRTG 1792.

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This research was supported by the Deutsche Forschungsgemeinschaft through the IRTG 1792.