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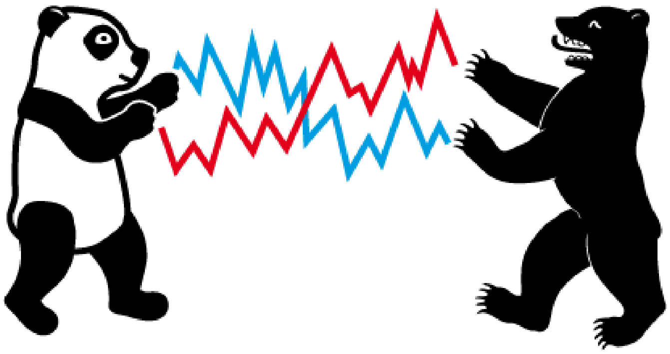

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On Complete Convergence in Marcinkiewicz-Zygmund Type SLLN for END Random Variables and its Applications[1]

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Abstract: In this paper, the complete convergence for maximal weighted sums of extended negatively dependent (END, for short) random variables is investigated. Some sufficient conditions for the complete convergence and some applications to a nonparametric model are provided. The results obtained in the paper generalise and improve the corresponding ones of Wang el al. (2014b) and Shen, Xue, and Wang (2017).

MSC (2010): 60F15.

Keywords: Complete convergence; Maximal weighted sums; Extended negatively dependent.

1 Introduction

In this paper, we consider a sequence $\{X_n, n \geq 1\}$ of random variables defined on some probability space $(\Omega, \mathscr{F}, P)$.

It is well known that the complete convergence plays an important role in establishing almost sure convergence of summation of random variables as well as weighted sums of random variables. The concept of the complete convergence was introduced by Hus and Robbins (1947) as follows.

Definition 1.1 *A sequence of random variables $\{X_n, n \geq 1\}$ is said to converge completely to a constant C if for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty.$$

For independent and identically distributed (i.i.d., in short) random variables $\{X, X_n,\ n \geq 1\}$, let $S_n = \sum_{k=1}^{n} X_k, n \geq 1$ be the partial sums, Hsu and Robbins (1947) proved that S_n/n converge completely to EX, provided $DX < \infty$. Erdős (1949) proved the converse theorem. This Hus-Robbins-Erdős's theorem was generalized in different ways. Kate (1963), Baum and Katz (1965), and Chow (1988) formed the following generalization of Marcinkiewicz-Zygmund type.

Theorem 1.1 *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables and let $\alpha p \geq 1, \alpha > 1/2$. The following statements are equivalent:*

(i) $E|X|^p < \infty$ *and* $EX = 0$ *if* $p \geq 1$*;*

(ii) $\sum_{n=1}^{\infty} n^{\alpha p-2} P(|S_n| > \varepsilon n^{\alpha}) < \infty$ *for all* $\varepsilon > 0$*;*

(iii) $\sum_{n=1}^{\infty} n^{\alpha p-2} P(\max_{1\leq i\leq n} |S_i| > \varepsilon n^{\alpha}) < \infty$ *for all* $\varepsilon > 0$*;*

If $\alpha p > 1, \alpha > 1/2$ the above are also equivalent to

(iv) $\sum_{n=1}^{\infty} n^{\alpha p-2} P(\sup_{i\geq n} i^{-\alpha}|S_i| > \varepsilon) < \infty$ *for all* $\varepsilon > 0$.

In convenient, we call item $n^{\alpha p-2}$ as the weight function of the tail probabilities. Many useful linear statistics are weighted sums of independent and identically distributed random variables, such as, least-squares estimators, nonparametric regression function estimators and jackknife estimates, and so on. However, in many stochastic model, the assumption that random variables are independent is not plausible. Increases in some random variables are often related to decreases in other random variables, so an assumption of dependence is more appropriate than an assumption of independence. One of the important dependence structure is the extended negatively dependent structure, which was introduced by Liu (2009) as follows.

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Definition 1.2 *Random variables $X_k, k = 1, \cdots, n$ are said to be lower extended negatively dependent (LEND) if there is some $M > 0$ such that, for all real numbers $x_k, k = 1, \cdots, n$,*

$$P\left\{\bigcap_{k=1}^{n}(X_k \leq x_k)\right\} \leq M \prod_{k=1}^{n} P\{X_k \leq x_k\}; \tag{1.1}$$

they are said to be upper extended negatively dependent (UEND) if there is some $M > 0$ such that, for all real numbers $x_k, k = 1, \cdots, n$,

$$P\left\{\bigcap_{k=1}^{n}(X_k > x_k)\right\} \leq M \prod_{k=1}^{n} P\{X_k > x_k\}; \tag{1.2}$$

and they are said to be extended negatively dependent (END) if they are both LEND and UEND. A sequence of infinitely many random variables $\{X_k, k = 1, 2, \cdots\}$ is said to be LEND/UEND/END if for each positive integer n, the random variables $X_1, X_2, \cdots, X_n$ are LEND/UEND/END, respectively.

The END structure covers all negative dependence structures and, more interestingly, it covers certain positive dependence structures. Some results for END sequence have been found. See, for example, Liu (2009) obtained the precise large deviations for dependent random variables with heavy tails. Liu (2010) studied the sufficient and necessary conditions of moderate deviations for dependent random variables with heavy tails. Shen (2011) gave the probability inequalities. Chen, Chen, and Ng (2010) considered the SLLN for END random variables and applications to risk theory and renewal theory, and so on.

Recall that the sequence $\{X_n, n \geq 1\}$ is stochastically dominated by a random variable X if

$$\sup_{n\geq 1} P\{|X_n| > t\} \leq CP\{|X| > t\}, \tag{1.3}$$

for some positive constant C and all $t \geq 0$. A real valued function $l(x)$, positive and measurable on $[A, \infty)$ for some $A > 0$, is said to be slowly varying if

$$\lim_{x\to\infty} \frac{l(\lambda x)}{l(x)} = 1 \quad \text{for each } \lambda > 0.$$

Motivated by applications in statistics, many researchers extended the Baum-Katz type result for independent random variables to some dependent random variables, such as some mixing random variables, NA sequences, NOD random variables, END random variables and even WOD random variables. Recently, Wang et al. (2014b) established the following result on complete convergence for weighted sums of END random variables.

Theorem 1.2 *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed END random variables with $EX = 0$ and $E|X|^p < \infty$ for some $\alpha p > 1$ and $1/2 < \alpha \leq 1$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers such that*

$$\sum_{i=1}^{n} |a_{ni}|^q = O(n), \quad \textit{for some } q > p. \tag{1.4}$$

Then for any $\varepsilon > 0$,

$$\sum_{i=1}^{\infty} n^{\alpha p-2} P\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right) < \infty. \tag{1.5}$$

Based on the above work, Shen, Xue, and Wang (2017) generalised the weight functions of tail probabilities in Theorem 1.2 for non-identical distributed END random variables, and obtained the following result about complete convergence:

Theorem 1.3 *Let $0 < p < 2, \alpha > 0, \alpha p > 1$, and $\{X_n, n \geq 1\}$ be a sequence of END random variables, which is stochastically dominated by a random variable X. Let $l(x) > 0$ be a slowly varying function. Assume further that $EX_n = 0$ for each $n \geq 1$ if $p > 1$ and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers such that*

$$\sum_{i=1}^{n} a_{ni}^2 = O(n).$$

If

$$E[|X|^p l(|X|^{1/\alpha})] < \infty, \tag{1.6}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\left|\sum_{i=1}^{n} a_{ni} X_i\right| > \varepsilon n^{\alpha}\right) < \infty, \tag{1.7}$$

Motivated by the above works, we will further study the complete convergence for weighted sums of END random variables, the purpose of this article is threefold:

(1) generalise the condition of weight $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ in Theorem 1.3 to one in Theorem 1.2;

(2) take into account the maximum of weighted sums.

(3) the values of p and α are extended to $\alpha > \frac{1}{2}, \alpha p > 1$.

This work is organized as follows: main results on complete convergence for maximal weighted sums of END random variables and the applications to a nonparametric model are provided in Section 2. Some lemmas and the proofs of the main results are presented in Section 3.

Before we present our main results, we note that C will be numerical constants whose value are without importance, and, in addition, may change between appearances. $a_n = O(b_n)$ stands for $a_n \leq Cb_n$ for all $n \geq 1$ and $I(A)$ is the indicator function on the set A. Denote $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$.

2 Main Results

Our main results on complete convergence for maximal weighted sums of END random variables are as follows.

Theorem 2.1 *Let $\alpha > \frac{1}{2}, \alpha p > 1$ and $\ell(x)$ be a slowly varying function at infinity. Let $\{X_n, n \geq 1\}$ be a sequence of END random variables, which is stochastically dominated by a random variable X. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers such that $a_{ni} = 0$ or $|a_{ni}| > 1$ and,*

$$\sum_{i=1}^{n} |a_{ni}|^{\xi} = O(n), \quad for\ some\ \xi > p. \tag{2.1}$$

Moreover, additionally assume that $EX_n = 0$ for $\alpha \leq 1$. If

$$E|X|^p \ell(|X|^{1/\alpha}) < \infty, \tag{2.2}$$

then, for arbitrary $\varepsilon > 0$,

$$\sum_{n=n_0}^{\infty} n^{\alpha p-2} \ell(n) P\left(\max_{1\leq j\leq n} |T_j| \geq \varepsilon n^{\alpha}\right) < \infty, \tag{2.3}$$

where $T_k = \sum_{i=1}^{k} a_{ni} X_i, 1 \leq k \leq n, n \geq 1$ and, $n_0 \geq 1$ be a suitable constant such that $\ell(n) \geq 0$ well defined.

In the case of $a_{ni} = 0$ or $|a_{ni}| > 1$, we give the following two Remarks on Theorem 2.1.

Remark 2.1 *Comparing Theorem* 2.1 *with Theorem* 1.2*, the identical distribution in Theorem* 1.2 *is extended to the case of non-identical distribution. In addition, there are many examples of slowly varying functions which are positive and monotone non-decreasing, such as* $\ell(x) = 1, \ell(x) = \log x, \ell(x) = (\log x)^2$*, and so on. on the other hand, it is obvious that the complete convergence of maximal weighted sums imply* (1.5)*. Hence, the main result of this paper generalise the corresponding one of Wang el al. (2014b).*

Remark 2.2 *Comparing Theorem* 2.1 *with Theorem* 1.3*, we extend the condition* (2.1) *on weight* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *from* $\xi = 2$ *to some* $\xi > p$ *with the same moment condition. In addition,* $0 < p < 2, \alpha > 0, \alpha p > 1$ *in Theorem* 1.3 *implies* $\alpha > \frac{1}{p} > \frac{1}{2}$.

Theorem 2.2 *Let* $\alpha > \frac{1}{2}, \alpha p > 1$ *and* $\ell(x)$ *be a slowly varying function at infinity. Let* $\{X_n, n \ge 1\}$ *be a sequence of END random variables, which is stochastically dominated by a random variable* X*. Let* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *be an array of real numbers such that* $|a_{ni}| \le 1$, $1 \le i \le n, n \ge 1$*. Moreover, additionally assume that* $EX_n = 0$ *for* $\alpha \le 1$*. If there exists a constant* $\theta > \max\left\{p, \frac{\alpha p - 1}{\alpha - \frac{1}{2}}, \frac{2(\alpha p - 1)}{\alpha(p-\delta)-1}\right\}$*, such that*

$$E|X|^p \ell(|X|^{1/\alpha})(\log|X|)^\theta < \infty, \tag{2.4}$$

where $0 < \delta < p, \alpha(p-\delta) > 1$*, and* $p - \delta > 1$ *for* $p > 1$*, then* (2.3) *holds.*

Remark 2.3 *Comparing Theorem* 2.2 *with Theorem* 2.1*, the moment condition* (2.4) *is a little stronger than* (2.2)*. It is reasonable since we consider the complete convergence of tail probability for weighted partial sums* $\sum_{i=1}^n a_{ni}X_i$*, where decreases in weights are related to increases in random variables.*

Combining Theorem 2.1 and Theorem 2.2, we can give the following result.

Corollary 2.1 *Let* $\alpha > \frac{1}{2}, \alpha p > 1$ *and* $\ell(x)$ *be a slowly varying function at infinity. Let* $\{X_n, n \ge 1\}$ *be a sequence of END random variables, which is stochastically dominated by a random variable* X*. Let* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *be an array of real numbers satisfying* (2.1)*. Moreover, additionally assume that* $EX_n = 0$ *for* $\alpha \le 1$*. Then* (2.4) *implies* (2.3)*.*

Remark 2.4 *For* $\alpha p > 1, 1/2 < \alpha \le 1$, $\{X, X_n.n \ge 1\}$ *be a sequence of identically distributed END random variables with* $EX = 0$ *and* $E|X|^p < \infty$*, and* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *is an array of real numbers satisfying* (2.1)*, Zhang (2014) obtained* (2.3) *for* $\ell(x) \equiv 1$*. Although we sacrificed a little bit of the moment condition, we generalized not only the non-identical distribution and the slowly varying function, but also the parameter* α *to* $\alpha > 1/2$*.*

In what follows, we apply the result of Corollary 2.1 to a nonparametric regression model and investigate the strong convergence for the nonparametric regression estimator based on END errors.

Consider the following nonparametric regression model:

$$Y_{ni} = f(x_{ni}) + \varepsilon_{ni}, \quad i = 1, 2, \cdots, n,\ n \ge 1, \tag{2.5}$$

where x_{nk} are known fixed design points from a given compact set $A \subset \mathbb{R}^m$ for some $m \ge 1$, $f(\cdot)$ is an unknown regression function defined on A and ε_{ni} are random errors. Assume that for each $n \ge 1$, $(\varepsilon_{n1}, \varepsilon_{n2}, \cdots, \varepsilon_{nn})$ have the same distribution as $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$. As an estimator of $f(\cdot)$, the following weighted regression estimator will be considered:

$$f_{nk}(x) = \sum_{i=1}^k W_{ni}(x)Y_{ni}, \quad k = 1, 2, \cdots, n, \quad x \in A \subset \mathbb{R}^m, \tag{2.6}$$

where $W_{ni}(x) = W_{ni}(x; x_{n1}, x_{n2}, \cdots, x_{nn})$, $i = 1, 2, \cdots, n$ are the weight functions.

The class of estimator (2.6) was first introduced by Stone (1977) and next adapted by Georgiev (1983) to the fixed design case. Until now, the estimator (2.6) has been studied by many authors, such as Georgiev and Greblicki (1986), Müller (1987) and Georgiev (1988) among others for independent errors, and Roussas (1989), Fan (1990), Tran et al. (1996), Hu et al. (2002), Liang and Jing (2005), Yang et al. (2012), Wang et al. (2014a), Wang et al. (2014c), Shen (2016) and Yan (2017) among others for dependent errors. The aim of this section is to investigate the strong convergence of the nonparametric regression estimator $f_{nk}(x)$ by using the complete convergence that we established in Section 2.

For any function $f(x)$, we denote $c(f)$ as all continuity points of the function f on A. The norm $\|x\|$ is the Euclidean norm. For any fixed point $x \in A$, the following assumptions on weight functions $W_{ni}(x)$ will be used:

$\mathbf{H}_1$: $\max\limits_{1\le k\le n}\left|\sum\limits_{i=1}^{k} W_{ni}(x)-1\right| \to 0$ as $n\to\infty$;

$\mathbf{H}_2$: $\sum\limits_{i=1}^{n}|W_{ni}(x)| \le C < \infty$ for all $n \ge 1$;

$\mathbf{H}_3$: $\sum\limits_{i=1}^{n}|W_{ni}(x)|\cdot|f(x_{ni})-f(x)|I(\|x_{ni}-x\|>a) \to 0$ as $n\to\infty$ for all $a>0$.

Based on the assumptions above, we can get the following results on complete convergence of the maximum of the nonparametric regression estimator $f_{nk}(x)$.

Theorem 2.3 *Let $\alpha > 1/2, p > 2/\alpha$ and $\ell(x)$ be a slowly varying function at infinity. Let $\{\varepsilon_n, n \ge 1\}$ be a sequence of zero mean END random variables, which is stochastically dominated by a random variable ϵ and $E|\epsilon|^p\ell(|\epsilon|^{1/\alpha})(\log|\epsilon|)^\theta < \infty$, where θ as one in Theorem 2.2. Assume that assumptions $(\mathbf{H_1}) - (\mathbf{H_3})$ hold. If*

$$\max_{1\le k\le n}|W_{nk}| = O\left(n^{-\alpha}\right) \quad as \quad n\to\infty, \tag{2.7}$$

then for any $x \in c(f)$,

$$\max_{1\le k\le n}|f_{nk}(x)-f(x)| \to 0, \ completely, \ as\ n\to\infty. \tag{2.8}$$

Theorem 2.4 *Let $\alpha > 1/2$ and $\{\varepsilon_n, n\ge 1\}$ be a sequence of zero mean END random variables, which is stochastically dominated by a random variable ϵ and $E|\epsilon|^{2/\alpha}(\log|\epsilon|)^\theta < \infty$, where θ as one in Theorem 2.2. Assume that assumptions $(\mathbf{H_1}) - (\mathbf{H_3})$ hold. Then for any $x \in c(f)$, (2.7) implies (2.8).*

3 Some Lemmas and Proofs

To prove the main results of the paper, we need the following important lemmas.

Lemma 3.1 (*cf.Liu* (2009)) *Let $\{X_n, n\ge 1\}$ be a sequence of END random variables.*
(*i*) *For each $n \ge 1$, if $f_1, f_2, \cdots, f_n$ are all nondecreasing (or nonincreasing) functions, then random variables $f_1(X_1)$, $f_2(X_2)$, $\cdots$, $f_n(X_n)$ are also END.*
(*ii*) *For each $n\ge 1$, there exists a constant $M > 0$ such that*

$$E\left(\prod_{i=1}^{n} X_i^+\right) \le M\prod_{i=1}^{n} EX_i^+.$$

Lemma 3.2 (*cf.Wang et al.* (2014b)) *Let $p \ge 2$ and $\{X_n, n \ge 1\}$ be a sequence of END random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for each $n \ge 1$. Then there exists a positive constant C_p depending only on p such that*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \le C_p\left[\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{p/2}\right], \quad for\ all\ n\ge 1.$$

and

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \le C_p(\log n)^p\left[\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{p/2}\right], \quad for\ all\ n\ge 1.$$

Lemma 3.3 *Let $p \geq 2$ and $\{X_n, n \geq 1\}$ be a sequence of END random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for each $n \geq 1$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers. Then there exists a positive constant C_p depending only on p such that*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right|^p\right) \leq C_p(\log n)^p\left[\sum_{i=1}^{n} E|a_{ni}X_i|^p + \left(\sum_{i=1}^{n} E|a_{ni}X_i|^2\right)^{p/2}\right],$$

for all $n \geq 1$.

Proof of Lemma 3.3 By Lemma 3.1, it's easy to see that $\{a_{ni}^+X_i, 1 \leq i \leq n\}$ and $\{a_{ni}^-X_i, 1 \leq i \leq n\}$ are still END random variables respectively for each $n \geq 1$. It follows from Lemma 3.2 that

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}^+X_i\right|^p\right) \leq C_p'(\log n)^p\left[\sum_{i=1}^{n} E|a_{ni}^+X_i|^p + \left(\sum_{i=1}^{n} E|a_{ni}^+X_i|^2\right)^{p/2}\right],\ n \geq 1, \tag{3.1}$$

and

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}^-X_i\right|^p\right) \leq C_p''(\log n)^p\left[\sum_{i=1}^{n} E|a_{ni}^-X_i|^p + \left(\sum_{i=1}^{n} E|a_{ni}^-X_i|^2\right)^{p/2}\right],\ n \geq 1, \tag{3.2}$$

where C_p', C_p'' are positive constants depending only on p. Let

$$\widetilde{C}_p = \max\{C_p', C_p''\},$$

then by C_r inequality, (3.1) and (3.2) we have

$$\begin{aligned}
&E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right|^p\right) = E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}^+X_i - \sum_{i=1}^{j} a_{ni}^-X_i\right|^p\right) \\
\leq\ & 2^{p-1}E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}^+X_i\right|^p + \max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}^-X_i\right|^p\right) \\
\leq\ & 2^{p-1}C_p'(\log n)^p\left[\sum_{i=1}^{n} E|a_{ni}^+X_i|^p + \left(\sum_{i=1}^{n} E|a_{ni}^+X_i|^2\right)^{p/2}\right] \\
&+2^{p-1}C_p''(\log n)^p\left[\sum_{i=1}^{n} E|a_{ni}^-X_i|^p + \left(\sum_{i=1}^{n} E|a_{ni}^-X_i|^2\right)^{p/2}\right] \\
\leq\ & 2^{p-1}\widetilde{C}_p(\log n)^p\left[\sum_{i=1}^{n} E|a_{ni}X_i|^p + \left(\sum_{i=1}^{n} E|a_{ni}X_i|^2\right)^{p/2}\right].
\end{aligned}$$

Therefore, the result holds for $C_p = 2^{p-1}\widetilde{C}_p$, which depends only on p. □

Lemma 3.4 (*Adler and Rosalsky* (1987); *Adler, Rosalsky, and Taylor et al*. (1989)) *Let $\{X_n, n \geq 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $\beta > 0$, the following two statements hold:*

$$E|X_n|^\alpha I(|X_n| \leq \beta) \leq C_1\left[E|X|^\alpha I(|X| \leq \beta) + \beta^\alpha P\{|X| > \beta\}\right];$$
$$E|X_n|^\alpha I(|X_n| > \beta) \leq C_2 E|X|^\alpha I(|X| > \beta),$$

where C_1 and C_2 are positive constants. Consequently, $E|X_n|^\alpha \leq CE|X|^\alpha$.

Lemma 3.5 (*Yan J.G.* (2018)) *Let* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *be an array of real numbers such that*

$$\sum_{i=1}^{n} |a_{ni}|^{\xi} \le n \quad \textit{for some } \xi > 0, \tag{3.3}$$

then

(*i*) *for every* $0 < \zeta < \xi$,

$$\sum_{i=1}^{n} |a_{ni}|^{\zeta} \le n. \tag{3.4}$$

(*ii*) *for all* $m \ge 1$, $\kappa > \xi$,

$$\sum_{j=m}^{\infty} \sharp(I_{nj})(j+1)^{-\kappa/\xi} \le C(m+1)^{1-\frac{\kappa}{\xi}}. \tag{3.5}$$

and then for all $n > m \ge 1$, $\kappa > \xi$,

$$\sum_{j=m}^{n-1} \sharp(I_{nj})(j+1)^{-\kappa/\xi} \le C(m+1)^{1-\frac{\kappa}{\xi}}, \tag{3.6}$$

where $I_{nj} = \left\{1 \le i \le n : n^{1/\xi}(j+1)^{-1/\xi} < |a_{ni}| \le n^{1/\xi} j^{-1/\xi}\right\}$, *for each* $n \ge 1, j \ge 1$.

(*iii*) *for arbitrary* $s \ge 1$,

$$\sum_{j=1}^{s} \sharp(I_{nj}) \le \min\{n, s+1\}. \tag{3.7}$$

We will use the following properties of slowly varying functions.

Lemma 3.6 (*cf.Seneta* (1976)) *If* $\ell(x)$ *is a function slowly varying at infinity, then for any* $s > 0$,

$$C_1 n^{-s}\ell(n) \le \sum_{i=n}^{\infty} i^{-1-s}\ell(i) \le C_2 n^{-s}\ell(n),$$

and

$$C_3 n^{s}\ell(n) \le \sum_{i=1}^{n} i^{-1+s}\ell(i) \le C_4 n^{s}\ell(n),$$

where $C_i, i = 1, 2, 3, 4$ *are positive constants depending only on* s.

Proof of Theorem 2.1. Without loss of generality, we may assume that $a_{ni} \ge 0$ for $1 \le i \le n, n \ge 1$ and

$$\sum_{i=1}^{n} |a_{ni}|^{\xi} \le n, \quad \text{for some } \xi > p.$$

We choose a number q such that

$$\frac{1}{\alpha p} < q < 1.$$

For $1 \le i \le n, n \ge 1$, denote

$$\begin{aligned}
&Y_{ni} = a_{ni}X_i;\\
&Y_{ni}^{(1)} = -n^{\alpha q} I(Y_{ni} < -n^{\alpha q}) + Y_{ni} I(|Y_{ni}| \le n^{\alpha q}) + n^{\alpha q} I(Y_{ni} > n^{\alpha q});\\
&Y_{ni}^{(2)} = (Y_{ni} - n^{\alpha q}) I(Y_{ni} > n^{\alpha q});\\
&Y_{ni}^{(3)} = -(Y_{ni} + n^{\alpha q}) I(Y_{ni} < -n^{\alpha q}).
\end{aligned}$$

Obviously,

$$Y_{ni} = Y_{ni}^{(1)} + Y_{ni}^{(2)} - Y_{ni}^{(3)},\ Y_{ni}^{(2)} \ge 0,\ Y_{ni}^{(3)} \ge 0,$$

and

$$\begin{aligned}
&\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\max_{1\le j\le n}|T_j| \ge \varepsilon n^{\alpha}\right) = \sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_{ni}\right| \ge \varepsilon n^{\alpha}\right)\\
\le\ & \sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_{ni}^{(1)}\right| \ge \frac{\varepsilon n^{\alpha}}{3}\right) + \sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_{ni}^{(2)}\right| \ge \frac{\varepsilon n^{\alpha}}{3}\right)\\
&\quad + \sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_{ni}^{(3)}\right| \ge \frac{\varepsilon n^{\alpha}}{3}\right)\\
=\ & \sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_{ni}^{(1)}\right| \ge \frac{\varepsilon n^{\alpha}}{3}\right) + \sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\sum_{i=1}^{n} Y_{ni}^{(2)} \ge \frac{\varepsilon n^{\alpha}}{3}\right)\\
&\quad + \sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\sum_{i=1}^{n} Y_{ni}^{(3)} \ge \frac{\varepsilon n^{\alpha}}{3}\right)\\
\triangleq\ & I_1 + I_2 + I_3.
\end{aligned}$$

To prove (2.3), it suffices to show that $I_k < \infty$ for $k = 1, 2, 3$. Since the convergence of I_2 and I_3 are the same, we will just present the proof of $I_2 < \infty$.

For I_1, we first show that

$$n^{-\alpha}\max_{1\le j\le n}\left|\sum_{i=1}^{j} EY_{ni}^{(1)}\right| \to 0, \quad n \to \infty. \tag{3.8}$$

By (2.2), it is easy to see that

$$E|X|^{p-\eta} < \infty, \quad \text{for all } 0 < \eta < p.$$

We choose η such that

$$0 < \eta < p,\ \alpha(p-\eta) > \alpha(p-\eta)q > 1 \text{ and, } p-\eta > 1 \text{ if } p > 1.$$

We prove (3.8) through the following three cases.

Case I. $\underline{\alpha \le 1}$. By $\alpha p > 1$ we have $p > 1$. Therefore, $EX_n = 0, n \ge 1$, and then $EY_{ni} = 0$ for $1 \le i \le n, n \ge 1$. By Lemma 3.4 and Lemma 3.5, we get

$$\begin{aligned}
& n^{-\alpha}\max_{1\le j\le n}\left|\sum_{i=1}^{j} EY_{ni}^{(1)}\right|\\
\le\ & n^{-\alpha}\max_{1\le j\le n}\left[\left|\sum_{i=1}^{j} EY_{ni}I(|Y_{ni}| \le n^{\alpha q})\right| + n^{\alpha q}\sum_{i=1}^{j} P(|Y_{ni}| > n^{\alpha q})\right]\\
=\ & n^{-\alpha}\max_{1\le j\le n}\left[\left|\sum_{i=1}^{j} EY_{ni}I(|Y_{ni}| > n^{\alpha q})\right| + n^{\alpha q}\sum_{i=1}^{j} P(|Y_{ni}| > n^{\alpha q})\right]\\
\le\ & 2n^{-\alpha}\sum_{i=1}^{n} E|Y_{ni}|I(|Y_{ni}| > n^{\alpha q}) \le Cn^{-\alpha}\sum_{i=1}^{n} E|a_{ni}X|I(|a_{ni}X| > n^{\alpha q})\\
\le\ & Cn^{-\alpha}(n^{\alpha q})^{1-(p-\eta)}\sum_{i=1}^{n} E|a_{ni}X|^{p-\eta} \le Cn^{-\alpha}(n^{\alpha q})^{1-(p-\eta)}\sum_{i=1}^{n} |a_{ni}|^{p-\eta}\\
\le\ & Cn^{-\alpha}(n^{\alpha q})^{1-(p-\eta)}\cdot n = Cn^{-[\alpha q(p-\eta)-1]}\cdot n^{-\alpha(1-q)} \to 0,\ n \to \infty.
\end{aligned} \tag{3.9}$$

Case II. $\underline{\alpha > 1, p > 1}$. By $\xi > p > 1$, Lemma 3.4 and Lemma 3.5, we have

$$\begin{aligned}
& n^{-\alpha} \max_{1\le j\le n} \left| \sum_{i=1}^{j} EY_{ni}^{(1)} \right| \le n^{-\alpha} \max_{1\le j\le n} \sum_{i=1}^{j} [E|Y_{ni}|I(|Y_{ni}| \le n^{\alpha q}) + n^{\alpha q} P(|Y_{ni}| > n^{\alpha q})] \\
\le \quad & Cn^{-\alpha} \left[\sum_{i=1}^{n} E|a_{ni}X|I(|a_{ni}X| \le n^{\alpha q}) + \sum_{i=1}^{n} n^{\alpha q} P(|a_{ni}X| > n^{\alpha q}) \right] \\
\le \quad & Cn^{-\alpha} \sum_{i=1}^{n} E|a_{ni}X| \le Cn^{-\alpha} \sum_{i=1}^{n} |a_{ni}| \le Cn^{-\alpha} \cdot n = Cn^{-(\alpha-1)} \to 0, \ n \to \infty. \qquad (3.10)
\end{aligned}$$

Case III. $\underline{\alpha > 1, p \le 1}$. By Lemma 3.4, Lemma 3.5 and Markov's inequality we have

$$\begin{aligned}
& n^{-\alpha} \max_{1\le j\le n} \left| \sum_{i=1}^{j} EY_{ni}^{(1)} \right| \le n^{-\alpha} \sum_{i=1}^{n} E|Y_{ni}^{(1)}| \\
\le \quad & n^{-\alpha} \sum_{i=1}^{n} [E|a_{ni}X_i|I(|a_{ni}X_i| \le n^{\alpha q}) + n^{\alpha q} P(|a_{ni}X_i| > n^{\alpha q})] \\
\le \quad & Cn^{-\alpha} \left[\sum_{i=1}^{n} E|a_{ni}X|I(|a_{ni}X| \le n^{\alpha q}) + \sum_{i=1}^{n} n^{\alpha q} P(|a_{ni}X| > n^{\alpha q}) \right] \\
\le \quad & Cn^{-\alpha} \left[\sum_{i=1}^{n} n^{\alpha q[1-(p-\eta)]} E|a_{ni}X|^{p-\eta} + \sum_{i=1}^{n} n^{\alpha q} (n^{-\alpha q})^{p-\eta} E|a_{ni}X|^{p-\eta} \right] \\
\le \quad & Cn^{-\alpha} \cdot n^{\alpha q[1-(p-\eta)]} \sum_{i=1}^{n} |a_{ni}|^{p-\eta} \le Cn^{-[\alpha q(p-\eta)-1]} \cdot n^{-\alpha(1-q)} \to 0, \ n \to \infty. \qquad (3.11)
\end{aligned}$$

By (3.9), (3.10) and (3.11), we get (3.8).

Therefore, in order to prove $I_1 < \infty$, it suffices to show that

$$I_1^* \triangleq \sum_{n=n_0}^{\infty} n^{\alpha p-2} \ell(n) P\left(\max_{1\le j\le n} \left| \sum_{i=1}^{j} \left(Y_{ni}^{(1)} - EY_{ni}^{(1)} \right) \right| \ge \frac{\varepsilon n^{\alpha}}{6} \right) < \infty. \qquad (3.12)$$

By the expression of $Y_{ni}^{(1)}$ and Lemma 3.1, for each fixed $n \ge 1$, $\{Y_{ni}^{(1)} - EY_{ni}^{(1)}, 1 \le i \le n\}$ remain END random variables. From $\alpha(p-\eta)q > 1$ and $0 < q < 1$, we get

$$\alpha - \frac{1}{2} - \alpha(1 - \frac{p-\eta}{2})q > \alpha - \frac{1}{2} - \alpha(1 - \frac{p-\eta}{2}) = \frac{\alpha(p-\eta)-1}{2} > 0 \text{ if } p \le 2.$$

We choose a constant τ such that

$$\tau > \max\left\{ 2, \ p, \ \frac{\alpha p - 1}{\alpha - \frac{1}{2}}, \ \frac{\alpha p - 1}{\alpha - \frac{1}{2} - \alpha(1 - \frac{p-\eta}{2})q}, \ \frac{p-(p-\eta)q}{1-q} \right\}.$$

Then, by Markov's inequality, Lemma 3.3 and C_r inequality we have

$$\begin{aligned}
I_1^* \quad \le \quad & C \sum_{n=n_0}^{\infty} n^{\alpha p-2} \ell(n) n^{-\alpha\tau} E\left(\max_{1\le j\le n} \left| \sum_{i=1}^{j} \left(Y_{ni}^{(1)} - EY_{ni}^{(1)} \right) \right| \right)^{\tau} \\
\le \quad & C \sum_{n=n_0}^{\infty} n^{\alpha p-2} \ell(n) n^{-\alpha\tau} (\log n)^{\tau} \sum_{i=1}^{n} E\left| Y_{ni}^{(1)} \right|^{\tau} \\
& + C \sum_{n=n_0}^{\infty} n^{\alpha p-2} \ell(n) n^{-\alpha\tau} (\log n)^{\tau} \left(\sum_{i=1}^{n} E\left| Y_{ni}^{(1)} \right|^{2} \right)^{\frac{\tau}{2}} \\
\triangleq \quad & I_{11}^* + I_{12}^*.
\end{aligned}$$

To prove (3.12), it needs only to prove $I_{11}^* < \infty$ and $I_{12}^* < \infty$ respectively.

By C_r inequality, Markov's inequality, Lemma 3.4 and Lemma 3.5, we have

$$\begin{aligned}
I_{11}^* &\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\tau}\ell(n)(\log n)^{\tau}\sum_{i=1}^{n}[E|Y_{ni}|^{\tau}I(|Y_{ni}|\leq n^{\alpha q})+n^{\alpha q\tau}P(|Y_{ni}|>n^{\alpha q})] \\
&\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\tau}\ell(n)(\log n)^{\tau}\left[\sum_{i=1}^{n}E|a_{ni}X|^{\tau}I(|a_{ni}X|\leq n^{\alpha q})+\sum_{i=1}^{n}n^{\alpha q\tau}P(|a_{ni}X|>n^{\alpha q})\right] \\
&\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\tau}\ell(n)(\log n)^{\tau}\left[\sum_{i=1}^{n}n^{\alpha q[\tau-(p-\eta)]}E|a_{ni}X|^{p-\eta}+\sum_{i=1}^{n}n^{\alpha q\tau}(n^{-\alpha q})^{p-\eta}E|a_{ni}X|^{p-\eta}\right] \\
&\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\tau}\ell(n)(\log n)^{\tau}n^{\alpha q[\tau-(p-\eta)]}\cdot\sum_{i=1}^{n}|a_{ni}|^{p-\eta} \\
&\leq C\sum_{n=n_0}^{\infty} n^{-\alpha(1-q)[\tau-\frac{p-(p-\eta)q}{1-q}]-1}\ell(n)(\log n)^{\tau}<\infty. \qquad (3.13)
\end{aligned}$$

Next for I_{12}^*. By C_r inequality and Lemma 3.4 we have

$$\begin{aligned}
I_{12}^* &\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\tau}\ell(n)(\log n)^{\tau}\left[\sum_{i=1}^{n}E|Y_{ni}|^{2}I(|Y_{ni}|\leq n^{\alpha q})+\sum_{i=1}^{n}n^{2\alpha q}P(|Y_{ni}|>n^{\alpha q})\right]^{\frac{\tau}{2}} \\
&\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\tau}\ell(n)(\log n)^{\tau}\left[\sum_{i=1}^{n}E|a_{ni}X|^{2}I(|a_{ni}X|\leq n^{\alpha q})+\sum_{i=1}^{n}n^{2\alpha q}P(|a_{ni}X|>n^{\alpha q})\right]^{\frac{\tau}{2}}.
\end{aligned}$$

If $p > 2$, by Markov's inequality and Lemma 3.5,

$$\begin{aligned}
I_{12}^* &\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\tau}\ell(n)(\log n)^{\tau}\cdot\left(\sum_{i=1}^{n}a_{ni}^2EX^2\right)^{\frac{\tau}{2}} \\
&\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\tau}\ell(n)(\log n)^{\tau}\cdot\left(\sum_{i=1}^{n}a_{ni}^2\right)^{\frac{\tau}{2}} \\
&\leq C\sum_{n=n_0}^{\infty} n^{-(\alpha-\frac{1}{2})[\tau-\frac{\alpha p-1}{\alpha-1/2}]-1}\ \ell(n)\ (\log n)^{\tau}<\infty. \qquad (3.14)
\end{aligned}$$

If $0 < p \leq 2$, again by Markov's inequality and Lemma 3.5,

$$\begin{aligned}
I_{12}^* &\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\tau}\ell(n)(\log n)^{\tau}\left[\sum_{i=1}^{n}(n^{\alpha q})^{2-(p-\eta)}E|a_{ni}X|^{p-\eta}+\sum_{i=1}^{n}n^{2\alpha q}(n^{-\alpha q})^{(p-\eta)}E|a_{ni}X|^{p-\eta}\right]^{\frac{\tau}{2}} \\
&\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\tau}\ell(n)(\log n)^{\tau}n^{\alpha q[2-(p-\eta)]\cdot\frac{\tau}{2}}\cdot\left(\sum_{i=1}^{n}|a_{ni}|^{p-\eta}\right)^{\frac{\tau}{2}} \\
&\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\tau[\alpha-\frac{1}{2}-\alpha q(1-\frac{p-\eta}{2})]}\ \ell(n)\ (\log n)^{\tau}<\infty. \qquad (3.15)
\end{aligned}$$

By (3.14) and (3.15), we see that $I_{12}^* < \infty$. Thus, combine (3.13), (3.12) holds, that is $I_1 < \infty$.

Next for I_2. Let

$$Z_{ni}^{(2)} = (Y_{ni}-n^{\alpha q})I(n^{\alpha q}<Y_{ni}\leq n^{\alpha}+n^{\alpha q})+n^{\alpha}I(Y_{ni}>n^{\alpha}+n^{\alpha q}),\quad 1\leq i\leq n,\ n\geq 1.$$

Thus, $Z_{ni}^{(2)} \geq 0$ and

$$Y_{ni}^{(2)} = Z_{ni}^{(2)}+(Y_{ni}-n^{\alpha}-n^{\alpha q})I(Y_{ni}>n^{\alpha}+n^{\alpha q}),\quad 1\leq i\leq n,\ n\geq 1.$$

Therefore,

$$
\begin{aligned}
I_2 &\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\sum_{i=1}^{n} Z_{ni}^{(2)} > \frac{\varepsilon n^{\alpha}}{6}\right) \\
&\quad +C\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\sum_{i=1}^{n}(Y_{ni}-n^{\alpha}-n^{\alpha q})I(Y_{ni}>n^{\alpha}+n^{\alpha q}) > \frac{\varepsilon n^{\alpha}}{6}\right) \\
&\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\sum_{i=1}^{n} Z_{ni}^{(2)} > \frac{\varepsilon n^{\alpha}}{6}\right)+C\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)\sum_{i=1}^{n}P(Y_{ni}>n^{\alpha}+n^{\alpha q}) \\
&\triangleq I_{21}+I_{22}.
\end{aligned}
\tag{3.16}
$$

To prove $I_2<\infty$, it suffices to show that $I_{21}<\infty$ and $I_{22}<\infty$.

For I_{21}, we first show that

$$
n^{-\alpha}\sum_{i=1}^{n} EZ_{ni}^{(2)} \to 0, \quad n\to\infty. \tag{3.17}
$$

If $p>1$. By Lemma 3.5, we have

$$
\begin{aligned}
0 &\leq n^{-\alpha}\sum_{i=1}^{n} EZ_{ni}^{(2)} \leq n^{-\alpha}\sum_{i=1}^{n} EY_{ni}I(Y_{ni}>n^{\alpha q}) \\
&\leq n^{-\alpha}(n^{\alpha q})^{1-(p-\eta)}\sum_{i=1}^{n} E|a_{ni}X|^{p-\eta} \\
&\leq n^{-\alpha}(n^{\alpha q})^{1-(p-\eta)}\sum_{i=1}^{n} |a_{ni}|^{p-\eta} \leq Cn^{-[\alpha q(p-\eta)-1]-\alpha(1-q)} \to 0, \quad n\to\infty.
\end{aligned}
\tag{3.18}
$$

If $0<p\leq 1$. By Markov's inequality, Lemma 3.4 and Lemma 3.5, we have

$$
\begin{aligned}
0 &\leq n^{-\alpha}\sum_{i=1}^{n} EZ_{ni}^{(2)} \leq n^{-\alpha}\sum_{i=1}^{n}[E|Y_{ni}|I(|Y_{ni}|\leq 2n^{\alpha})+n^{\alpha}P(|Y_{ni}|>2n^{\alpha q})] \\
&\leq Cn^{-\alpha}\sum_{i=1}^{n}[E|a_{ni}X|I(|a_{ni}X|\leq 2n^{\alpha})+2n^{\alpha}P(|a_{ni}X|>2n^{\alpha})+n^{\alpha}P(|a_{ni}X|>2n^{\alpha q})] \\
&\leq Cn^{-\alpha}\sum_{i=1}^{n}[E|a_{ni}X|I(|a_{ni}X|\leq 2n^{\alpha})+n^{\alpha}P(|a_{ni}X|>2n^{\alpha q})] \\
&\leq Cn^{-\alpha}\sum_{i=1}^{n}\left[(n^{\alpha})^{1-(p-\eta)}|a_{ni}|^{p-\eta}+n^{\alpha}(n^{-\alpha q})^{p-\eta}|a_{ni}|^{p-\eta}\right] \\
&\leq Cn^{-\alpha}n^{\alpha}n^{-\alpha q(p-\eta)}\sum_{i=1}^{n}|a_{ni}|^{p-\eta} \leq Cn^{-[\alpha q(p-\eta)-1]} \to 0, \quad n\to\infty.
\end{aligned}
\tag{3.19}
$$

Obviously, (3.17) follows from (3.18) and (3.19). Therefore, to prove $I_{21}<\infty$, it needs only to prove

$$
I_{21}^{*} \triangleq \sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\left|\sum_{i=1}^{n}(Z_{ni}^{(2)}-EZ_{ni}^{(2)})\right| > \frac{\varepsilon n^{\alpha}}{12}\right) < \infty. \tag{3.20}
$$

From Lemma 3.1, for each fixed $n\geq 1$, $\{Z_{ni}^{(2)}-EZ_{ni}^{(2)}, 1\leq i\leq n\}$ are still END random variables. We choose constant κ such that

$$
\kappa > \max\left\{\xi,\ \frac{\alpha p-1}{\alpha-\frac{1}{2}},\ \frac{\alpha p-1}{\alpha-\frac{1}{\xi}},\ \frac{2(\alpha p-1)}{\alpha(p-\eta)-1}\right\}.
$$

And by $\alpha p-1>\alpha(p-\eta)-1>0$, we see that $\frac{2(\alpha p-1)}{\alpha(p-\eta)-1}>2$, therefore, $\kappa>2$.

By Markov's inequality, C_r inequality and Lemma 3.3, we have

$$\begin{aligned}
I_{21}^* &\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)n^{-\alpha\kappa}E\left|\sum_{i=1}^{n}(Z_{ni}^{(2)}-EZ_{ni}^{(2)})\right|^{\kappa}\\
&\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa}\ell(n)\sum_{i=1}^{n}E\left|Z_{ni}^{(2)}\right|^{\kappa}+C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa}\ell(n)\left(\sum_{i=1}^{n}E\left(Z_{ni}^{(2)}\right)^2\right)^{\frac{\kappa}{2}}\\
&\triangleq I_{211}^*+I_{212}^*.
\end{aligned}$$

To prove (3.20), it suffices to show that $I_{211}^*<\infty$ and $I_{212}^*<\infty$.

We will proceed with two aspects.

(i) If $p\ge 2$. First for I_{212}^*. If $p>2$, then by C_r inequality, Lemma 3.5 and $\xi>p$, we have

$$\begin{aligned}
I_{212}^* &\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa}\ell(n)\left(\sum_{i=1}^{n}E\left|Y_{ni}\right|^2\right)^{\frac{\kappa}{2}}\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa}\ell(n)\left(\sum_{i=1}^{n}a_{ni}^2\right)^{\frac{\kappa}{2}}\\
&\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa}\ell(n)\cdot n^{\frac{\kappa}{2}}=C\sum_{n=n_0}^{\infty} n^{-(\alpha-\frac{1}{2})\left(\kappa-\frac{\alpha p-1}{\alpha-\frac{1}{2}}\right)-1}\ell(n)<\infty. \qquad (3.21)
\end{aligned}$$

If $p=2$. Similarly we have

$$\begin{aligned}
I_{212}^* &\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa}\ell(n)\left(\sum_{i=1}^{n}E\left|Y_{ni}\right|^2 I(|Y_{ni}|\le 2n^{\alpha})+n^{2\alpha}P(|Y_{ni}|>n^{\alpha})\right)^{\frac{\kappa}{2}}\\
&\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa}\ell(n)\left(\sum_{i=1}^{n}n^{\alpha\eta}E\left|Y_{ni}\right|^{2-\eta} I(|Y_{ni}|\le 2n^{\alpha})+n^{2\alpha}\cdot(n^{-\alpha})^{2-\eta}E|Y_{ni}|^{2-\eta}\right)^{\frac{\kappa}{2}}\\
&\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa}\ell(n)\left(\sum_{i=1}^{n}n^{\alpha\eta}E\left|Y_{ni}\right|^{2-\eta}\right)^{\frac{\kappa}{2}}\\
&\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa}\ell(n)\left(n^{\alpha\eta+1}\right)^{\frac{\kappa}{2}}\\
&\le C\sum_{n=n_0}^{\infty} n^{-\frac{\alpha(p-\eta)-1}{2}\left(\kappa-\frac{2(\alpha p-1)}{\alpha(p-\eta)-1}\right)-1}\ell(n)<\infty. \qquad (3.22)
\end{aligned}$$

Next for I_{211}^*. By C_r inequality, we get

$$\begin{aligned}
I_{211}^* &\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa}\ell(n)\sum_{i=1}^{n}[E|Y_{ni}|^{\kappa}I(n^{\alpha q}<Y_{ni}\le n^{\alpha}+n^{\alpha q})\ +\ n^{\alpha\kappa}P(|Y_{ni}|>n^{\alpha}+n^{\alpha q})]\\
&\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa}\ell(n)\sum_{i=1}^{n}[E|Y_{ni}|^{\kappa}I(|Y_{ni}|\le 2n^{\alpha})+n^{\alpha\kappa}P(|Y_{ni}|>n^{\alpha})]\\
&= C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa}\ell(n)\sum_{i=1}^{n}E|Y_{ni}|^{\kappa}I(|Y_{ni}|\le 2n^{\alpha})+C\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)\sum_{i=1}^{n}P(|Y_{ni}|>n^{\alpha})\\
&\triangleq I_{2111}^*+I_{2112}^*. \qquad (3.23)
\end{aligned}$$

Thus, to prove $I_{211}^*<\infty$, we need to prove $I_{2111}^*<\infty$ and $I_{2112}^*<\infty$.

Firstly, in view of I_{nj} and $a_{ni}=0$ or $|a_{ni}|>1$, it is easy to see that

$$\bigcup_{j=1}^{n-1} I_{nj}=\{1\le i\le n:|a_{ni}|>1\}.$$

We thus by Lemma 3.5 that

$$\begin{aligned}
I^*_{2112} &\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)\sum_{j=1}^{n-1}\sum_{i\in I_{nj}} P(|a_{ni}X|>n^{\alpha}) \\
&\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)\sum_{j=1}^{n-1} P(|X|>n^{\alpha}(\frac{j}{n})^{\frac{1}{\xi}})\ \sharp(I_{nj}) \\
&= C\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)\sum_{j=1}^{n-1} P(|X|^t>n\cdot j^{\frac{t}{\xi}})\ \sharp(I_{nj}) \qquad\qquad (\frac{1}{t}=\alpha-\frac{1}{\xi}) \\
&\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)\sum_{j=1}^{n-1}\sharp(I_{nj})\sum_{m=[n\cdot j^{\frac{t}{\xi}}]}^{\infty} P(m<|X|^t\le m+1) \\
&\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)\sum_{m=n}^{\infty} P(m<|X|^t\le m+1)\sum_{j=1}^{(n-1)\wedge[(\frac{m+1}{n})^{\frac{\xi}{t}}]}\sharp(I_{nj}) \\
&\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)\sum_{m=n}^{\infty} P(m<|X|^t\le m+1)\left(n\wedge\left[\left(\frac{m+1}{n}\right)^{\frac{\xi}{t}}\right]+1\right) \\
&\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)\sum_{m=n}^{[n^{1+\frac{\xi}{t}}]} P(m<|X|^t\le m+1)\left(\left[\left(\frac{m+1}{n}\right)^{\frac{\xi}{t}}\right]+1\right) \\
&\qquad +C\sum_{n=n_0}^{\infty} n^{\alpha p-1}\ell(n)\sum_{m=[n^{1+\frac{\xi}{t}}]+1}^{\infty} P(m<|X|^t\le m+1) \\
&\triangleq I^*_{21121}+I^*_{21122}.
\end{aligned}$$

We will prove $I^*_{21121}<\infty$ and $I^*_{21122}<\infty$.

From $\frac{1}{t}=\alpha-\frac{1}{\xi}$, it is easy to see that

$$t(1-\frac{1}{\alpha\xi})=\frac{1}{\alpha},\ \frac{\xi}{\xi+t}=\frac{1}{\alpha t},\ \frac{\xi}{t}=\alpha\xi-1,$$

and then

$$\begin{aligned}
\frac{\xi}{t}+\frac{\xi}{\xi+t}(\alpha p-1-\frac{\xi}{t}) &= \alpha\xi-1+\frac{\alpha\xi-1}{\alpha\xi}(\alpha p-1-\alpha\xi+1) \\
&= (\alpha\xi-1)(1+\frac{p-\xi}{\xi})=(\alpha\xi-1)\cdot\frac{p}{\xi}=\frac{p}{t}.
\end{aligned}$$

By Lemma 3.6 and (2.2), we have

$$\begin{aligned}
I^*_{21121} &\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\frac{\xi}{t}}\ell(n)\sum_{m=n}^{[n^{1+\frac{\xi}{t}}]} P(m<|X|^t\le m+1)\cdot m^{\frac{\xi}{t}} \\
&\le C\sum_{m=n_0}^{\infty} P(m<|X|^t\le m+1)\cdot m^{\frac{\xi}{t}}\sum_{n=[m^{\frac{\xi}{\xi+t}}]}^{m} n^{\alpha p-2-\frac{\xi}{t}}\ \ell(n) \\
&\le C\sum_{m=n_0}^{\infty} P(m<|X|^t\le m+1)\cdot m^{\frac{p}{t}}\ \ell(m^{1-\frac{1}{\alpha\xi}}) \\
&\le C+CE|X|^p\ell(|X|^{\frac{1}{\alpha}})<\infty. \qquad (3.24)
\end{aligned}$$

Similarly,

$$\begin{aligned}
I^*_{21122} &\le C \sum_{m=[n_0^{1+\frac{\xi}{t}}]+1}^{\infty} P(m < |X|^t \le m+1) \sum_{n=n_0}^{[m^{\frac{\xi}{\xi+t}}]} n^{\alpha p-1}\ \ell(n) \\
&\le C \sum_{m=[n_0^{1+\frac{\xi}{t}}]+1}^{\infty} P(m < |X|^t \le m+1) \cdot m^{\frac{p}{t}}\ \ell(m^{\frac{t}{\alpha}}) \\
&\le C + CE|X|^p \ell(|X|^{\frac{1}{\alpha}}) < \infty. \qquad (3.25)
\end{aligned}$$

From (3.24) and (3.25), we get $I^*_{2112} < \infty$.

Next for I^*_{2111}. By Lemma 3.4 and C_r inequality, we have

$$\begin{aligned}
I^*_{2111} &= C \sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa} \ell(n) \sum_{i=1}^{n} E|a_{ni}X_i|^\kappa I(|a_{ni}X_i| \le 2n^\alpha) \\
&\le C \sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa} \ell(n) \sum_{i=1}^{n} E|a_{ni}X|^\kappa I(|a_{ni}X| \le 2n^\alpha) \\
&\quad + C \sum_{n=n_0}^{\infty} n^{\alpha p-2} \ell(n) \sum_{i=1}^{n} P(|a_{ni}X| > 2n^\alpha) \\
&\triangleq J_1 + J_2.
\end{aligned}$$

Similarly to the process of $I^*_{2112} < \infty$, we can get that $J_2 < \infty$. It remains to show that $J_1 < \infty$.

For simplicity, we will prove

$$J^*_1 \triangleq C \sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa} \ell(n) \sum_{i=1}^{n} E|a_{ni}X|^\kappa I(|a_{ni}X| \le n^\alpha) < \infty.$$

In fact,

$$\begin{aligned}
J^*_1 &= C \sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa} \ell(n) \sum_{j=1}^{n-1} \sum_{i\in I_{nj}} E|a_{ni}X|^\kappa I(|a_{ni}X| \le n^\alpha) \\
&\le C \sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa} \ell(n) \sum_{j=1}^{n-1} (\frac{n}{j})^{\frac{\kappa}{\xi}}\ E|X|^\kappa I\left(|X| \le n^\alpha \cdot (\frac{j+1}{n})^{\frac{1}{\xi}}\right)\ \sharp(I_{nj}) \\
&\le C \sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa+\frac{\kappa}{\xi}} \ell(n) \sum_{j=1}^{n-1} j^{-\frac{\kappa}{\xi}}\ \sharp(I_{nj}) \sum_{m=0}^{[n\cdot(j+1)^{\frac{t}{\xi}}]} E|X|^\kappa I(m < |X|^t \le m+1) \\
&\le C \sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa+\frac{\kappa}{\xi}} \ell(n) \sum_{j=1}^{n-1} j^{-\frac{\kappa}{\xi}}\ \sharp(I_{nj}) \sum_{m=0}^{2n} E|X|^\kappa I(m < |X|^t \le m+1) \\
&\quad + C \sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa+\frac{\kappa}{\xi}} \ell(n) \sum_{j=1}^{n-1} j^{-\frac{\kappa}{\xi}}\ \sharp(I_{nj}) \sum_{m=2n+1}^{[n\cdot(j+1)^{\frac{t}{\xi}}]} E|X|^\kappa I(m < |X|^t \le m+1) \\
&\triangleq J^*_{11} + J^*_{12}. \qquad (3.26)
\end{aligned}$$

It remains to show that $J^*_{11} < \infty$ and $J^*_{12} < \infty$.

We have by Lemma 3.5 and Lemma 3.6 that

$$J^*_{11} = C \sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa+\frac{\kappa}{\xi}} \ell(n) \sum_{m=0}^{2n} E|X|^\kappa I(m < |X|^t \le m+1) \sum_{j=1}^{n-1} j^{-\frac{\kappa}{\xi}}\ \sharp(I_{nj})$$

$$
\begin{aligned}
&\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa+\frac{\kappa}{\xi}}\ell(n)\sum_{m=0}^{2n} E|X|^{\kappa}I(m<|X|^t\le m+1)\\
&\le C\sum_{m=1}^{\infty} E|X|^{\kappa}I(m<|X|^t\le m+1)\sum_{n=[m/2]}^{\infty} n^{\alpha p-2-\alpha\kappa+\frac{\kappa}{\xi}}\ell(n)\\
&\le C\sum_{m=1}^{\infty} E|X|^{\kappa}I(m<|X|^t\le m+1)m^{\alpha p-1-\alpha\kappa+\frac{\kappa}{\xi}}\ell(m)\\
&\le C\sum_{m=1}^{\infty} P(m<|X|^t\le m+1)m^{\alpha p-1}\ell(m)\\
&\le C+CE|X|^{p-t(1-\frac{p}{\xi})}\ell(|X|^t)\le C+CE|X|^p\ell(|X|^{1/\alpha}). \qquad (3.27)
\end{aligned}
$$

On the other hand, again by Lemma 3.5 and Lemma 3.6, we have

$$
\begin{aligned}
J_{12}^* &\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa+\frac{\kappa}{\xi}}\ell(n)\sum_{m=2n+1}^{[n^{1+\frac{t}{\xi}}]} E|X|^{\kappa}I(m<|X|^t\le m+1)\sum_{j=[(m/n)^{\frac{\xi}{t}}]-1}^{n-1} j^{-\frac{\kappa}{\xi}}\,\sharp(I_{nj})\\
&\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\kappa+\frac{\kappa}{\xi}}\ell(n)\sum_{m=2n+1}^{[n^{1+\frac{t}{\xi}}]} E|X|^{\kappa}I(m<|X|^t\le m+1)\left([(m/n)^{\frac{\xi}{t}}]-1\right)^{1-\frac{\kappa}{\xi}}\\
&\le C\sum_{m=2n_0+1}^{\infty} E|X|^{\kappa}I(m<|X|^t\le m+1)m^{-\frac{\kappa-\xi}{t}}\sum_{n=[m^{\frac{\xi}{\xi+t}}]}^{\infty} n^{\alpha p-2-\alpha\kappa+\frac{\kappa}{\xi}-\frac{\xi}{t}(1-\frac{\kappa}{\xi})}\ell(n)\\
&\le C\sum_{m=2n_0+1}^{\infty} E|X|^{\kappa}I(m<|X|^t\le m+1)m^{-\frac{\kappa-\xi}{t}}\cdot m^{\frac{\xi}{\xi+t}\,(\alpha p-1-\frac{\xi}{t})}\ell(m^{\frac{\xi}{\xi+t}})\\
&\le C\sum_{m=2n_0+1}^{\infty} P(m<|X|^t\le m+1)\cdot m^{\frac{p}{t}}\,\ell(m^{\frac{t}{\alpha}})\le C+CE|X|^p\ell(|X|^{1/\alpha}). \qquad (3.28)
\end{aligned}
$$

From (3.27) and (3.28), we have $J_1<\infty$, and then $I_{211}^*<\infty$.

The case of $0<p<2$ is similarly to the process of $p\ge 2$, so we omit it. Thus, (2.3) holds. This ends the proof. $\square$

Proof of Theorem 2.2. Without loss of generality, for $1\le i\le n, n\ge 1$, we assume that $a_{ni}\ge 0$ and denote

$$
\begin{aligned}
X_{ni}^{(1)} &= -a_{ni}n^{\alpha}I(X_i<-n^{\alpha})+a_{ni}X_iI(|X_i|\le n^{\alpha})+a_{ni}n^{\alpha}I(X_i>n^{\alpha});\\
X_{ni}^{(2)} &= a_{ni}(X_i-n^{\alpha})I(X_i>n^{\alpha});\\
X_{ni}^{(3)} &= -a_{ni}(X_i+n^{\alpha})I(X_i<-n^{\alpha}).
\end{aligned}
$$

Then

$$
a_{ni}X_i=X_{ni}^{(1)}+X_{ni}^{(2)}-X_{ni}^{(3)},\quad X_{ni}^{(2)}\ge 0, X_{ni}^{(3)}\ge 0,
$$

and

$$
\begin{aligned}
&\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\max_{1\le j\le n}|T_j|\ge\varepsilon n^{\alpha}\right)\\
\le&\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_{ni}^{(1)}\right|\ge\frac{\varepsilon n^{\alpha}}{3}\right)+\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\sum_{i=1}^{n}X_{ni}^{(2)}\ge\frac{\varepsilon n^{\alpha}}{3}\right)\\
&+\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\sum_{i=1}^{n}X_{ni}^{(3)}\ge\frac{\varepsilon n^{\alpha}}{3}\right)\\
\triangleq&\ D_1+D_2+D_3.
\end{aligned}
$$

In the sequel, we will prove that $D_i < \infty$, $i = 1, 2, 3$. Since the proof of $D_2 < \infty$ and $D_3 < \infty$ are the same, we will only present the convergence of D_2. In fact, by Lemma 3.4, Lemma 3.6 and (2.4)

$$\begin{aligned} D_2 &\le \sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)\sum_{i=1}^{n} P(|X_i| > n^\alpha) \le C\sum_{n=n_0}^{\infty} n^{\alpha p-1}\ell(n)P(|X| > n^\alpha) \\ &= C\sum_{n=n_0}^{\infty} n^{\alpha p-1}\ell(n)\sum_{m=n}^{\infty} P(m^\alpha < |X| \le (m+1)^\alpha) \\ &\le C\sum_{m=n_0}^{\infty} P(m^\alpha < |X| \le (m+1)^\alpha)\sum_{n=n_0}^{m} n^{\alpha p-1}\ell(n) \\ &\le C\sum_{m=n_0}^{\infty} m^{\alpha p}\ell(m)P(m^\alpha < |X| \le (m+1)^\alpha) \le CE|X|^p\ell(|X|^{1/\alpha}) < \infty. \end{aligned} \tag{3.29}$$

For D_1, we first prove that

$$n^{-\alpha}\max_{1\le j\le n}\left|\sum_{i=1}^{j} EX_{ni}^{(1)}\right| \to 0, n \to \infty. \tag{3.30}$$

When $\alpha \le 1$, it follows from $EX_i = 0$, $|a_{ni}| \le 1$, Lemma 3.4 and (2.4) that

$$\begin{aligned} & n^{-\alpha}\max_{1\le j\le n}\left|\sum_{i=1}^{j} EX_{ni}^{(1)}\right| \\ \le\ & n^{-\alpha}\max_{1\le j\le n}\left[\left|\sum_{i=1}^{j} Ea_{ni}X_iI(|X_i| \le n^\alpha)\right| + \left|\sum_{i=1}^{j} a_{ni}n^\alpha P(|X_i| > n^\alpha)\right|\right] \\ =\ & n^{-\alpha}\max_{1\le j\le n}\left[\left|\sum_{i=1}^{j} Ea_{ni}X_iI(|X_i| > n^\alpha)\right| + \left|\sum_{i=1}^{j} a_{ni}n^\alpha P(|X_i| > n^\alpha)\right|\right] \\ \le\ & Cn^{-\alpha}\sum_{i=1}^{n} E|X_i|I(|X_i| > n^\alpha) \le Cn^{1-\alpha(p-\delta)}E|X|^{p-\delta}I(|X| > n^\alpha) \to 0, n \to \infty. \end{aligned} \tag{3.31}$$

When $\alpha > 1$ and $p > 1$, we have from Lemma 3.4 and (2.4) that $E|X| < \infty$, and

$$\begin{aligned} n^{-\alpha}\max_{1\le j\le n}\left|\sum_{i=1}^{j} EX_{ni}^{(1)}\right| &\le n^{-\alpha}\sum_{i=1}^{n}[E|X_i|I(|X_i| \le n^\alpha) + n^\alpha P(|X_i| > n^\alpha)] \\ &\le Cn^{1-\alpha}E|X| \to 0, n \to \infty. \end{aligned} \tag{3.32}$$

When $\alpha > 1$ and $p \le 1$, by Lemma 3.4, Chebyshev's inequality and (2.4) we get

$$\begin{aligned} & n^{-\alpha}\max_{1\le j\le n}\left|\sum_{i=1}^{j} EX_{ni}^{(1)}\right| \\ \le\ & n^{-\alpha}\sum_{i=1}^{n}[E|X_i|I(|X_i| \le n^\alpha) + n^\alpha P(|X_i| > n^\alpha)] \\ \le\ & Cn^{1-\alpha}\left[n^{\alpha(1-(p-\delta))}E|X|^{p-\delta}I(|X| \le n^\alpha) + n^\alpha n^{-\alpha(p-\delta)}E|X|^{p-\delta}\right] \\ =\ & Cn^{1-\alpha(p-\delta)}E|X|^{p-\delta} \to 0, n \to \infty. \end{aligned} \tag{3.33}$$

Thus, formula (3.30) follows from (3.31) $\sim$(3.33). To get $D_1 < \infty$, it suffices to show that

$$D_1^* \triangleq \sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(X_{ni}^{(1)} - EX_{ni}^{(1)}\right)\right| \ge \frac{\varepsilon n^\alpha}{6}\right) < \infty.$$

By Lemma 3.1, it is easy to see that $\{X_{ni}^{(1)} - EX_{ni}^{(1)}\}$ remain END random variables for every $n \geq 1$. We have from Chebyshev's inequality and Lemma 3.3 that

$$\begin{aligned}
D_1^* &\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\theta}\ell(n)E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\left(X_{ni}^{(1)} - EX_{ni}^{(1)}\right)\right|\right)^{\theta} \\
&\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\theta}\ell(n)(\log n)^{\theta}\sum_{i=1}^{n}E\left|X_{ni}^{(1)}\right|^{\theta} + C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\theta}\ell(n)(\log n)^{\theta}\left(\sum_{i=1}^{n}E\left|X_{ni}^{(1)}\right|^{2}\right)^{\theta/2} \\
&\triangleq D_{11}^* + D_{12}^*,
\end{aligned}$$

since $\theta > \frac{2(\alpha p-1)}{\alpha(p-\delta)-1} > 2$. We will prove that $D_{11}^* < \infty$ and $D_{12}^* < \infty$, respectively. By C_r inequality and Lemma 3.4, we have

$$\begin{aligned}
D_{11}^* &\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\theta}\ell(n)(\log n)^{\theta}\sum_{i=1}^{n}\left[E|a_{ni}X_i|^{\theta}I(|X_i| \leq n^{\alpha}) + (a_{ni}n^{\alpha})^{\theta}P(|X_i| > n^{\alpha})\right] \\
&\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\theta}\ell(n)(\log n)^{\theta}\sum_{i=1}^{n}\left[E|X_i|^{\theta}I(|X_i| \leq n^{\alpha}) + n^{\alpha\theta}P(|X_i| > n^{\alpha})\right] \\
&\leq C\sum_{n=n_0}^{\infty} n^{\alpha p-1-\alpha\theta}\ell(n)(\log n)^{\theta}\left[E|X|^{\theta}I(|X| \leq n^{\alpha}) + n^{\alpha\theta}P(|X| > n^{\alpha})\right] \\
&= C\sum_{n=n_0}^{\infty} n^{\alpha p-1-\alpha\theta}\ell(n)(\log n)^{\theta}E|X|^{\theta}I(|X| \leq n^{\alpha}) + C\sum_{n=n_0}^{\infty} n^{\alpha p-1}\ell(n)(\log n)^{\theta}P(|X| > n^{\alpha}) \\
&\triangleq D_{111}^* + D_{112}^*.
\end{aligned}$$

Similarly to the proof of (3.29) we have

$$D_{112}^* \leq CE|X|^p\ell(|X|^{1/\alpha})(\log|X|)^{\theta} < \infty.$$

For D_{111}^*, we get from Lemma 3.6 and (2.4) that

$$\begin{aligned}
D_{111}^* &= C\sum_{n=n_0}^{\infty} n^{\alpha p-1-\alpha\theta}\ell(n)(\log n)^{\theta}\sum_{m=1}^{n}E|X|^{\theta}I((m-1)^{\alpha} < |X| \leq m^{\alpha}) \\
&\leq C\sum_{m=1}^{\infty}E|X|^{\theta}I((m-1)^{\alpha} < |X| \leq m^{\alpha})\sum_{n=m}^{\infty} n^{\alpha p-1-\alpha\theta}\ell(n)(\log n)^{\theta} \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha p-\alpha\theta}\ell(m)(\log m)^{\theta}E|X|^{\theta}I((m-1)^{\alpha} < |X| \leq m^{\alpha}) \\
&\leq CE|X|^p\ell(|X|^{1/\alpha})(\log|X|)^{\theta} < \infty.
\end{aligned}$$

We thus get $D_{11}^* < \infty$. Finally, for D_{12}^*, it follows from C_r inequality and Lemma 3.4 that

$$D_{12}^* \leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\theta+\frac{\theta}{2}}\ell(n)(\log n)^{\theta}\left(EX^2I(|X| \leq n^{\alpha}) + n^{2\alpha}P(|X| > n^{\alpha})\right)^{\theta/2}.$$

If $p > 2$, then we have $EX^2 < \infty$. Since $\theta > \frac{\alpha p-1}{\alpha-\frac{1}{2}}$, we get

$$D_{12}^* \leq C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\theta+\frac{\theta}{2}}\ell(n)(\log n)^{\theta} < \infty.$$

If $p\le 2$, it follows from Chebyshev's inequality that

$$\begin{aligned}
D_{12}^* &\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\alpha\theta+\frac{\theta}{2}}\ell(n)(\log n)^{\theta}\left(n^{\alpha(2-(p-\delta))}E|X|^{p-\delta}I(|X|\le n^{\alpha})+n^{2\alpha}n^{-\alpha(p-\delta)}E|X|^{p-\delta}\right)^{\theta/2}\\
&\le C\sum_{n=n_0}^{\infty} n^{\alpha p-2-\frac{\theta}{2}(\alpha(p-\delta)-1)}\ell(n)(\log n)^{\theta}<\infty,
\end{aligned}$$

since $\theta>\frac{2(\alpha p-1)}{\alpha(p-\delta)-1}$. This ends the proof of Theorem 2.2. □

Proof of Corollary 2.1. Without loss of generality, we may assume that $\sum_{i=1}^{n}|a_{ni}|^{\xi}\le n$ for some $\xi>p$. For fixed $n\ge 1$, let

$$A_{n1}=\{1\le i\le n,|a_{ni}|\le 1\},\quad A_{n2}=\{1\le i\le n,|a_{ni}|>1\},$$

and let

$$a_{ni}^{(1)}=a_{ni}I(i\in A_{n1})\quad a_{ni}^{(2)}=a_{ni}I(i\in A_{n2}).$$

Then

$$\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|\le\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}^{(1)}X_i\right|+\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}^{(2)}X_i\right|.$$

It follows from Theorem 2.1 and Theorem 2.2 that

$$\begin{aligned}
&\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|\ge\varepsilon n^{\alpha}\right)\\
\le\ &\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}^{(1)}X_i\right|\ge\frac{\varepsilon n^{\alpha}}{2}\right)+\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}^{(2)}X_i\right|\ge\frac{\varepsilon n^{\alpha}}{2}\right)\\
<\ &\infty,
\end{aligned}$$

which ends the proof of Corollary 2.1. □

Proof of Theorem 2.3 Let $a_{ni}=W_{ni}\cdot n^{\alpha}$. For any $\xi>p$, it follows by (2.7) that

$$\sum_{i=1}^{n}|a_{ni}|^{\xi}=\sum_{i=1}^{n}|W_{ni}|^{\xi}\cdot n^{\alpha\xi}=O(n\cdot n^{-\alpha\xi})\cdot n^{\alpha\xi}=O(n).$$

It can be found that condition (2.1) in Theorem 2.1 is satisfied. For any $x\in c(f)$, it is easy to see that

$$f_{nk}(x)-Ef_{nk}(x)=\sum_{i=1}^{k}W_{ni}(x)\varepsilon_{ni}.$$

Hence, it follows by (2.3) of Corollary 2.1 that

$$\begin{aligned}
&\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}W_{ni}(x)\varepsilon_{ni}\right|\ge\varepsilon\right)\\
=\ &\sum_{n=n_0}^{\infty} n^{\alpha p-2}\ell(n)P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}(x)\varepsilon_{i}\right|\ge\varepsilon n^{\alpha}\right)<\infty,
\end{aligned}$$

since $(\varepsilon_{n1},\varepsilon_{n2},\cdots,\varepsilon_{nn})$ have the same distribution as $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)$ for each $n\ge 1$. By the property of the slowly varying function, it follows that for any $x\in c(f)$,

$$\max_{1\le k\le n}|f_{nk}(x)-Ef_{nk}(x)|\to 0,\ \text{completely},\ \text{ as } n\to\infty. \tag{3.34}$$

Hence, to prove (2.8), it suffices to show that for any $x \in c(f)$,

$$\max_{1\le k\le n} |Ef_{nk}(x) - f(x)| \to 0, \ as\ n \to \infty. \tag{3.35}$$

For any $x \in c(f)$ and $a > 0$, it follows from (2.5) and (2.6) that

$$\begin{aligned}
\max_{1\le k\le n} |Ef_{nk}(x) - f(x)| &\le \max_{1\le k\le n} \left|\sum_{i=1}^{k} W_{ni}(x)\cdot(f(x_{ni}) - f(x))\right| \\
&\quad + |f(x)| \max_{1\le k\le n} \left|\sum_{i=1}^{k} W_{ni}(x) - 1\right| \\
&\le \sum_{i=1}^{n} |W_{ni}(x)|\cdot|f(x_{ni}) - f(x)|\cdot I(||x_{nk} - x|| \le a) \\
&\quad + \sum_{i=1}^{n} |W_{ni}(x)|\cdot|f(x_{ni}) - f(x)|\cdot I(||x_{nk} - x|| > a) \\
&\quad + |f(x)| \max_{1\le k\le n} \left|\sum_{i=1}^{k} W_{ni}(x) - 1\right|.
\end{aligned} \tag{3.36}$$

Since $x \in c(f)$, for any $\upsilon > 0$, there exists a $\vartheta > 0$, such that $|f(x') - f(x)| < \upsilon$ when $\|x' - x\| < \vartheta$. Select $0 < a < \vartheta$ in formula (3.36), we have

$$\begin{aligned}
\max_{1\le k\le n} |Ef_{nk}(x) - f(x)| &\le \theta \sum_{i=1}^{n} |W_{ni}(x)| + |f(x)| \cdot \max_{1\le k\le n} \left|\sum_{i=1}^{k} W_{ni}(x) - 1\right| \\
&\quad + \sum_{i=1}^{n} |W_{ni}(x)|\cdot|f(x_{ni}) - f(x)| I(||x_{ni} - x|| > a),
\end{aligned}$$

which together with conditions $(H_1) - (H_3)$ yields that (3.35) holds. This completes the proof. □

Proof of Theorem 2.4 The proof is similar to that of Theorem 2.3. We will apply Corollary 2.1 with $p = 2/\alpha$ and $\ell(x) \equiv 1$. Hence, the desired result (2.8) follows by Corollary 2.1 immediately. The proof is ended. □

References

[1] Adler, A., Rosalsky, A. 1987. Some general strong laws for weighted sums of stochastically dominated random variables. *Stoch. Anal. Appl.* 5, 1-16.

[2] Adler, A., Rosalsky, A., Taylor, R.L. 1989. Strong laws of large numbers for weighted sums of random elements in normed linear spaces. *Int. J. Math. Math. Sci.* 12, 507-530.

[3] Baum, L.E., Katz, M. 1965. Convergence rates in the law of large numbers. *Trans. Am. Math. Soc.* 120, 1, 108-123.

[4] Chow, Y.S. 1973. Delayed sums and Borel summability of independent, identically distributed random variables. *Bull. Inst. Math. Acad. Sinica* 1, 207-220.

[5] Birkel, T. 1989. A note on the strong law of large numbers for positively dependent random variables, *Statist. Probab. Lett.* 7, 17-20.

[6] Chen, Y., Chen, A., Ng, K. 2010. The strong law of large numbers for extended negatively dependent random variables, *J. Appl. Prob.* 47, 4, 908-922.

[7] Chow, Y.S. 1988. On the rate of moment convergence of sample sums and extremes, *Bulletin of the Institute of Math.* 16, 3, 177-201.

[8] Erdős, P. 1949. On a theorem of Hsu and Robbins. *Ann. Math. Stat.* 20, 286-291.

[9] Etemadi, N. 1981. An elementary proof of the strong law of large numbers, *Z. Wahrsch. Verw. Gebiete.* 55, 119-122.

[10] Etemadi, N. 1983. On the strong law of large numbers for nonnegative random variables, *J. Multivariate Anal.* 13, 187-193.

[11] Fan, Y. 1990. Consistent nonparametric multiple regression for dependent heterogeneous processes: the fixed design case. *J. Multivariate Analysis*, 33, 72-88.

[12] Georgiev, A.A. 1983. Local properties of function fitting estimates with applications to system identification, in *Math. Stat. and Appl.* eds. W.Grossmann et al., *Proceedings 4th Pannonian Sump. Math. Stat.* 4-10 September 1983, Bad Tatzmannsdorf, Austria: Reidel, Dordrecht, 141-151.

[13] Georgiev, A.A. 1988. Consistent nonparametric multiple regression: the fixed design case. *J. Multivariate Analysis*, 25, 100-110.

[14] Georgiev, A.A., Greblicki, W. 1986. Nonparametric function recovering from noisy observations. *J. Stat. Planning and Inference* 13, 1-14.

[15] Heyde, C.C. 1975. A supplement to the strong law of large numbers. J. Appl. Probab. 12, 173-175.

[16] Hsu, P.L., Robbins, H. 1947. Complete convergence and the law of large numbers. Proc. Natl. Acad. Sci. USA 33, 25-31.

[17] Hu, S.H., Zhu, C.H., Chen, Y.B., and Wang, L.C. 2002. Fixed-design Regression for Linear Time Series. *Acta Mathematica Scientia*, 22B, 9-18.

[18] Katz, M. 1963. The probability in the tail of a distribution. *Ann. Math. Stat.* 34, 312-318.

[19] Liang, H.Y., Jing, B.Y. 2005. Asymptotic properties for estimates of nonparametric regression models based on negatively associated sequences. *J. Multivariate Analysis*, 95, 227-245.

[20] Liu, L. 2009. Precise large deviations for dependent random variables with heavy tails. Stat Probab Lett. 79, 1290-1298. doi:10.1016/j.spl.2009.02.001.

[21] Liu, L. 2010. Necessary and sufficient conditions for moderate deviations of dependent random variables with heavy tails. Sci China Ser A Math. 53, 6, 1421-1434. doi:10.1007/s11425-010-4012-9.

[22] Müller, H.G. 1987. Weak and universal consistency of moving weighted averages. *Periodica Math. Hungarica*, 18, 241-250.

[23] Qiu, D.H., Chen, P.Y. 2011. Limiting behavior of weighted sums of NOD random variables. *J. Math. Res. Exposition* 31, 1081-1091.

[24] Qiu, D.H., Chen, P.Y. 2014. Complete and complete moment convergence for weighted sums fo widely orthant dependent random variables. *Acta Math. Sin. Engl. Ser.* 30, 9, 1539-1548.

[25] Qiu, D.H., Hu, T.C. 2014. Strong limit theorems for weighted sums fo widely orthant dependent random variables. *J. Math. Res. Appl.* 34, 1, 105-113.

[26] Roussas, G.G. 1989. Consistent regression estimation with fixed design points under dependence conditions. *Stat. Prob. Letters*, 8, 41-50.

[27] Seneta, E. 1976. *Regularly varying function.* Lecture notes in Math., Vol. 508. Springer, Berlin.

[28] Shen, A. 2011. Probability inequalities for END sequence and their applications, *J. Inequal. Appl.* 98, 1-12.

[29] Shen, A. 2016. Complete convergence for weighted sums of END random variables and its applications to nonparametric regression models. *J. Nonparametric Stat.* 28, 4, 702-715.

[30] Shen, A., Xue, M., Wang, W. 2017. Complete convergence for weighted sums of extended negatively dependent random variables. *Communications in Stat.- Theory and Methods* 46, 3, 1433-1444.

[31] Stone, C.J. 1977. Consistent nonparametric regression. *The Annals of Stat.* 5, 595-620.

[32] Tran, L., Roussas, G., Yakowitz. S., Van, B.T. 1996. Fixed -design regression for linear time series. *Annal. Stat.* 24, 975-991.

[33] Wang, X.J., Deng, X., Zheng, L.L., Hu, S.H. 2014a. Complete convergence for arrays of rowwise negatively superadditive-dependent random variables and its applications. *Statistics: J. Theor. Appl. Stat.* 48, 4, 834-850.

[34] Wang, X.J., Hu, S.H. 2015. The consistency of the nearest neighbor estimator of the density function based on WOD samples. *J. Math. Anal. Appl.* 429, 497-512.

[35] Wang, X.J., Li, X.Q., Hu, S.H., Wang, X.H. 2014b. On complete convergence for an extended negatively dependent sequence. *Communications in Stat.- Theory and Methods* 43, 2923-2937.

[36] Wang, X.J., Xu, C., Hu, T.C., Volodin, A., Hu, S.H. 2014c. On complete convergence for Widely Orthant-dependent random variables and its applications in nonparametric regression models. *TEST*, 23, 607-629.

[37] Wang, X.J., Zheng, L.L., Xu, C., Hu, S.H. 2015. Complete consistency for the estimator of nonparameter regression models based on extended negatively dependent errors. *Statistics: A Journal of Theoretical and Applied Stat.* 49, 396-407.

[38] Wu, Q.Y. 2006. *Probability limit theory for mixing sequences*. Science Press of China, Beijing.

[39] Wu, Y.F., Song, M.Z., Wang, C.H. 2014. Complete moment convergence and mean convergence for arrays of rowwise extended negatively dependent random variables. *The Scientific World Journal*, Artical ID 478612, 7 pages. doi: 10.1155/2014/478612.

[40] Yan, J.G. 2017. Strong Stability of a type of Jamison Weighted Sums for END Random Variables. *J. Korean Math. Soc.* 54, 3, 897-907.

[41] Yan, J.G. 2017. Almost sure convergence for weighted sums of WNOD random variables and its applications in nonparametric regression models1. *Communications in Stat.-Theory and Methods.* 47, 16, 3893-3909.

[42] Yan, J.G. 2018. Complete convergence and complete moment convergence for maximal weighted sums of extended negatively dependent random variables. *Acta Math. Sin. Engl. Ser.* https://doi.org/10.1007/s10114-018-7133-7.

[43] Yang, W.Z., Liu, T.T., Wang, X.J., Hu, S.H. 2014. On the Bahadur representation of sample quantitles for widely orthant dependent sequences. *Filomat* 28, 7, 1333-1343.

[44] Yang, W.Z., Wang, X.J., Wang, X.H., Hu, S.H. 2012. The consistency for estimator of nonparametric regression model based on NOD errors. *J. Inequalities and Applications* 13 pages, Article ID 140.

[45] Zhang, G.H. 2014. Complete convergence for Sung's type weighted sums of END random variables. *J. Inequalities and Applications* 2014:353.

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