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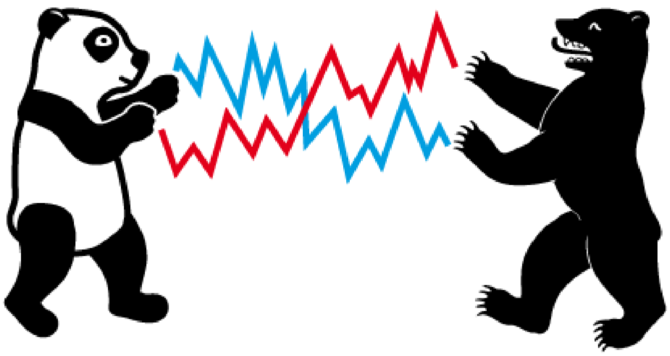

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Complete Convergence and Complete Moment Convergence for Maximal Weighted Sums of Extended Negatively Dependent Random Variables

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Abstract In this paper, the complete convergence and complete moment convergence for maximal weighted sums of extended negatively dependent random variables are investigated. Some sufficient conditions for the convergence are provided. In addition, the Marcinkiewicz–Zygmund type strong law of large numbers for weighted sums of extended negatively dependent random variables is obtained. The results obtained in the article extend the corresponding ones for independent random variables and some dependent random variables.

Keywords Extended negatively dependent, complete convergence, complete moment convergence, maximal weighted sums, strong law of large numbers

MR(2010) Subject Classification 60F15

1 Introduction

In this paper, we consider a sequence $\{X_n, n \geq 1\}$ of random variables defined on some probability space $(\Omega, \mathscr{F}, P)$. It is well known that the complete convergence plays an important role in establishing almost sure convergence of summation of random variables as well as weighted sums of random variables. The concept of the complete convergence was introduced by Hus and Robbins [11] as follows.

Definition 1.1 *A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant θ if for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \varepsilon) < \infty.$$

Chow [7] presented the following more general concept of the complete moment convergence.

Definition 1.2 *Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0, b_n > 0$, and $q > 0$, if*

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \quad \text{for some or all } \varepsilon > 0,$$

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then the above result was called the complete moment convergence.

As we know, the complete moment convergence implies complete convergence. Moreover, the complete moment convergence can more exactly describe the convergence rate of a sequence of random variables than the complete convergence. So, a study on complete moment convergence is of interest.

When $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables, Baum and Katz [3] proved the following remarkable result concerning the convergence rate of the tail probabilities $P(|S_n| > \varepsilon n^r)$ for all $\varepsilon > 0$, where $S_n = \sum_{i=1}^n X_i$ are the partial sums.

Theorem 1.3 *Let $r > 1/2$ and $p > 1$. Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identical distributed random variables. Assume further that $EX = 0$ if $r \leq 1$. Then the following statements are equivalent*:

(i) $E|X|^{p/r} < \infty$;

(ii) $\sum_{n=1}^\infty n^{p-2} P(|S_n| > \varepsilon n^r) < \infty$, *for all* $\varepsilon > 0$.

In convenient, we call n^{p-2} as the weight function of the tail probabilities, and n^r as the boundary function in the tail probabilities. Many useful linear statistics are weighted sums of independent and identically distributed random variables, such as, least-squares estimators, nonparametric regression function estimators and jackknife estimates, and so on. However, in many stochastic model, the assumption that random variables are independent is not plausible. Increases in some random variables are often related to decreases in other random variables, so an assumption of dependence is more appropriate than an assumption of independence. One of the important dependence structure is the extended negatively dependent structure, which was introduced by Liu [12] as follows.

Definition 1.4 *Random variables $X_k, k = 1, \ldots, n$ are said to be lower extended negatively dependent* (LEND) *if there is some $M > 0$ such that, for all real numbers $x_k, k = 1, \ldots, n$,*

$$P\left\{\bigcap_{k=1}^n (X_k \leq x_k)\right\} \leq M \prod_{k=1}^n P\{X_k \leq x_k\}; \tag{1.1}$$

they are said to be upper extended negatively dependent (UEND) *if there is some $M > 0$ such that, for all real numbers $x_k, k = 1, \ldots, n$,*

$$P\left\{\bigcap_{k=1}^n (X_k > x_k)\right\} \leq M \prod_{k=1}^n P\{X_k > x_k\}; \tag{1.2}$$

and they are said to be extended negatively dependent (END) *if they are both* LEND *and* UEND. *A sequence of infinitely many random variables $\{X_k, k = 1, 2, \ldots\}$ is said to be* LEND/ UEND/ END *if there is some $M > 0$ such that for each positive integer n, the random variables $X_1, X_2, \ldots, X_n$ are* LEND/ UEND/ END, *respectively.*

The END structure covers all negative dependence structures and, more interestingly, it covers certain positive dependence structures. Some applications for END sequence have been found. See, for example, Liu [12] obtained the precise large deviations for dependent random variables with heavy tails. Liu [13] studied the sufficient and necessary conditions of moderate deviations for dependent random variables with heavy tails. Shen [18] gave the probability inequalities. Chen [5], Yan [30] considered the SLLN for END random variables and applications to risk theory and renewal theory, and so on.

Recall that the sequence $\{X_n, n \geq 1\}$ is stochastically dominated by a random variable X if

$$\sup_{n\geq 1} P\{|X_n| > t\} \leq CP\{|X| > t\}, \tag{1.3}$$

for some positive constant C and all $t \geq 0$. A real valued function $l(x)$, positive and measurable on $[A, \infty)$ for some $A > 0$, is said to be slowly varying if

$$\lim_{x\to\infty} \frac{l(\lambda x)}{l(x)} = 1 \quad \text{for each } \lambda > 0.$$

Recently, Shen et al. [20] discussed the complete convergence of weighted sums of extended negatively dependent random variables and obtained the following result:

Theorem 1.5 *Let $1/2 < r \leq 1$ and $\{X_n, n \geq 1\}$ be a sequence of mean zero* END *random variables, which is stochastically dominated by a random variable X. Let $l(x) > 0$ be a slowly varying and monotone non decreasing function. Assume further that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers such that*

$$\sum_{i=1}^{n} a_{ni}^2 = O(n).$$

If

$$E[|X|^{1/r} l(|X|^{1/r})] < \infty,$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} P\left(\left| \sum_{i=1}^{n} a_{ni} X_i \right| > \varepsilon n^r \right) < \infty.$$

The above result extended Theorem 1.3 in three aspects: independent and identically distributed random variables to END random variables without identical distribution; partial sums to weighted sums and changed the weight function of tail probability.

Motivated by the above works, we will further study the complete convergence for weighted sums of END random variables, the purpose of this article is threefold:

(1) generalise the condition of weight $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ to $\sum_{i=1}^n |a_{ni}|^\alpha = O(n)$ for some $\alpha > 0$;

(2) change the boundary function in the tail probabilities;

(3) take into account the complete moment convergence for the maximal of weighted sums as well.

This work is organized as follows: some lemmas and main results on complete convergence and complete moment convergence for maximal weighted sums of END random variables are provided in Section 2. The proofs for the main results are presented in Section 3.

Before we present our main results, we note that C will be numerical constants whose value are without importance, and, in addition, may change between appearances. $a_n = O(b_n)$ stands for $a_n \leq Cb_n$ for all $n \geq 1$ and $I(A)$ is the indicator function on the set A. Denote $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$.

2 Some Lemmas and Main Results

To proceed the main results, we need some important lemmas.

Lemma 2.1 ([12]) *Let $\{X_n, n \geq 1\}$ be a sequence of* END *random variables.*

(i) *For each $n \geq 1$, if $f_1, f_2, \ldots, f_n$ are all nondecreasing* (*or nonincreasing*) *functions, then $f_1(X_1)$, $f_2(X_2)$, $\ldots$, $f_n(X_n)$ are also* END *random variables.*

(ii) *For each $n \geq 1$, there exists a constant $M > 0$ such that*

$$E\Big(\prod_{i=1}^{n} X_i^+\Big) \leq M \prod_{i=1}^{n} EX_i^+.$$

Lemma 2.2 ([24]) *Let $p \geq 2$ and $\{X_n, n \geq 1\}$ be a sequence of* END *random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for each $n \geq 1$. Then there exists a positive constant C_p depending only on p such that*

$$E\Big|\sum_{i=1}^{n} X_i\Big|^p \leq C_p\Big[\sum_{i=1}^{n} E|X_i|^p + \Big(\sum_{i=1}^{n} E|X_i|^2\Big)^{p/2}\Big], \quad \text{for all } n \geq 1$$

and

$$E\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j} X_i\Big|^p\Big) \leq C_p(\log n)^p\Big[\sum_{i=1}^{n} E|X_i|^p + \Big(\sum_{i=1}^{n} E|X_i|^2\Big)^{p/2}\Big], \quad \text{for all } n \geq 1.$$

Lemma 2.3 ([1, 2]) *Let $\{X_n, n \geq 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $\beta > 0$, the following two statements hold*:

$$E|X_n|^\alpha I(|X_n| \leq \beta) \leq C_1[E|X|^\alpha I(|X| \leq \beta) + \beta^\alpha P\{|X| > \beta\}];$$
$$E|X_n|^\alpha I(|X_n| > \beta) \leq C_2 E|X|^\alpha I(|X| > \beta),$$

where C_1 and C_2 are positive constants. Consequently, $E|X_n|^\alpha \leq CE|X|^\alpha$.

Lemma 2.4 *Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers such that*

$$\sum_{i=1}^{n} |a_{ni}|^\xi \leq n \quad \text{for some } \xi > 0. \tag{2.1}$$

Then

(i) *for every $0 < \zeta < \xi$,*

$$\sum_{i=1}^{n} |a_{ni}|^\zeta \leq n; \tag{2.2}$$

(ii) *for all $m \geq 1$, $\kappa > \xi$,*

$$\sum_{j=m}^{\infty} \sharp(I_{nj})(j+1)^{-\kappa/\xi} \leq C(m+1)^{1-\frac{\kappa}{\xi}}, \tag{2.3}$$

where $I_{nj} = \{1 \leq i \leq n : n^{1/\xi}(j+1)^{-1/\xi} < |a_{ni}| \leq n^{1/\xi}j^{-1/\xi}\}$, for each $n \geq 1, j \geq 1$. Obviously, it holds that for all $n > m \geq 1$, $\kappa > \xi$,

$$\sum_{j=m}^{n-1} \sharp(I_{nj})(j+1)^{-\kappa/\xi} \leq C(m+1)^{1-\frac{\kappa}{\xi}}; \tag{2.4}$$

(iii) *for arbitrary* $s \geq 1$,

$$\sum_{j=1}^{s} \sharp(I_{nj}) \leq s+1. \tag{2.5}$$

We will use the following properties of slowly varying functions.

Lemma 2.5 ([17]) *If $\ell(x)$ is a function slowly varying at infinity, then for any $r > 0$,*

$$C_1 n^{-r}\ell(n) \leq \sum_{i=n}^{\infty} i^{-1-r}\ell(i) \leq C_2 n^{-r}\ell(n),$$

and

$$C_3 n^{r}\ell(n) \leq \sum_{i=1}^{n} i^{-1+r}\ell(i) \leq C_4 n^{r}\ell(n),$$

where $C_i, i = 1, 2, 3, 4$ are positive constants depending only on r.

Lemma 2.6 *Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n) \quad \textit{for some } \alpha > 0 \tag{2.6}$$

and X be a random variable. Let $b_n = n^{1/\alpha}(\log n)^{3/\gamma}$ for some $\gamma > 0$.

(i) *If $p > \max\{\alpha, \gamma(\beta+1)/3\}$ for some β, then*

$$\sum_{n=2}^{\infty} \frac{(\log n)^{\beta}}{n b_n^{p}} \sum_{i=1}^{n} E|a_{ni}X|^{p} I(|a_{ni}X| \leq b_n)$$
$$\leq \begin{cases} CE|X|^{\alpha}, & \textit{for } \alpha > \gamma(\beta+1)/3, \\ CE|X|^{\alpha}\log(1+|X|), & \textit{for } \alpha = \gamma(\beta+1)/3, \\ CE|X|^{\gamma(\beta+1)}, & \textit{for } \alpha < \gamma(\beta+1)/3. \end{cases}$$

(ii) *If $p = \alpha, \beta = 2$, then*

$$\sum_{n=2}^{\infty} \frac{(\log n)^{2}}{n b_n^{\alpha}} \sum_{i=1}^{n} E|a_{ni}X|^{\alpha} I(|a_{ni}X| > b_n)$$
$$\leq \begin{cases} CE|X|^{\alpha}, & \textit{for } \alpha > \gamma, \\ CE|X|^{\alpha}\log(1+|X|), & \textit{for } \alpha = \gamma, \\ CE|X|^{\gamma}, & \textit{for } \alpha < \gamma. \end{cases}$$

Lemma 2.7 *Let $0 < \alpha < 2$ and $\gamma > 0$. Suppose that the sequence $\{X_{ni}, i \geq 1, n \geq 1\}$ is stochastically dominated by a random variable X with $E|X|^{\alpha} < \infty$. And $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers satisfying* (2.6)*. Additionally, assume that $EX_{ni} = 0$ for $1 < \alpha < 2$ and b_n as in Lemma* 2.6*. Then*

$$\frac{1}{b_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} EY_{ni} \right| \leq C(\log n)^{-3\alpha/\gamma} E|X|^{\alpha} \to 0 \quad \textit{as } n \to \infty, \tag{2.7}$$

where $Y_{ni} = a_{ni}X_{ni}I(|a_{ni}X_{ni}| \leq b_n) + b_n I(a_{ni}X_{ni} > b_n) - b_n I(a_{ni}X_{ni} < -b_n)$ for each $1 \leq i \leq n, n \geq 1$.

Now, we present our main results as follows.

Theorem 2.8 *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise* END *random variables, which was stochastically dominated by a random variable X. For some $0 < \alpha < 2$, $\gamma > 0$, $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers satisfying* (2.6) *and b_n as in Lemma* 2.6. *Additionally, assume that $EX_{ni} = 0$ for $1 < \alpha < 2$. If*

$$\begin{cases} E|X|^\alpha < \infty, & \text{for } \alpha > \gamma, \\ E|X|^\alpha \log(1 + |X|) < \infty, & \text{for } \alpha = \gamma, \\ E|X|^\gamma < \infty, & \text{for } \alpha < \gamma, \end{cases} \tag{2.8}$$

then

$$\sum_{n=2}^{\infty} \frac{1}{n} P\bigg(\max_{1 \leq j \leq n} \bigg| \sum_{i=1}^{j} a_{ni} X_{ni} \bigg| > \varepsilon b_n \bigg) < \infty \quad \text{for all } \varepsilon > 0. \tag{2.9}$$

Theorem 2.9 *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise* END *random variables, which was stochastically dominated by a random variable X. For some $0 < \alpha < 2$, $\gamma > 0$, $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers satisfying* (2.6) *and b_n as in Lemma* 2.6. *Additionally, assume that $EX_{ni} = 0$ for $1 < \alpha < 2$. If* (2.8) *holds, then for $0 < \tau < \alpha$,*

$$\sum_{n=2}^{\infty} \frac{1}{n} E\bigg(\frac{1}{b_n} \max_{1 \leq j \leq n} \bigg| \sum_{i=1}^{j} a_{ni} X_{ni} \bigg| - \varepsilon \bigg)_+^\tau < \infty \quad \text{for all } \varepsilon > 0. \tag{2.10}$$

Remark 2.10 Under the conditions of Theorem 2.9, we can obtain that

$$\begin{aligned} \infty &> \sum_{n=2}^{\infty} \frac{1}{n} E\bigg(\frac{1}{b_n} \max_{1 \leq j \leq n} \bigg| \sum_{i=1}^{j} a_{ni} X_{ni} \bigg| - \varepsilon \bigg)_+^\tau \\ &= \sum_{n=2}^{\infty} \frac{1}{n} \int_0^\infty P\bigg(\frac{1}{b_n} \max_{1 \leq j \leq n} \bigg| \sum_{i=1}^{j} a_{ni} X_{ni} \bigg| - \varepsilon > t^{1/\tau} \bigg) dt \\ &\geq \int_0^\varepsilon \sum_{n=2}^{\infty} \frac{1}{n} P\bigg(\max_{1 \leq j \leq n} \bigg| \sum_{i=1}^{j} a_{ni} X_{ni} \bigg| > b_n(\varepsilon + \varepsilon^{1/\tau}) \bigg) dt \\ &\geq \varepsilon \sum_{n=2}^{\infty} \frac{1}{n} P\bigg(\max_{1 \leq j \leq n} \bigg| \sum_{i=1}^{j} a_{ni} X_{ni} \bigg| > b_n(\varepsilon + \varepsilon^{1/\tau}) \bigg) \quad \text{for all } \varepsilon > 0. \end{aligned} \tag{2.11}$$

Hence, from (2.11) we get that the complete moment convergence implies the complete convergence.

Theorem 2.11 *Conditions as those of Theorem* 2.8. *Let $T_n = \sum_{i=1}^n a_{ni} X_{ni}$ for each $n \geq 1$, then*

$$\lim_{n \to \infty} \frac{T_n}{b_n} = 0, \quad \text{a.s.} \tag{2.12}$$

3 Proofs of Main Results

Proof of Lemma 2.4 (i) By Jensen's inequality and (2.1), we get

$$\sum_{i=1}^{n} |a_{ni}|^\zeta \leq \bigg(\sum_{i=1}^{n} |a_{ni}|^\xi \bigg)^{\frac{\zeta}{\xi}} \cdot n^{1-\frac{\zeta}{\xi}} \leq n.$$

(ii) By the denotes of I_{nj}, we see that for every $n \geq 1$, $\{I_{nj}, j \geq 1\}$ are pairwise disjoint and

$$\bigcup_{j\geq 1} I_{nj} = \{1 \leq i \leq n : 0 < |a_{ni}| \leq n^{1/\xi}\}$$

since $\sum_{i=1}^{n} |a_{ni}|^{\xi} \leq n < \infty$.

For all $m \geq 1$, by $\kappa > \xi$ we have

$$\begin{aligned} n &\geq \sum_{i=1}^{n} |a_{ni}|^{\xi} = \sum_{j=1}^{\infty} \sum_{i\in I_{nj}} |a_{ni}|^{\xi} \\ &\geq \sum_{j=1}^{\infty} \sharp(I_{nj}) n (j+1)^{-1} \geq \sum_{j=m}^{\infty} \sharp(I_{nj}) n (j+1)^{-1} \\ &= \sum_{j=m}^{\infty} \sharp(I_{nj}) n (j+1)^{-\kappa/\xi} (j+1)^{\kappa/\xi - 1} \\ &\geq \sum_{j=m}^{\infty} \sharp(I_{nj}) n (j+1)^{-\kappa/\xi} (m+1)^{\kappa/\xi - 1}. \end{aligned}$$

Hence, (2.3) holds.

(iii) On the other hand, by (2.1), for arbitrary $s \geq 1$,

$$n \geq \sum_{k=1}^{n} |a_{nk}|^{\xi} = \sum_{j=1}^{\infty} \sum_{k\in I_{nj}} |a_{nk}|^{\xi} \geq n \sum_{j=1}^{s} \frac{1}{j+1} \sharp(I_{nj}) \geq \frac{n}{s+1} \sum_{j=1}^{s} \sharp(I_{nj}),$$

which implies (2.5).

Proof of Lemma 2.6 Without loss of generality, we may assume that

$$\sum_{i=1}^{n} |a_{ni}|^{\alpha} \leq n, \qquad \text{for some } \alpha > 0.$$

Then, by Lemma 2.4,

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{(\log n)^{\beta}}{n b_n^p} \sum_{i=1}^{n} E|a_{ni}X|^p I(|a_{ni}X| \leq b_n) \\ &= \sum_{n=2}^{\infty} n^{-1-\frac{p}{\alpha}} (\log n)^{\beta - \frac{3p}{\gamma}} \sum_{i=1}^{n} |a_{ni}|^p E|X|^p I(|a_{ni}X| \leq n^{1/\alpha} (\log n)^{3/\gamma}) \\ &= \sum_{n=2}^{\infty} n^{-1-\frac{p}{\alpha}} (\log n)^{\beta - \frac{3p}{\gamma}} \sum_{j=1}^{\infty} \sum_{i\in I_{nj}} |a_{ni}|^p E|X|^p I(|a_{ni}X| \leq n^{1/\alpha} (\log n)^{3/\gamma}) \\ &\leq \sum_{n=2}^{\infty} n^{-1-\frac{p}{\alpha}} (\log n)^{\beta - \frac{3p}{\gamma}} \sum_{j=1}^{\infty} \sharp(I_{nj}) n^{p/\alpha} j^{-p/\alpha} E|X|^p I(|X| \leq (j+1)^{1/\alpha} (\log n)^{3/\gamma}) \\ &= \sum_{n=2}^{\infty} n^{-1} (\log n)^{\beta - \frac{3p}{\gamma}} \sum_{j=1}^{\infty} \sharp(I_{nj}) j^{-p/\alpha} E|X|^p I(|X| \leq (\log n)^{3/\gamma}) \\ &\quad + \sum_{n=2}^{\infty} n^{-1} (\log n)^{\beta - \frac{3p}{\gamma}} \sum_{j=1}^{\infty} \sharp(I_{nj}) j^{-p/\alpha} \\ &\quad \times \sum_{k=1}^{j} E|X|^p I(k^{1/\alpha} (\log n)^{3/\gamma} < |X| \leq (k+1)^{1/\alpha} (\log n)^{3/\gamma}) \end{aligned}$$

$$\triangleq I_1 + I_2.$$

First for I_1, if $\alpha > \frac{\gamma(\beta+1)}{3}$, by Lemma 2.4 and $p > \alpha$, we have

$$\begin{aligned}
I_1 &\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-\frac{3p}{\gamma}} E|X|^p I(|X| \le (\log n)^{3/\gamma}) \\
&\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-\frac{3p}{\gamma}} (\log n)^{3(p-\alpha)/\gamma} E|X|^\alpha I(|X| \le (\log n)^{3/\gamma}) \\
&\le CE|X|^\alpha \sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-\frac{3\alpha}{\gamma}} \\
&\le CE|X|^\alpha.
\end{aligned}$$

If $\alpha \le \frac{\gamma(\beta+1)}{3}$, by Lemma 2.4 and $p > \frac{\gamma(\beta+1)}{3}$, we have

$$\begin{aligned}
I_1 &\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-\frac{3p}{\gamma}} E|X|^p I(|X| \le (\log n)^{3/\gamma}) \\
&= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-\frac{3p}{\gamma}} \sum_{m=2}^{n} E|X|^p I(\log(m-1) < |X|^{\gamma/3} \le \log m) \\
&\le C\sum_{m=2}^{\infty} E|X|^p I(\log(m-1) < |X|^{\gamma/3} \le \log m) \sum_{n=m}^{\infty} n^{-1}(\log n)^{\beta-\frac{3p}{\gamma}} \\
&\le C\sum_{m=2}^{\infty} (\log m)^{\beta-\frac{3p}{\gamma}+1} E|X|^p I(\log(m-1) < |X|^{\gamma/3} \le \log m) \\
&= C\sum_{m=2}^{\infty} (\log m)^{\beta-\frac{3p}{\gamma}+1} E|X|^{\frac{\gamma(\beta+1)}{3}} E|X|^{p-\frac{\gamma(\beta+1)}{3}} I(\log(m-1) < |X|^{\gamma/3} \le \log m) \\
&\le C\sum_{m=2}^{\infty} (\log m)^{\beta-\frac{3p}{\gamma}+1} (\log m)^{\frac{3}{\gamma}(p-\frac{\gamma(\beta+1)}{3})} E|X|^{\frac{\gamma(\beta+1)}{3}} I(\log(m-1) < |X|^{\gamma/3} \le \log m) \\
&\le CE|X|^{\frac{\gamma(\beta+1)}{3}}.
\end{aligned}$$

Next for I_2, by Lemma 2.4 and $p > \alpha$, we have

$$\begin{aligned}
I_2 &\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-\frac{3p}{\gamma}} \sum_{k=1}^{\infty} E|X|^p I\left(k^{\frac{1}{\alpha}} < \frac{|X|}{(\log n)^{3/\gamma}} \le (k+1)^{\frac{1}{\alpha}}\right) \sum_{j=k}^{\infty} \sharp(I_{nj}) j^{-p/\alpha} \\
&\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-\frac{3p}{\gamma}} \sum_{k=1}^{\infty} (k+1)^{1-\frac{p}{\alpha}} E|X|^p I\left(k^{\frac{1}{\alpha}} < \frac{|X|}{(\log n)^{3/\gamma}} \le (k+1)^{\frac{1}{\alpha}}\right) \\
&= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-\frac{3p}{\gamma}} \sum_{k=1}^{\infty} (k+1)^{1-\frac{p}{\alpha}} E|X|^\alpha E|X|^{p-\alpha} I\left(k^{\frac{1}{\alpha}} < \frac{|X|}{(\log n)^{3/\gamma}} \le (k+1)^{\frac{1}{\alpha}}\right) \\
&\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-\frac{3\alpha}{\gamma}} \sum_{k=1}^{\infty} E|X|^\alpha I\left(k^{\frac{1}{\alpha}} < \frac{|X|}{(\log n)^{3/\gamma}} \le (k+1)^{\frac{1}{\alpha}}\right) \\
&= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-\frac{3\alpha}{\gamma}} E|X|^\alpha I(|X| > (\log n)^{3/\gamma}) \\
&= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-\frac{3\alpha}{\gamma}} \sum_{m=n}^{\infty} E|X|^\alpha I(\log m < |X|^{\gamma/3} \le \log(m+1))
\end{aligned}$$

$$= C\sum_{m=2}^{\infty} E|X|^{\alpha} I(\log m < |X|^{\gamma/3} \le \log(m+1)) \sum_{n=2}^{m} n^{-1}(\log n)^{\beta-\frac{3\alpha}{\gamma}}.$$

Observing that

$$\sum_{n=2}^{m} n^{-1}(\log n)^{\beta-\frac{3\alpha}{\gamma}} \le \begin{cases} C, & \text{for } \alpha > \dfrac{\gamma(\beta+1)}{3}, \\ C\log\log m, & \text{for } \alpha = \dfrac{\gamma(\beta+1)}{3}, \\ C(\log m)^{\beta-\frac{3\alpha}{\gamma}+1}, & \text{for } \alpha < \dfrac{\gamma(\beta+1)}{3}, \end{cases}$$

we get

$$I_2 \le \begin{cases} CE|X|^{\alpha}, & \text{for } \alpha > \dfrac{\gamma(\beta+1)}{3}, \\ CE|X|^{\alpha}\log(1+|X|), & \text{for } \alpha = \dfrac{\gamma(\beta+1)}{3}, \\ CE|X|^{\gamma(\beta+1)/3}, & \text{for } \alpha < \dfrac{\gamma(\beta+1)}{3}. \end{cases}$$

Therefore,

$$I = I_1 + I_2 \le \begin{cases} CE|X|^{\alpha}, & \text{for } \alpha > \dfrac{\gamma(\beta+1)}{3}, \\ CE|X|^{\alpha}\log(1+|X|), & \text{for } \alpha = \dfrac{\gamma(\beta+1)}{3}, \\ CE|X|^{\gamma(\beta+1)/3}, & \text{for } \alpha < \dfrac{\gamma(\beta+1)}{3}. \end{cases}$$

The proof of Lemma 2.6 is completed. □

Proof of Lemma 2.7 We will discuss (2.7) into two cases.

Case I $0 < \alpha \le 1$.

By Lemma 2.3, Markov's inequality and $E|X|^{\alpha} < \infty$ we have

$$\begin{aligned}
\frac{1}{b_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j} EY_{ni}\right| &\le \frac{1}{b_n}\sum_{i=1}^{n}|EY_{ni}| \\
&\le \frac{C}{b_n}\sum_{i=1}^{n} E|a_{ni}X_{ni}|I(|a_{ni}X_{ni}| \le b_n) + C\sum_{i=1}^{n} P(|a_{ni}X_{ni}| > b_n) \\
&\le \frac{C}{b_n}\sum_{i=1}^{n} E|a_{ni}X|I(|a_{ni}X| \le b_n) + C\sum_{i=1}^{n} P(|a_{ni}X| > b_n) \\
&\le \frac{C}{b_n}\sum_{i=1}^{n} E|a_{ni}X|^{\alpha}|a_{ni}X|^{1-\alpha}I(|a_{ni}X| \le b_n) + \frac{C}{b_n^{\alpha}}\sum_{i=1}^{n}|a_{ni}|^{\alpha}E|X|^{\alpha} \\
&\le \frac{C}{b_n^{\alpha}}\sum_{i=1}^{n}|a_{ni}|^{\alpha}E|X|^{\alpha} \\
&\le C(\log n)^{-3\alpha/\gamma}E|X|^{\alpha} \longrightarrow 0 \quad \text{as } n\to\infty. \qquad (3.1)
\end{aligned}$$

Case II $1 < \alpha < 2$.

For all $1 \le i \le n, n \ge 1$, denote

$$\begin{aligned}
Z_{ni} &= a_{ni}X_{ni} - Y_{ni} \\
&= (a_{ni}X_{ni} + b_n)I(a_{ni}X_{ni} < -b_n) + (a_{ni}X_{ni} - b_n)I(a_{ni}X_{ni} > b_n).
\end{aligned}$$

Then

$$0 < Z_{ni} = a_{ni}X_{ni} - b_n < a_{ni}X_{ni} \quad \text{for } a_{ni}X_{ni} > b_n,$$

and

$$a_{ni}X_{ni} < a_{ni}X_{ni} + b_n = Z_{ni} < 0 \quad \text{for } a_{ni}X_{ni} < -b_n.$$

Hence,

$$|Z_{ni}| < |a_{ni}X_{ni}|I(|a_{ni}X_{ni}| > b_n).$$

By $EX_{ni} = 0$ for $1 < \alpha < 2$, Lemma 2.3 and $E|X|^\alpha < \infty$ we get

$$\begin{aligned}
\frac{1}{b_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j} EY_{ni}\right| &= \frac{1}{b_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j} EZ_{ni}\right| \\
&\le \frac{1}{b_n}\sum_{i=1}^{n} E|Z_{ni}| \\
&\le \frac{1}{b_n}\sum_{i=1}^{n} E|a_{ni}X_{ni}|I(|a_{ni}X_{ni}| > b_n) \\
&\le \frac{C}{b_n}\sum_{i=1}^{n} E|a_{ni}X|I(|a_{ni}X| > b_n) \\
&\le \frac{C}{b_n^\alpha}\sum_{i=1}^{n} |a_{ni}|^\alpha E|X|^\alpha I(|a_{ni}X| > b_n) \\
&\le C(\log n)^{-3\alpha/\gamma}E|X|^\alpha \longrightarrow 0 \quad \text{as } n\to\infty. \qquad (3.2)
\end{aligned}$$

From (3.1) and (3.2), we see that (2.7) holds. □

Proof of Theorem 2.8 By (2.6) and

$$a_{ni} = a_{ni}^+ - a_{ni}^-,$$

without loss of generality, we may assume that $a_{ni} \ge 0$ and $\sum_{i=1}^n a_{ni}^\alpha \le n$.

For fixed $n \ge 1$, let Z_{ni} as those in the proof of Lemma 2.7 and

$$\begin{aligned}
A &= \bigcap_{i=1}^{n}\{Y_{ni} = a_{ni}X_{ni}\}, \\
B &= \overline{A} = \bigcup_{i=1}^{n}\{Y_{ni} \ne a_{ni}X_{ni}\} = \bigcup_{i=1}^{n}\{|a_{ni}X_{ni}| > b_n\}, \\
E_{ni} &= \left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right\}.
\end{aligned}$$

It is easy to check that for all $\varepsilon > 0$,

$$\begin{aligned}
E_{ni} &= E_{ni}A \cup E_{ni}B \\
&\subset \left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_{ni}\right| > \varepsilon b_n\right\} \cup \left\{\bigcup_{i=1}^{n}\{|a_{ni}X_{ni}| > b_n\}\right\}.
\end{aligned}$$

Then, by Lemma 2.7, for n large enough, we have

$$P(E_{ni}) \le P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_{ni}\right| > \varepsilon b_n\right) + P\left(\bigcup_{i=1}^{n}\{|a_{ni}X_{ni}| > b_n\}\right)$$

$$\le P\bigg(\max_{1\le j\le n}\bigg|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\bigg|>\varepsilon b_n-\max_{1\le j\le n}\bigg|\sum_{i=1}^{j}EY_{ni}\bigg|\bigg)+\sum_{i=1}^{n}P(|a_{ni}X_{ni}|>b_n)$$

$$\le P\bigg(\max_{1\le j\le n}\bigg|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\bigg|>\frac{\varepsilon b_n}{2}\bigg)+\sum_{i=1}^{n}P(|a_{ni}X_{ni}|>b_n). \tag{3.3}$$

To prove (2.9), it suffices to show that

$$J_1\triangleq\sum_{n=2}^{\infty}\frac{1}{n}P\bigg(\max_{1\le j\le n}\bigg|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\bigg|>\frac{\varepsilon b_n}{2}\bigg)<\infty, \tag{3.4}$$

$$J_2\triangleq\sum_{n=2}^{\infty}\frac{1}{n}\sum_{i=1}^{n}P(|a_{ni}X_{ni}|>b_n)<\infty. \tag{3.5}$$

First for J_1, by Lemma 2.1, it is easy to see that $\{Y_{ni}-EY_{ni},1\le i\le n,n\ge 1\}$ is an array of rowwise END random variables. Hence, it follows from Markov's inequality, Lemmas 2.2 and 2.3 that

$$\begin{aligned}
J_1&\le C\sum_{n=2}^{\infty}\frac{1}{nb_n^2}E\bigg(\max_{1\le j\le n}\bigg|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\bigg|^2\bigg)\\
&\le C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\sum_{i=1}^{n}E(Y_{ni}-EY_{ni})^2\\
&\le C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\sum_{i=1}^{n}E(Y_{ni})^2\\
&\le C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\sum_{i=1}^{n}E|a_{ni}X_{ni}|^2I(|a_{ni}X_{ni}|\le b_n)+C\sum_{n=2}^{\infty}\frac{\log^2 n}{n}\sum_{i=1}^{n}P(|a_{ni}X_{ni}|>b_n)\\
&\le C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\sum_{i=1}^{n}E|a_{ni}X|^2I(|a_{ni}X|\le b_n)+C\sum_{n=2}^{\infty}\frac{\log^2 n}{n}\sum_{i=1}^{n}P(|a_{ni}X|>b_n)\\
&\le C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\sum_{i=1}^{n}E|a_{ni}X|^2I(|a_{ni}X|\le b_n)+C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^\alpha}\sum_{i=1}^{n}E|a_{ni}X|^\alpha I(|a_{ni}X|>b_n)\\
&\triangleq J_{11}+J_{12}.
\end{aligned}\tag{3.6}$$

From Lemma 2.6 (i) (for $p=\beta=2$), (ii) and (2.8) we obtain that $J_{11}<\infty$, $J_{12}<\infty$ and then $J_1<\infty$.

Finally, we have

$$J_2\le\sum_{n=2}^{\infty}\frac{1}{nb_n^\alpha}\sum_{i=1}^{n}E|a_{ni}X_{ni}|^\alpha I(|a_{ni}X_{ni}|>b_n)\le J_{12}<\infty.$$

The proof of Theorem 2.8 is completed. □

Proof of Theorem 2.9 For all $\varepsilon>0$, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{1}{n}E\bigg(\frac{1}{b_n}\max_{1\le j\le n}\bigg|\sum_{i=1}^{j}a_{ni}X_{ni}\bigg|-\varepsilon\bigg)_+^\tau\\
&\quad=\sum_{n=2}^{\infty}\frac{1}{n}\int_0^\infty P\bigg(\frac{1}{b_n}\max_{1\le j\le n}\bigg|\sum_{i=1}^{j}a_{ni}X_{ni}\bigg|-\varepsilon>t^{\frac{1}{\tau}}\bigg)dt
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=2}^{\infty} \frac{1}{n} \int_0^1 P\Big(\frac{1}{b_n} \max_{1\le j\le n} \Big| \sum_{i=1}^{j} a_{ni} X_{ni} \Big| > \varepsilon + t^{\frac{1}{\tau}} \Big) dt \\
&\quad + \sum_{n=2}^{\infty} \frac{1}{n} \int_1^{\infty} P\Big(\frac{1}{b_n} \max_{1\le j\le n} \Big| \sum_{i=1}^{j} a_{ni} X_{ni} \Big| > \varepsilon + t^{\frac{1}{\tau}} \Big) dt \\
&\le \sum_{n=2}^{\infty} \frac{1}{n} P\Big(\max_{1\le j\le n} \Big| \sum_{i=1}^{j} a_{ni} X_{ni} \Big| > \varepsilon b_n \Big) \\
&\quad + \sum_{n=2}^{\infty} \frac{1}{n} \int_1^{\infty} P\Big(\max_{1\le j\le n} \Big| \sum_{i=1}^{j} a_{ni} X_{ni} \Big| > b_n t^{\frac{1}{\tau}} \Big) dt \\
&\triangleq K_1 + K_2. \qquad (3.7)
\end{aligned}
$$

To prove (2.10), it suffices to show that $K_1 < \infty$ and $K_2 < \infty$. By Theorem 2.8, we have $K_1 < \infty$. Next for K_2. For each $1 \le i \le n, n \ge 1$ and all $t \ge 1$, let

$$
\begin{aligned}
&Y'_{ni} = a_{ni} X_{ni} I(|a_{ni} X_{ni}| \le b_n t^{1/\tau}) + b_n t^{1/\tau} I(a_{ni} X_{ni} > b_n t^{1/\tau}) - b_n t^{1/\tau} I(a_{ni} X_{ni} < -b_n t^{1/\tau}); \\
&Z'_{ni} = a_{ni} X_{ni} - Y'_{ni}; \\
&A' = \bigcap_{i=1}^{n} \{Y'_{ni} = a_{ni} X_{ni}\}; \\
&B' = \overline{A'} = \bigcup_{i=1}^{n} \{Y'_{ni} \ne a_{ni} X_{ni}\} = \bigcup_{i=1}^{n} \{|a_{ni} X_{ni}| > b_n t^{1/\tau}\}; \\
&E'_{ni} = \Big\{ \max_{1\le j\le n} \Big| \sum_{i=1}^{j} a_{ni} X_{ni} \Big| > b_n t^{1/\tau} \Big\}.
\end{aligned}
$$

By Lemma 2.7, for all $t \ge 1$, and n large enough

$$
\begin{aligned}
P(E'_{ni}) &\le P\Big(\max_{1\le j\le n} \Big| \sum_{i=1}^{j} Y'_{ni} \Big| > b_n t^{1/\tau} \Big) + P\Big(\bigcup_{i=1}^{n} \{|a_{ni} X_{ni}| > b_n t^{1/\tau}\} \Big) \\
&\le P\Big(\max_{1\le j\le n} \Big| \sum_{i=1}^{j} (Y'_{ni} - EY'_{ni}) \Big| > b_n t^{1/\tau} - \max_{1\le j\le n} \Big| \sum_{i=1}^{j} EY'_{ni} \Big| \Big) \\
&\quad + \sum_{i=1}^{n} P(|a_{ni} X_{ni}| > b_n t^{1/\tau}) \\
&\le P\Big(\max_{1\le j\le n} \Big| \sum_{i=1}^{j} (Y'_{ni} - EY'_{ni}) \Big| > \frac{1}{2} b_n t^{1/\tau} \Big) + \sum_{i=1}^{n} P(|a_{ni} X_{ni}| > b_n t^{1/\tau}). \qquad (3.8)
\end{aligned}
$$

To prove $K_2 < \infty$, it suffices to show that

$$
K_{21} \triangleq \sum_{n=2}^{\infty} \frac{1}{n} \int_1^{\infty} P\Big(\max_{1\le j\le n} \Big| \sum_{i=1}^{j} (Y'_{ni} - EY'_{ni}) \Big| > \frac{1}{2} b_n t^{1/\tau} \Big) dt < \infty; \qquad (3.9)
$$

$$
K_{22} \triangleq \sum_{n=2}^{\infty} \frac{1}{n} \int_1^{\infty} \sum_{i=1}^{n} P(|a_{ni} X_{ni}| > b_n t^{1/\tau}) dt < \infty. \qquad (3.10)
$$

By Lemma 2.1, $\{Y'_{ni} - EY'_{ni}, 1 \le i \le n, n \ge 1\}$ is still an array of rowwise END random variables. Hence, it follows from Markov's inequality, Lemmas 2.2 and 2.3 that

$$\begin{aligned}
K_{21} &\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_{1}^{\infty}\frac{1}{b_n^2 t^{2/\tau}}E\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j}(Y'_{ni}-EY'_{ni})\Big|^2\Big)dt\\
&\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_{1}^{\infty}\frac{\log^2 n}{b_n^2 t^{2/\tau}}\sum_{i=1}^{n}E(Y'_{ni}-EY'_{ni})^2 dt\\
&\leq C\sum_{n=2}^{\infty}\frac{\log^2 n}{n}\int_{1}^{\infty}\sum_{i=1}^{n}P(|a_{ni}X_{ni}|>b_n t^{1/\tau})dt\\
&\quad + C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\int_{1}^{\infty}t^{-2/\tau}\sum_{i=1}^{n}E|a_{ni}X_{ni}|^2(|a_{ni}X_{ni}|\leq b_n t^{1/\tau})dt\\
&\leq C\sum_{n=2}^{\infty}\frac{\log^2 n}{n}\int_{1}^{\infty}\sum_{i=1}^{n}P(|a_{ni}X|>b_n t^{1/\tau})dt\\
&\quad + C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\int_{1}^{\infty}t^{-2/\tau}\sum_{i=1}^{n}E|a_{ni}X|^2(|a_{ni}X|\leq b_n)dt\\
&\quad + C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\int_{1}^{\infty}t^{-2/\tau}\sum_{i=1}^{n}E|a_{ni}X|^2(b_n<|a_{ni}X|\leq b_n t^{1/\tau})dt\\
&\triangleq CK_{211}+CK_{212}+CK_{213}.
\end{aligned}\tag{3.11}$$

For $0<\tau<\alpha$ and (2.6), we get

$$\begin{aligned}
K_{211} &= C\sum_{n=2}^{\infty}\frac{\log^2 n}{n}\int_{1}^{\infty}\sum_{i=1}^{n}P\Big(\frac{|a_{ni}X|^\tau}{b_n^\tau}>t\Big)dt\\
&\leq C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^\tau}\sum_{i=1}^{n}E|a_{ni}X|^\tau I(|a_{ni}X|>b_n)\\
&\leq C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^\alpha}\sum_{i=1}^{n}E|a_{ni}X|^\alpha I(|a_{ni}X|>b_n)\\
&\leq CJ_{12}<\infty.
\end{aligned}\tag{3.12}$$

For $0<\tau<\alpha<2$, it follows from Lemma 2.6 and (2.8) that

$$K_{212}\leq C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\sum_{i=1}^{n}E|a_{ni}X|^2(|a_{ni}X|\leq b_n)<\infty.\tag{3.13}$$

Taking $t=x^\tau$, by Markov's inequality and Lemma 2.3 if follows that

$$\begin{aligned}
K_{213} &\leq C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\int_{1}^{\infty}x^{\tau-3}\sum_{i=1}^{n}E|a_{ni}X|^2(b_n<|a_{ni}X|\leq b_n x)dx\\
&= C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\sum_{m=1}^{\infty}\int_{m}^{m+1}x^{\tau-3}\sum_{i=1}^{n}E|a_{ni}X|^2(b_n<|a_{ni}X|\leq b_n x)dx\\
&\leq C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\sum_{m=1}^{\infty}m^{\tau-3}\sum_{i=1}^{n}E|a_{ni}X|^2(b_n<|a_{ni}X|\leq b_n(m+1))\\
&\leq C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\sum_{i=1}^{n}\sum_{m=1}^{\infty}\sum_{s=1}^{m}m^{\tau-3}E|a_{ni}X|^2(b_n s<|a_{ni}X|\leq b_n(s+1))
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\sum_{i=1}^{n}\sum_{s=1}^{\infty}E|a_{ni}X|^2(b_ns<|a_{ni}X|\le b_n(s+1))\sum_{m=s}^{\infty}m^{\tau-3}\\
&= C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\sum_{i=1}^{n}\sum_{s=1}^{\infty}s^{\tau-2}E|a_{ni}X|^2(b_ns<|a_{ni}X|\le b_n(s+1))\\
&= C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^2}\sum_{i=1}^{n}\sum_{s=1}^{\infty}s^{\tau-2}E|a_{ni}X|^{\tau}|a_{ni}X|^{2-\tau}(b_ns<|a_{ni}X|\le b_n(s+1))\\
&\le C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^{\tau}}\sum_{i=1}^{n}E|a_{ni}X|^{\tau}I(|a_{ni}X|>b_n)\\
&\le C\sum_{n=2}^{\infty}\frac{\log^2 n}{nb_n^{\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^{\alpha}I(|a_{ni}X|>b_n)\\
&\le CJ_{12}<\infty. \qquad (3.14)
\end{aligned}
$$

By (3.11)–(3.14), we get (3.9). On the other hand, it is obviously to see that

$$K_{22}\le CK_{211}<\infty,$$

which completes the proof of Theorem 2.9. □

Proof of Theorem 2.11 For all $\varepsilon>0$, from (2.9) we know that

$$
\begin{aligned}
\infty &> \sum_{n=2}^{\infty}n^{-1}P\Big(\max_{1\le j\le n}|T_j|>\varepsilon b_n\Big)\\
&\ge \sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1}(2^{i+1}-1)^{-1}P\Big(\max_{1\le j\le n}|T_j|>\varepsilon b_n\Big)\\
&\ge C\sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1}(2^{i+1}-1)^{-1}P\Big(\max_{1\le j\le 2^i}|T_j|>\varepsilon b_{2^{i+1}}\Big)\\
&\ge C\sum_{i=1}^{\infty}P\Big(\max_{1\le j\le 2^i}|T_j|>\varepsilon b_{2^{i+1}}\Big). \qquad (3.15)
\end{aligned}
$$

It follows from Borel–Cantelli lemma that

$$P\Big(\max_{1\le j\le 2^i}|T_j|>\varepsilon b_{2^{i+1}}, i.o.\Big)=0. \qquad (3.16)$$

Hence

$$\frac{\max_{1\le j\le 2^i}|T_j|}{b_{2^{i+1}}}\longrightarrow 0, \quad \text{a.s. as } i\to\infty. \qquad (3.17)$$

For all positive integers n, there exists a non-negative integer i, such that $2^{i-1}\le n<2^i$. Thus

$$
\begin{aligned}
\max_{2^{i-1}\le n\le 2^i}\frac{|T_n|}{b_n} &\le \frac{\max_{1\le n\le 2^i}|T_j|}{b_{2^{i-1}}}\\
&= \frac{\max_{1\le n\le 2^i}|T_j|}{b_{2^{i+1}}}\cdot 2^{2/\alpha}\cdot\left(\frac{i+1}{i-1}\right)^{3/\gamma}\to 0, \quad \text{a.s. as } i\to\infty. \qquad (3.18)
\end{aligned}
$$

We have

$$\lim_{n\to\infty}\frac{|T_n|}{b_n}=0, \quad \text{a.s.} \qquad (3.19)$$

The proof of Theorem 2.11 is completed. □

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