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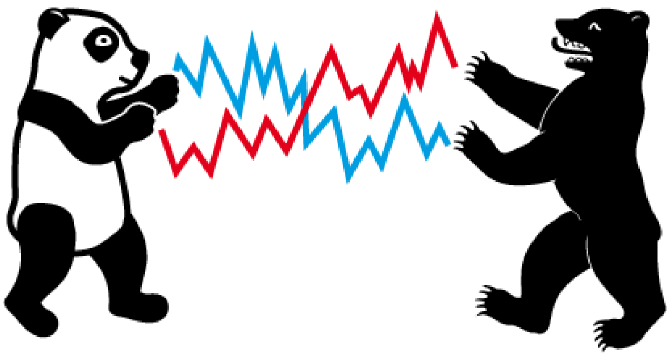

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On complete convergence in Marcinkiewicz-Zygmund type SLLN for random variables[1]

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Abstract: We consider a generalization of Baum-Katz theorem for random variables satisfying some cover conditions. Consequently, we get the result for many dependent structure, such as END, ϱ^*-mixing, ϱ^--mixing and φ-mixing, etc.
MSC (2010): 60F15.
Keywords: Complete convergence; Marcinkiewicz-Zygmund type SLLN; Extended negatively dependent; Mixing dependency; Weakly mean bounded.

1 Introduction

In this paper, we consider a sequence $\{X_n, n \geq 1\}$ of random variables defined on some probability space $(\Omega, \mathscr{F}, P)$. Hsu and Robbins [10] introduced the following concept of complete convergence. A sequence $\{X_n, n \geq 1\}$ is said to converge completely to a constant C if

$$\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty,$$

for all $\varepsilon > 0$. For independent and identically distributed (i.i.d., in short) random variables $\{X, X_n, n \geq 1\}$, let $S_n = \sum_{k=1}^{n} X_k, n \geq 1$ be the partial sums, Hsu and Robbins [10] proved that S_n/n converge completely to EX, provided $DX < \infty$. Erdős [8] proved the converse theorem. This Hsu-Robbins-Erdős's theorem was generalized in different ways. Katz [13], Baum and Katz [1], and Chow [7] formed the following generalization of Marcinkiewicz-Zygmund type.

Theorem 1.1 *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables and let $\alpha p \geq 1, \alpha > 1/2$. Then the following statements are equivalent:*
(i) $E|X|^p < \infty$ *and* $EX = 0$ *if* $p \geq 1$*;*
(ii) $\sum_{n=1}^{\infty} n^{\alpha p-2} P(|S_n| > \varepsilon n^{\alpha}) < \infty$ *for all* $\varepsilon > 0$*;*
(iii) $\sum_{n=1}^{\infty} n^{\alpha p-2} P(\max_{1\leq i\leq n} |S_i| > \varepsilon n^{\alpha}) < \infty$ *for all* $\varepsilon > 0$*;*
If $\alpha p > 1, \alpha > 1/2$ *the above are also equivalent to*
(iv) $\sum_{n=1}^{\infty} n^{\alpha p-2} P(\sup_{i\geq n} i^{-\alpha}|S_i| > \varepsilon) < \infty$ *for all* $\varepsilon > 0$.

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In many stochastic models, the assumption that random variables are independent is not plausible. Increases in some random variables are often related to decreases in other random variables, so an assumption of dependence is more appropriate than an assumption of independence. One of the important dependence structure is the extended negatively dependent structure, which was introduced by Liu [18] as follows.

Definition 1.1 *Random variables $X_k, k = 1, \cdots, n$ are said to be lower extended negatively dependent (LEND) if there is some $M > 0$ such that, for all real numbers $x_k, k = 1, \cdots, n$,*

$$P\left\{\bigcap_{k=1}^{n}(X_k \le x_k)\right\} \le M \prod_{k=1}^{n} P\{X_k \le x_k\}; \tag{1.1}$$

they are said to be upper extended negatively dependent (UEND) if there is some $M > 0$ such that, for all real numbers $x_k, k = 1, \cdots, n$,

$$P\left\{\bigcap_{k=1}^{n}(X_k > x_k)\right\} \le M \prod_{k=1}^{n} P\{X_k > x_k\}; \tag{1.2}$$

and they are said to be extended negatively dependent (END) if they are both LEND and UEND. A sequence of infinitely many random variables $\{X_k, k = 1, 2, \cdots\}$ is said to be LEND/UEND/END if there is some $M > 0$ such that, for each positive integer n, the random variables $X_1, X_2, \cdots, X_n$ are LEND/UEND/END, respectively.

In the case $M = 1$, the formula of END random variables reduces to the notion of negatively orthant dependent (NOD, in short) random variables which was introduced by Joag-Dev and Proschan [12]. They also pointed out that negatively associated (NA, in short) random variables are NOD random variables and then END.

As pointed out in Liu [18], the END structure covers many negative dependence structures and, more interestingly, it covers certain positive dependence structures. Hence, studying the limiting behavior of END random variables is of great significance. There are more and more literatures appeared. See, for example, Liu [18] obtained the precise large deviations for dependent random variables with heavy tails. Shen [25] presented some probability inequalities and gave some applications. Wu et al [33] considered some complete moment convergence and mean convergence theorems for the partial sums of END random variables. Qiu et al [24], Wang et al [31, 32] and Shen et al [27] investigated some results on complete convergence of END random variables. Chen et al [6], Yan [36, 37] considered the SLLN for END random variables, and so on.

An another group of dependencies is formed by mixing type structures defined by special sequences of mixing coefficients. Some of them, however defined in a way that is significantly different from the negative dependence structures, have many similar properties which allow us to use in consideration methods and tools similar to those used in END case. In further consideration we will deal with three types of mixing dependencies: ϱ^*-mixing, φ-mixing and ϱ^--mixing.

Definition 1.2 *A sequence of random variable $\{X_n, n \geq 1\}$ is said to be a ϱ^*-mixing sequence if there exists $k \in \mathbb{N}$ such that*

$$\varrho^*(k) = \sup_{S,T} \left(\sup_{X \in L^2(\mathcal{F}_S), Y \in L^2(\mathcal{F}_T)} \frac{cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}} \right) < 1,$$

where S, T are the finite subsets of positive integers such that dist $(S,T) \geq k$ and $\mathcal{F}_W$ is the σ-field generated by the random variable $\{X_i, i \in W \subset \mathbb{N}\}$.

Bradley [2] and Miller [19] studied various limit properties of random fields under the assumption $\varrho^*(k) \to 0$, $k \to \infty$. We refer to the results obtained under the condition $\varrho^*(k) < 1$ for some $k \in \mathbb{N}$ which is important in estimating the moments of partial and maxima of partial sums, see Bryc and Smolenski [3] and Peligrad [20]. Peligrad [20], Utev and Peligrad [29] investigated the properties of the maximum of partial sums and used them to obtain an invariance principle, Peligrad and Gut [21] presented Rosenthal-type maximal inequality and rate convergence for the Marcinkiewicz-Zygmund type SLLN, Cai [4] obtained SLLN and complete convergence for random variables with different distributions.

Definition 1.3 *A sequence of random variables $\{X_n, n \geq 1\}$ is called to be φ-mixing (or uniformly strong mixing) if*

$$\varphi(n) = \sup_{k \geq 1, A \in \mathcal{F}_1^k, P(A) > 0, B \in \mathcal{F}_{k+n}^\infty} |P(B|A) - P(B)| \to 0 \quad as \quad n \to \infty,$$

where $\mathcal{F}_n^m$ is the σ-field generated by random variables $X_n, X_{n+1}, \ldots, X_m$.

A concept of φ-mixing dependence was introduced independently by Rozanov and Volkonski [23] and Ibragimov [11]. A number of limit theorems for φ-mixing random variables have been established by many authors. We refer to Wang *at al* [30] (Rosenthal type maximal inequality, Hájek-Rényi type inequality, SLLN), Tuyen [28] (SLLN), and Chen *at al* [5] (complete convergence and Marcinkiewicz-Zygmund type SLLN of moving averages processes) and Kuczmaszewska [16] (complete convergence for NA, ϱ^*-mixing and φ-mixing sequences satisfying Petrov's condition).

Definition 1.4 *A sequence $\{X_n, n \geq 1\}$ is called ϱ^--mixing, if*

$$\varrho^-(n) = \sup\{\varrho^-(S,T) : S, T \subset \mathbb{N}, dist(S,T) \geq n\} \to 0, as\ n \to \infty,$$

where

$$\varrho^-(S,T) = 0 \vee \sup\{corr(f(X_i, i \in S), g(X_j, j \in T))\},$$

and the supremum is taken over all coordinatewise increasing real functions f on R^S and g on R^T.

Some results concerning the complete moment convergence, the complete convergence and strong law of large numbers of Marcinkiewicz-Zygmund type for moving average process generated by ϱ^--mixing seguences one can find in Zhang [38]. We also refer to Wang and Lu [34].

Most of results concerning limit theorems are formulated for identically distributed random variables. Pruss [22] introduced the following concept of regular cover which allowed to consider sequences without identical distribution.

Definition 1.5 *Let $X_1, X_2, \cdots, X_n$ be random variables, and X be a random variable possibly defined on a different probability space. Then, $X_1, X_2, \cdots, X_n$ are said to be a regular cover of (the distribution of) X provided we have*

$$E(G(X)) = \frac{1}{n}\sum_{k=1}^{n} E(G(X_k)), \tag{1.3}$$

for any measurable function G for which both sides make sense.

In this paper, we are interested in generalizations of the Baum-Katz result. Under some cover condition weaker than (1.3), Kuczmaszewska [14] extended the result to the case of negatively associated (NA, in short) sequence. They got the following result

Theorem 1.2 (*Kuczmaszewska [14]*). *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables and X be a random variable possibly defined on a different space satisfying the condition*

$$\frac{1}{n}\sum_{k=1}^{n} P(|X_k| > x) = c \cdot P(|X| > x) \tag{1.4}$$

for all $x > 0$, all $n \geq 1$ and some positive constant c. Let $\alpha p > 1$ and $\alpha > 1/2$. Moreover, additionally assume that for $p \geq 1$ $EX_n = 0$ for all $n \geq 1$. Then the following statements are equivalent:

(i) $E|X|^p < \infty$;

(ii) $\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq i\leq n} |S_i| > \varepsilon n^{\alpha}\right) < \infty$ *for all* $\varepsilon > 0$.

Though condition (1.4) is weak in some sense, it remains a strong condition, we even get the same result for arbitrary random variables with some rough conditions. Gut [9] introduced the following concept of weakly mean dominated

Definition 1.6 *We say that the array $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ is weakly mean dominated (WMD, in short) by the random variable X if, for some $\gamma > 0$,*

$$\frac{1}{n}\sum_{k=1}^{n} P(|X_{nk}| > x) \leq \gamma P(|X| > x), \tag{1.5}$$

for all $x > 0$ and all n.

A. Kuczmaszewska and Z. A.Lagodowski [15] introduced another structure which can also be used to prove results for non-identically distributed random variables.

Definition 1.7 *Random variables* $\{X_k, k \geq 1\}$ *are weakly mean bounded (WMB, in short) by random variable* X *(possibly defined on a different probability space) iff there exist some constant* $\gamma_1, \gamma_2 > 0$ *such that for all* $x > 0$ *and* $n \geq 1$

$$\gamma_1 \cdot P(|X| > x) \leq \frac{1}{n} \sum_{k=1}^{n} P(|X_k| > x) \leq \gamma_2 \cdot P(|X| > x). \tag{1.6}$$

Obviously, if a sequence $\{X_k, k \geq 1\}$ and a random variable X satisfy WMB condition, they must satisfy WMD ones. The aim of this paper is to consider the analogous generalization of the Baum-Katz theorem for a sequence of random variables satisfying WMD or WMB sense and some usual conditions (Marcinkiewicz-Zygmund type inequality and Rosenthal type inequality). The main results are provided in Section 2. Some lemmas and the proofs of the main results are presented in Section 3.

As usual, we note that C will be numerical constants whose value are without importance, and, in addition, may change between appearances. $I(A)$ is the indicator function on the set A. Denote $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$.

2 Main Results

Before presenting our main results, we first give the following assumptions.

Hypothesis. Let $\{X_n, n \geq 1\}$ be an arbitrary sequence of random variables satisfying for every $\theta \geq 2$ and $n \geq 1$

$$E\left(\left|\sum_{i=1}^{n} f_i(X_i)\right|\right)^{\theta} \leq C\left[\sum_{i=1}^{n} E|f_i(X_i)|^{\theta} + \left(\sum_{i=1}^{n} E|f_i(X_i)|^2\right)^{\theta/2}\right] \tag{2.1}$$

and

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} f_i(X_i)\right|\right)^{\theta} \leq C\log^{\theta} n\left[\sum_{i=1}^{n} E|f_i(X_i)|^{\theta} + \left(\sum_{i=1}^{n} E|f_i(X_i)|^2\right)^{\theta/2}\right], \tag{2.2}$$

whenever $f_1, f_2, \cdots, f_n$ are all nondecreasing (or non-increasing) functions, $Ef_i(X_i) = 0$ and $E|f_i(X_i))|^{\theta} < \infty$, for all $1 \leq i \leq n$.

Remark 2.1 *A lot of dependent structures, for example such as* ρ^**-mixing,* φ*-mixing, NA, ND, END, etc., satisfy* (2.1) *and* (2.2) *in Hypothesis.*

Theorem 2.1 *Suppose* $\alpha p > 1$, $\alpha > 1/2$. *Let* $\{X_n, n \geq 1\}$ *be an arbitrary sequence of random variables with* $EX_n = 0$ *for all* $n \geq 1$ *if* $p > 1$ *and* X *be a random variable possibly defined on a different probability space satisfying* (1.5) *for all* $x > 0$, *all* $n \geq 1$ *and some positive constant* γ. *Assume that* $\{X_n, n \geq 1\}$ *satisfies the conditions of Hypothesis. Then* $E|X|^p < \infty$ *implies that for all* $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq k\leq n} |S_k| > \varepsilon n^{\alpha}\right) < \infty, \tag{2.3}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P(\sup_{i\geq n} i^{-\alpha}|S_i| > \varepsilon) < \infty, \tag{2.4}$$

where $S_k = \sum_{i=1}^{k} X_i$, $1 \leq k \leq n$.

The next theorem presents the necessary condition for (2.3) under assumption that random variables $\{X_n, n \geq 1\}$ and X satisfy WMB condition (1.6).

Example 2.1 *We give an example of* (1.6). *Suppose that* $P(X_k = 1-\frac{1}{k}) = P(X_k = 2-\frac{1}{k}) = 1/2$ *for* $k = 1, 2, \cdots,$. *Then*

$$\frac{1}{n}\sum_{k=1}^{n} P(|X_k| > x) \to \begin{cases} 1, & 0 \leq x < 1, \\ 1/2, & 1 \leq x < 2, \\ 0, & x \geq 2. \end{cases}$$

that is, if the mean dominating random variable, X*, is such that* $P(X = 1) = P(X = 2) = 1/2$*, then* (1.6) *is satisfied.*

Theorem 2.2 *Suppose* $\alpha p > 1$, $\alpha > 1/2$. *Let* $\{X_n, n \geq 1\}$ *be a sequence of random variables satisfying* (2.1) *for* $\theta = 2$ *and* X *be a random variable possibly defined on a different probability space satisfying* (1.6) *for all* $x > 0$, *all* $n \geq 1$ *and some positive constants* γ_1 *and* γ_2. *Then* (2.3) *implies* $E|X|^p < \infty$.

As a consequence of Theorem 2.1 and Theorem 2.2 by Lemma 3.2 and Lemma 3.3 we get the following result.

Corollary 2.1 *Suppose* $\alpha p > 1$, $\alpha > 1/2$. *Let* $\{X_n, n \geq 1\}$ *be a sequence of END,* ϱ^**-mixing,* ϱ^-*-mixing or* φ*-mixing random variables and* X *be a random variable possibly defined on a different probability space satisfying* (1.6) *for all* $x > 0$, *all* $n \geq 1$ *and some positive constants* γ_1 *and* γ_2. *Moreover, we assume* $EX_n = 0$ *for all* $n \geq 1$ *if* $p > 1$ *and* $\sum_{n=1}^{\infty} \varphi^{\frac{1}{2}}(n) < \infty$ *in case of* φ*-mixing sequence. Then the following statments are equivalent:*

(*i*) $E|X|^p < \infty$;

(*ii*) $\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq i\leq n} |S_i| > \varepsilon n^{\alpha}\right) < \infty$ *for all* $\varepsilon > 0$.

Remark 2.2 *Since END and* ϱ^-*- mixing random variables include NA random variables, our result also holds for NA case.*

3 Some Lemmas and Proofs

To prove the main results of the paper, we need the following important lemmas.

Lemma 3.1 ($cf.$*Liu* [*18*]) *Let* $\{X_n, n \geq 1\}$ *be a sequence of END random variables. For each* $n \geq 1$, *if* $f_1, f_2, \cdots, f_n$ *are all nondecreasing (or nonincreasing) functions, then random variables* $f_1(X_1)$, $f_2(X_2)$, $\cdots$, $f_n(X_n)$ *are also END.*

Lemma 3.2 (*cf.Wang et al.[31]*) *Let $p \geq 2$ and $\{X_n, n \geq 1\}$ be a sequence of END random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for each $n \geq 1$. Then there exists a positive constant C_p depending only on p such that*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq C_p \left[\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{p/2}\right], \quad \text{for all } n \geq 1.$$

and

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \leq C_p(\log n)^p \left[\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{p/2}\right], \quad \text{for all } n \geq 1.$$

Lemma 3.3 (*cf.Utev and Peligrad[29], Wang and Lu[34], Wang et al.[30]*) *Let $p \geq 2$ and $\{X_n, n \geq 1\}$ be a sequence of ϱ^*-mixing, ϱ^--mixing or φ-mixing random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for each $n \geq 1$. Moreover, if X_n, $n \geq 1$ are φ-mixing we assume that $\sum_{n=1}^{\infty} \varphi^{\frac{1}{2}}(n) < \infty$. Then there exists a positive constant C_p depending only on p such that*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \leq C_p \left[\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{p/2}\right], \quad \text{for all } n \geq 1. \tag{3.1}$$

Lemma 3.4 (*cf.Gut [9]*) *Let $\{X_n, n \geq 1\}$ be a sequence of random variables satisfying a weak dominating condition with mean dominating random variable X, i.e. for some $c > 0$*

$$\frac{1}{n}\sum_{i=1}^{n} P(|X_i| > x) \leq cP(|X| > x).$$

Let $r > 0$ and for some $A > 0$

$$X_i' = X_i I(|X_i| \leq A), \quad X_i'' = X_i I(|X_i| > A),$$
$$X_i^* = X_i I(|X_i| \leq A) - AI(X_i < -A) + AI(X_i > A),$$

and

$$X' = XI(|X| \leq A), \quad X'' = XI(|X| > A),$$
$$X^* = XI(|X| \leq A) - AI(X < -A) + AI(X > A).$$

Then

(*i*) *if $E|X|^r < \infty$, then $\frac{1}{n}\sum_{i=1}^{n} E|X_i|^r \leq CE|X|^r$;*
(*ii*) *$\frac{1}{n}\sum_{i=1}^{n} E|X_i'|^r \leq C\left(E|X'|^r + A^r P(|X| > A)\right)$ for any $A > 0$;*
(*iii*) *$\frac{1}{n}\sum_{i=1}^{n} E|X_i''|^r \leq CE|X''|^r$ for any $A > 0$;*
(*iv*) *$\frac{1}{n}\sum_{i=1}^{n} E|X_i^*|^r \leq CE|X^*|^r$ for any $A > 0$.*

Now, we present the proofs of the main results step by step.

Proof of Theorem 2.1. We first take

$$0 < p' < p, \qquad \frac{1}{\alpha p} < q < 1$$

such that

$$\alpha(p-p') > \alpha(p-p')q > 1, \quad \text{and} \quad p-p' > 1 \text{ if } p > 1.$$

For all $1 \le i \le n, n \ge 1$, denote that

$$\begin{aligned} X_{ni}^{(1)} &= -n^{\alpha q} I(X_i < -n^{\alpha q}) + X_i I(|X_i| \le n^{\alpha q}) + n^{\alpha q} I(X_i > n^{\alpha q}); \\ X_{ni}^{(2)} &= (X_i - n^{\alpha q}) I(X_i > n^{\alpha q}); \\ X_{ni}^{(3)} &= -(X_i + n^{\alpha q}) I(X_i < -n^{\alpha q}). \end{aligned}$$

Then, for every $1 \le i \le n$, $n \ge 1$

$$X_i = X_{ni}^{(1)} + X_{ni}^{(2)} - X_{ni}^{(3)} \quad \text{and} \quad X_{ni}^{(2)} \ge 0, \ X_{ni}^{(3)} \ge 0.$$

Thus,

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n} |S_k| > \varepsilon n^{\alpha}\right) \\ \le \; & \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n} \left|\sum_{i=1}^{k} X_{ni}^{(1)}\right| > \varepsilon n^{\alpha}/3\right) + \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n} \left|\sum_{i=1}^{k} X_{ni}^{(2)}\right| > \varepsilon n^{\alpha}/3\right) \\ & \quad + \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n} \left|\sum_{i=1}^{k} X_{ni}^{(3)}\right| > \varepsilon n^{\alpha}/3\right) \\ = \; & \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n} \left|\sum_{i=1}^{k} X_{ni}^{(1)}\right| > \varepsilon n^{\alpha}/3\right) + \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\sum_{i=1}^{n} X_{ni}^{(2)} > \varepsilon n^{\alpha}/3\right) \\ & \quad + \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\sum_{i=1}^{n} X_{ni}^{(3)} > \varepsilon n^{\alpha}/3\right) \\ \triangleq \; & I_1 + I_2 + I_3. \text{ (in say)} \end{aligned}$$

To prove (2.3), it suffices to show that $I_k < \infty$, $k = 1, 2, 3$.

For I_1, we first prove that

$$n^{-\alpha} \max_{1\le k\le n} \left|\sum_{i=1}^{k} E X_{ni}^{(1)}\right| \to 0, \quad n \to \infty. \tag{3.2}$$

We will do it in three cases.

Case I. Let $\alpha \le 1$. Then $\alpha p > 1$ implies $p > 1$ and, according to the assumption,

$EX_n = 0, n \geq 1$. It follows from Lemma 3.4 that

$$\begin{aligned}
& n^{-\alpha} \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} EX_{ni}^{(1)} \right| \\
\leq\ & n^{-\alpha} \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} [EX_i I(|X_i| \leq n^{\alpha q}) + n^{\alpha q} P(|X_i| > n^{\alpha q})] \right| \\
=\ & n^{-\alpha} \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} [EX_i I(|X_i| > n^{\alpha q}) + n^{\alpha q} P(|X_i| > n^{\alpha q})] \right| \\
\leq\ & 2n^{-\alpha} \sum_{i=1}^{n} E|X_i| I(|X_i| > n^{\alpha q}) \leq 2n^{1-\alpha} E|X| I(|X| > n^{\alpha q}) \\
\leq\ & Cn^{1-\alpha} \cdot n^{\alpha q(1-(p-p'))} E|X|^{p-p'} \leq Cn^{-[\alpha q(p-p')-1]-\alpha(1-q)} \to 0, \quad n \to \infty.
\end{aligned}$$

Case II. Let $\alpha > 1$ and $p > 1$. We have

$$\begin{aligned}
& n^{-\alpha} \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} EX_{ni}^{(1)} \right| \leq n^{-\alpha} \sum_{i=1}^{n} [E|X_i| I(|X_i| \leq n^{\alpha q}) + n^{\alpha q} P(|X_i| > n^{\alpha q})] \\
\leq\ & Cn^{1-\alpha} [E|X| I(|X| \leq n^{\alpha q}) + n^{\alpha q} P(|X| > n^{\alpha q})] \leq Cn^{1-\alpha} \to 0, \quad n \to \infty.
\end{aligned}$$

Case III. Let $\alpha > 1$ and $p \leq 1$. We have

$$\begin{aligned}
& n^{-\alpha} \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} EX_{ni}^{(1)} \right| \leq n^{-\alpha} \sum_{i=1}^{n} [E|X_i| I(|X_i| \leq n^{\alpha q}) + n^{\alpha q} P(|X_i| > n^{\alpha q})] \\
\leq\ & Cn^{1-\alpha} [E|X| I(|X| \leq n^{\alpha q}) + n^{\alpha q} P(|X| > n^{\alpha q})] \\
\leq\ & Cn^{1-\alpha} \cdot n^{\alpha q(1-(p-p'))} E|X|^{p-p'} \leq Cn^{-[\alpha q(p-p')-1]-\alpha(1-q)} \to 0, \quad n \to \infty.
\end{aligned}$$

By (3.2), to prove $I_1 < \infty$, we prove only that

$$I_1^* \triangleq \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} \left(X_{ni}^{(1)} - EX_{ni}^{(1)} \right) \right| > \varepsilon n^{\alpha}/6 \right) < \infty. \tag{3.3}$$

From Hypothesis, for each $n \geq 1$, $\{X_{ni}^{(1)} - EX_{ni}^{(1)}, 1 \leq i \leq n\}$ remain satisfy the inequalities in Hypothesis. By $\alpha q(p-p') > 1$ and $0 < q < 1$, we have for $p \leq 2$

$$\alpha - \frac{1}{2} - \alpha q \left(1 - \frac{p-p'}{2} \right) > \alpha - \frac{1}{2} - \alpha \left(1 - \frac{p-p'}{2} \right) = \frac{\alpha(p-p')-1}{2} > 0.$$

By taking

$$\tau > \max\left\{ 2, p, \frac{p-(p-p')q}{1-q}, \frac{\alpha p-1}{\alpha - \frac{1}{2}}, \frac{\alpha p-1}{\alpha - \frac{1}{2} - \alpha q\left(1 - \frac{p-p'}{2}\right)} \right\},$$

Chebyshev's inequality and Hypothesis we get

$$\begin{aligned}
I_1^* &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\tau} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\left(X_{ni}^{(1)}-EX_{ni}^{(1)}\right)\right|\right)^{\tau} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\tau}\log^{\tau} n\sum_{i=1}^{n} E\left|X_{ni}^{(1)}\right|^{\tau}+C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\tau}\log^{\tau} n\left(\sum_{i=1}^{n} E\left|X_{ni}^{(1)}\right|^{2}\right)^{\tau/2} \\
&\triangleq I_{11}^*+I_{12}^*. \text{ (in say)}
\end{aligned}$$

Again by Hypothesis

$$\begin{aligned}
I_{11}^* &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\tau}\log^{\tau} n\sum_{i=1}^{n}\left[E|X_i|^{\tau} I(|X_i|\le n^{\alpha q})+n^{\alpha q\tau}P(|X_i|>n^{\alpha q})\right] \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\tau}\log^{\tau} n\cdot n^{\alpha q(\tau-(p-p'))}E|X|^{p-p'} \\
&\le C\sum_{n=1}^{\infty} n^{-\alpha(1-q)\left(\tau-\frac{p-q(p-p')}{1-q}\right)-1}\log^{\tau} n<\infty,
\end{aligned}$$

and

$$I_{12}^*\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\tau}\log^{\tau} n\left\{\sum_{i=1}^{n}\left[E|X_i|^2 I(|X_i|\le n^{\alpha q})+n^{2\alpha q}P(|X_i|>n^{\alpha q})\right]\right\}^{\tau/2}.$$

Now, we prove $I_{12}^*<\infty$ in two cases: $p>2$ and $0<p\le 2$.

Let $p>2$.

$$I_{12}^*\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\tau}\log^{\tau} n\cdot n^{\tau/2}\left(EX^2\right)^{\tau/2}\le C\sum_{n=1}^{\infty} n^{-(\alpha-\frac{1}{2})\left(\tau-\frac{\alpha p-1}{\alpha-\frac{1}{2}}\right)-1}\log^{\tau} n<\infty.$$

For $0<p\le 2$ we have

$$\begin{aligned}
I_{12}^* &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\tau}\log^{\tau} n\cdot\left(n^{\alpha q(2-(p-p'))}\cdot n\right)^{\tau/2}\left(E|X|^{p-p'}\right)^{\tau/2} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\tau\left[\alpha-\frac{1}{2}-\alpha q\left(1-\frac{p-p'}{2}\right)\right]}\log^{\tau} n<\infty.
\end{aligned}$$

This ends the proof of $I_1<\infty$. Next for $I_2<\infty$ and each $1\le i\le n, n\ge 1$, let

$$Y_{ni}^{(2)}=(X_i-n^{\alpha q})I(n^{\alpha q}<X_i\le n^{\alpha q}+n^{\alpha})+n^{\alpha}I(X_i>n^{\alpha q}+n^{\alpha}).$$

Then $Y_{ni}^{(2)}\ge 0$ and

$$X_{ni}^{(2)}=Y_{ni}^{(2)}+(X_i-n^{\alpha q}-n^{\alpha})I(X_i>n^{\alpha q}+n^{\alpha}).$$

Thus,

$$
\begin{aligned}
I_2 &\leq \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\sum_{i=1}^{n} Y_{ni}^{(2)} > \frac{\varepsilon n^{\alpha}}{6}\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\sum_{i=1}^{n} (X_i - n^{\alpha q} - n^{\alpha}) I(X_i > n^{\alpha q} + n^{\alpha}) > \frac{\varepsilon n^{\alpha}}{6}\right) \\
&\leq \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\sum_{i=1}^{n} Y_{ni}^{(2)} > \frac{\varepsilon n^{\alpha}}{6}\right) + \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{i=1}^{n} P(X_i > n^{\alpha q} + n^{\alpha}) \\
&\triangleq I_{21} + I_{22}. \text{ (in say)}
\end{aligned}
$$

By Lemma 3.4,

$$
\begin{aligned}
I_{22} &\leq \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{i=1}^{n} P(|X_i| > n^{\alpha}) \leq C \sum_{n=1}^{\infty} n^{\alpha p-1} P(|X| > n^{\alpha}) \\
&= C \sum_{n=1}^{\infty} n^{\alpha p-1} \sum_{i=n}^{\infty} P(i^{\alpha} < |X| \leq (i+1)^{\alpha}) \\
&= C \sum_{i=1}^{\infty} P(i^{\alpha} < |X| \leq (i+1)^{\alpha}) \sum_{n=1}^{i} n^{\alpha p-1} \\
&\leq C \sum_{i=1}^{\infty} i^{\alpha p} P(i^{\alpha} < |X| \leq (i+1)^{\alpha}) \leq CE|X|^p < \infty.
\end{aligned}
\tag{3.4}
$$

Next we prove only that $I_{21} < \infty$. We first show that

$$
n^{-\alpha} \sum_{i=1}^{n} EY_{ni}^{(2)} \to 0, \quad n \to \infty. \tag{3.5}
$$

If $p > 1$, then by Lemma 3.4

$$
\begin{aligned}
0 &\leq n^{-\alpha} \sum_{i=1}^{n} EY_{ni}^{(2)} \leq n^{-\alpha} \sum_{i=1}^{n} EX_i I(X_i > n^{\alpha q}) \\
&\leq n^{1-\alpha} EXI(X > n^{\alpha q}) \leq n^{1-\alpha} \cdot n^{\alpha q(1-(p-p'))} E|X|^{p-p'} \\
&\leq Cn^{-[\alpha q(p-p')-1]-\alpha(1-q)} \to 0, \quad n \to \infty.
\end{aligned}
$$

If $0 < p \leq 1$, then by Lemma 3.4

$$
\begin{aligned}
0 &\leq n^{-\alpha} \sum_{i=1}^{n} EY_{ni}^{(2)} \leq n^{-\alpha} \sum_{i=1}^{n} \left[EX_i I(|X_i| \leq 2n^{\alpha}) + n^{\alpha} P(|X_i| > 2n^{\alpha q})\right] \\
&\leq Cn^{1-\alpha} \left[EXI(|X| \leq 2n^{\alpha}) + 2n^{\alpha} P(|X| > 2n^{\alpha}) + n^{\alpha} P(|X| > 2n^{\alpha q})\right] \\
&\leq Cn^{1-\alpha} \left[n^{\alpha(1-(p-p'))} + n^{\alpha-\alpha q(p-p')}\right] \\
&\leq Cn^{-[\alpha q(p-p')-1]} \to 0, \quad n \to \infty.
\end{aligned}
$$

By (3.5), to prove $I_{21} < \infty$, it is sufficient to show that

$$I_{21}^* = \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\left|\sum_{i=1}^{n}\left(Y_{ni}^{(2)} - EY_{ni}^{(2)}\right)\right| > \frac{\varepsilon n^{\alpha}}{12}\right) < \infty.$$

We will prove it in two cases: $0 < p < 2$ and $p \geq 2$.

For $0 < p < 2$, by Chebyshev's inequality, Hypothesis, Lemma 3.4 and (3.4) we have

$$\begin{aligned}
I_{21}^* &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} E\left|\sum_{i=1}^{n}\left(Y_{ni}^{(2)} - EY_{ni}^{(2)}\right)\right|^2 \leq C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} \sum_{i=1}^{n} E\left|Y_{ni}^{(2)}\right|^2 \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} \sum_{i=1}^{n}\left[E|X_i|^2 I(n^{\alpha q} < X_i \leq n^{\alpha q} + n^{\alpha}) + n^{2\alpha} P(|X_i| > n^{\alpha q} + n^{\alpha})\right] \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} \sum_{i=1}^{n} E|X_i|^2 I(|X_i| \leq 2n^{\alpha}) + C\sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{i=1}^{n} P(|X_i| > n^{\alpha}) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} E|X|^2 I(|X| \leq 2n^{\alpha}) + C\sum_{n=1}^{\infty} n^{\alpha p-1} P(|X| > n^{\alpha}) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} \sum_{i=1}^{n} E|X|^2 I(2(i-1)^{\alpha} < |X| \leq 2i^{\alpha}) + CE|X|^p \\
&= C\sum_{i=1}^{\infty} E|X|^2 I(2(i-1)^{\alpha} < |X| \leq 2i^{\alpha}) \sum_{n=i}^{\infty} n^{\alpha p-1-2\alpha} + CE|X|^p \\
&\leq C\sum_{i=1}^{\infty} i^{\alpha p-2\alpha} E|X|^2 I(2(i-1)^{\alpha} < |X| \leq 2i^{\alpha}) + CE|X|^p \\
&\leq CE|X|^p < \infty.
\end{aligned}$$

Let $p \geq 2$. By taking

$$\kappa > \max\left\{p, \frac{\alpha p - 1}{\alpha - \frac{1}{2}}, \frac{2(\alpha p - 1)}{\alpha(p - p') - 1}\right\}$$

Chebyshev's inequality and Hypothesis we have

$$\begin{aligned}
I_{21}^* &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\kappa} E\left|\sum_{i=1}^{n}\left(Y_{ni}^{(2)} - EY_{ni}^{(2)}\right)\right|^{\kappa} \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\kappa} \sum_{i=1}^{n} E\left|Y_{ni}^{(2)}\right|^{\kappa} + C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\kappa}\left(\sum_{i=1}^{n} E\left|Y_{ni}^{(2)}\right|^2\right)^{\kappa/2} \\
&\triangleq I_{211}^* + I_{212}^*. \text{ (in say)}
\end{aligned}$$

It is easy to see that

$$I_{212}^* \leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\kappa}\left(\sum_{i=1}^{n} EX_i^2\right)^{\kappa/2} \leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\kappa+\frac{\kappa}{2}} < \infty.$$

On the other hand, by Lemma 3.4 and (3.4) we get

$$\begin{aligned}
I_{211}^* &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\kappa}\sum_{i=1}^{n}[E|X_i|^{\kappa}I(n^{\alpha q}<X_i\leq n^{\alpha}+n^{\alpha q})+n^{\alpha\kappa}P(|X_i|>n^{\alpha}+n^{\alpha q})]\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\kappa}\sum_{i=1}^{n}[E|X_i|^{\kappa}I(|X_i|\leq 2n^{\alpha})+n^{\alpha\kappa}P(|X_i|>n^{\alpha})]\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\kappa}[E|X|^{\kappa}I(|X|\leq 2n^{\alpha})+n^{\alpha\kappa}P(|X|>n^{\alpha})]\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\kappa}E|X|^{\kappa}I(|X|\leq 2n^{\alpha})+CE|X|^p\\
&= C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\kappa}\sum_{i=1}^{n}E|X|^{\kappa}I(2(i-1)^{\alpha}<|X|\leq 2i^{\alpha})+CE|X|^p\\
&= C\sum_{i=1}^{\infty} E|X|^{\kappa}I(2(i-1)^{\alpha}<|X|\leq 2i^{\alpha})\sum_{n=i}^{\infty} n^{\alpha p-1-\alpha\kappa}+CE|X|^p\\
&\leq C\sum_{i=1}^{\infty} n^{\alpha p-\alpha\kappa}E|X|^{\kappa}I(2(i-1)^{\alpha}<|X|\leq 2i^{\alpha})+CE|X|^p\\
&\leq CE|X|^p<\infty.
\end{aligned}$$

To prove the second thesis of Theorem 2.1 it is enough to show that (2.3) implies (2.4). For $0<\varepsilon<1$ and $\alpha p>1$ we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2}P\left(\sup_{k\geq n}k^{-\alpha}|S_k|>\varepsilon\right)\\
=\ &\sum_{j=1}^{\infty}\sum_{n=2^{j-1}}^{2^j-1} n^{\alpha p-2}P\left(\sup_{k\geq n}k^{-\alpha}|S_k|>\varepsilon\right)\leq C\sum_{j=1}^{\infty}2^{j(\alpha p-1)}P\left(\sup_{k\geq 2^{j-1}}k^{-\alpha}|S_k|>\varepsilon\right)\\
\leq\ &C\sum_{j=1}^{\infty}2^{j(\alpha p-1)}\sum_{i=j}^{\infty}P\left(\max_{2^{i-1}\leq k<2^i}k^{-\alpha}|S_k|>\varepsilon\right)\\
\leq\ &C\sum_{i=1}^{\infty}P\left(\max_{2^{i-1}\leq k<2^i}k^{-\alpha}|S_k|>\varepsilon\right)\sum_{j=1}^{i}2^{j(\alpha p-1)}\leq C\sum_{i=1}^{\infty}2^{i(\alpha p-1)}P\left(\max_{2^{i-1}\leq k<2^i}k^{-\alpha}|S_k|>\varepsilon\right)\\
\leq\ &C\sum_{i=1}^{\infty}2^{i(\alpha p-1)}P\left(\max_{2^{i-1}\leq k<2^i}|S_k|>\varepsilon 2^{(i-1)\alpha}\right)\leq C\sum_{i=1}^{\infty}2^{i(\alpha p-1)}P\left(\max_{1\leq k\leq 2^i}|S_k|>\varepsilon 2^{-\alpha}2^{i\alpha}\right)\\
\leq\ &C\sum_{i=1}^{\infty}n^{\alpha p-2}P\left(\max_{1\leq k\leq n}|S_k|>\varepsilon 2^{-2\alpha}n^{\alpha}\right)<\infty.
\end{aligned}$$

This ends the proof of Theorem 2.1. □

Proof of Theorem 2.2. In order to prove the result, it suffices to show that

$$\sum_{n=1}^{\infty} n^{\alpha p-1} P(|X| > n^{\alpha}) < \infty. \tag{3.6}$$

By (2.3), we first note that for $0 < \varepsilon < 1$

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le i\le n} |X_i| > n^{\alpha}\right) \le \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le i\le n} |X_i| > \varepsilon n^{\alpha}\right) \\
= &\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le i\le n} |S_i - S_{i-1}| > \varepsilon n^{\alpha}\right) \qquad (S_0 = 0) \\
\le &\ 2\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le i\le n} |S_i| > \varepsilon n^{\alpha}/2\right) < \infty. \qquad (3.7)
\end{aligned}$$

Therefore,

$$P\left(\max_{1\le i\le n} |X_i| > n^{\alpha}\right) \to 0, \quad \text{as } n \to \infty. \tag{3.8}$$

Moreover, by Lemma **??** and (1.6)

$$\begin{aligned}
&\gamma_1 \cdot nP(|X| > n^{\alpha}) \le \sum_{k=1}^{n} P(|X_k| > n^{\alpha}) \\
= &\sum_{k=1}^{n} P\left(|X_k| > n^{\alpha}, \max_{1\le i<k} |X_i| > n^{\alpha}\right) + \sum_{k=1}^{n} P\left(|X_k| > n^{\alpha}, \max_{1\le i<k} |X_i| \le n^{\alpha}\right) \\
= &\sum_{k=1}^{n} P\left(X_k^{+} > n^{\alpha}, \max_{1\le i<k} |X_i| > n^{\alpha}\right) + \sum_{k=1}^{n} P\left(X_k^{-} > n^{\alpha}, \max_{1\le i<k} |X_i| > n^{\alpha}\right) \\
&+P\left(\bigcup_{k=1}^{n} \left\{|X_k| > n^{\alpha}, \max_{1\le i<k} |X_i| \le n^{\alpha}\right\}\right) \\
= &\sum_{k=1}^{n} P\left(X_k^{+} > n^{\alpha}, \max_{1\le i<k} |X_i| > n^{\alpha}\right) + \sum_{k=1}^{n} P\left(X_k^{-} > n^{\alpha}, \max_{1\le i<k} |X_i| > n^{\alpha}\right) \\
&+P\left(\max_{1\le i\le n} |X_i| > n^{\alpha}\right), \qquad (3.9)
\end{aligned}$$

since the sets $\left\{|X_k| > n^{\alpha}, \max_{1\le i<k} |X_i| \le n^{\alpha}\right\}, 1 \le k \le n$ are disjoint.

By Hypothesis, $\{X_n^{+}, n \ge 1\}$ still satisfy the inequality in Hypothesis. It follows from Cauchy-Schwarz inequality and Hypothesis that

$$\begin{aligned}
&\sum_{k=1}^{n} P\left(X_k^{+} > n^{\alpha}, \max_{1\le i<k} |X_i| > n^{\alpha}\right) = E\left[\sum_{k=1}^{n} I\left(X_k^{+} > n^{\alpha}, \max_{1\le i<k} |X_i| > n^{\alpha}\right)\right] \\
= &\ E\left[\sum_{k=1}^{n} I\left(X_k^{+} > n^{\alpha}\right) I\left(\max_{1\le i<k} |X_i| > n^{\alpha}\right)\right] \le E\left[\sum_{k=1}^{n} I\left(X_k^{+} > n^{\alpha}\right) I\left(\max_{1\le i\le n} |X_i| > n^{\alpha}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= E\left[\sum_{k=1}^{n}\left\{I\left(X_k^+ > n^\alpha\right) - P\left(X_k^+ > n^\alpha\right)\right\} I\left(\max_{1\le i\le n}|X_i| > n^\alpha\right)\right] \\
&\quad + E\left[\sum_{k=1}^{n} P\left(X_k^+ > n^\alpha\right) I\left(\max_{1\le i\le n}|X_i| > n^\alpha\right)\right] \\
&\le \sqrt{E\left(\sum_{k=1}^{n}\left\{I\left(X_k^+ > n^\alpha\right) - P\left(X_k^+ > n^\alpha\right)\right\}\right)^2 P\left(\max_{1\le i\le n}|X_i| > n^\alpha\right)} \\
&\quad + P\left(\max_{1\le i\le n}|X_i| > n^\alpha\right)\sum_{k=1}^{n} P\left(X_k^+ > n^\alpha\right) \\
&\le \sqrt{C_2\sum_{k=1}^{n} E\left\{I\left(X_k^+ > n^\alpha\right) - P\left(X_k^+ > n^\alpha\right)\right\}^2 P\left(\max_{1\le i\le n}|X_i| > n^\alpha\right)} \\
&\quad + P\left(\max_{1\le i\le n}|X_i| > n^\alpha\right)\sum_{k=1}^{n} P\left(X_k^+ > n^\alpha\right) \\
&\le \sqrt{C_2\sum_{k=1}^{n} P(X_k^+ > n^\alpha) P\left(\max_{1\le i\le n}|X_i| > n^\alpha\right)} + P\left(\max_{1\le i\le n}|X_i| > n^\alpha\right)\sum_{k=1}^{n} P\left(X_k^+ > n^\alpha\right) \\
&\le \sqrt{C_2\gamma_2 n P(|X| > n^\alpha) P\left(\max_{1\le i\le n}|X_i| > n^\alpha\right)} + \gamma_2 P\left(\max_{1\le i\le n}|X_i| > n^\alpha\right)\cdot nP\left(|X| > n^\alpha\right) \\
&\le \frac{\gamma_1 n}{4} P(|X| > n^\alpha) + \frac{\gamma_2 C_2}{\gamma_1} P\left(\max_{1\le i\le n}|X_i| > n^\alpha\right) \\
&\quad + \gamma_2 P\left(\max_{1\le i\le n}|X_i| > n^\alpha\right)\cdot nP\left(|X| > n^\alpha\right),
\end{aligned} \tag{3.10}
$$

where we used the following inequality

$$
\sqrt{ab} \le \frac{a\gamma_1}{4\gamma_2 C_2} + \frac{\gamma_2 C_2}{\gamma_1} b, \quad a \ge 0, b \ge 0.
$$

Similarly, we can have

$$
\begin{aligned}
&\sum_{k=1}^{n} P\left(X_k^- > n^\alpha, \max_{1\le i<k}|X_i| > n^\alpha\right) \\
&\le \frac{\gamma_1 n}{4} P(|X| > n^\alpha) + \frac{\gamma_2 C_2}{\gamma_1} P\left(\max_{1\le i\le n}|X_i| > n^\alpha\right) \\
&\quad + \gamma_2 P\left(\max_{1\le i\le n}|X_i| > n^\alpha\right)\cdot nP\left(|X| > n^\alpha\right),
\end{aligned} \tag{3.11}
$$

Now we see that (3.9), (3.10) and (3.11) lead to

$$\begin{aligned}\frac{\gamma_1 n}{2} P(|X| > n^\alpha) &\leq \frac{2\gamma_2 C_2 + \gamma_1}{\gamma_1} P\left(\max_{1\leq i\leq n} |X_i| > n^\alpha\right) \\ &\quad + 2\gamma_2 n P\left(\max_{1\leq i\leq n} |X_i| > n^\alpha\right) P\left(|X| > n^\alpha\right)\end{aligned}$$

By (3.8), for sufficiently large n we have

$$P\left(\max_{1\leq i\leq n} |X_i| > n^\alpha\right) < \frac{\gamma_1}{8\gamma_2},$$

and consequently

$$nP(|X| > n^\alpha) \leq \frac{4(2\gamma_2 C_2 + \gamma_1)}{\gamma_1} P\left(\max_{1\leq i\leq n} |X_i| > n^\alpha\right). \tag{3.12}$$

Relations (3.7) and (3.12) give (3.6) and in conclusion we get the desired condition $E|X|^p < \infty$. This ends the proof of Theorem 2.2. □

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