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Strict Stationarity Testing and GLAD Estimation of Double Autoregressive Models

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Strict Stationarity Testing and GLAD Estimation of Double Autoregressive Models

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Abstract

In this article we develop a tractable procedure for testing strict stationarity in a double autoregressive model and formulate the problem as testing if the top Lyapunov exponent is negative. Without strict stationarity assumption, we construct a consistent estimator of the associated top Lyapunov exponent and employ a random weighting approach for its variance estimation, which in turn are used in a t -type test. We also propose a GLAD estimation for parameters of interest, relaxing key assumptions on the commonly used QMLE. All estimators, except for the intercept, are shown to be consistent and asymptotically normal in both stationary and explosive situations. The finite-sample performance of the proposed procedures is evaluated via Monte Carlo simulation studies and a real dataset of interest rates is analyzed.

Keywords: DAR model, GLAD estimation, Nonstationarity, Random weighting, Strict stationarity testing.

JEL Classification: C15, C22.

1 Introduction

The assumption of strict stationarity is pivotal in nonlinear time series inference and forecasting. Testing stationarity in the context of linear time series models has been well documented, such as various unit root tests. However, this testing problem may pose considerably more challenges in a nonlinear setting. Recently, [Francq and Zakoïan \(2012, 2013\)](#)

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considered strict stationarity testing for GARCH models (Engle (1982), Bollerslev (1986)) based on the sign of the associated top Lyapunov exponent. To our best knowledge, they are the first to address such a testing issue under the GARCH framework.

Besides GARCH, double autoregressive (DAR) model is another important conditional heteroscedastic one. The first-order DAR (hereafter DAR (1)) model is defined as

$$y_t = \phi_0 y_{t-1} + \eta_t \sqrt{\omega_0 + \alpha_0 y_{t-1}^2}, \quad t = 0, \pm 1, \pm 2, \dots, \quad (1)$$

where $\phi_0 \in R$, $\omega_0 > 0$, $\alpha_0 > 0$, $\{\eta_t\}$ is a sequence of independent and identically distributed (i.i.d.) random variables and independent of $\{y_j, j < t\}$. Model (1) is a special case of the ARMA-ARCH models in Weiss (1984) and of the nonlinear AR models in Cline and Pu (2004). It is different from Engel's ARCH model when $\phi_0 \neq 0$. There has been considerable work on DAR models and its generalizations, see, e.g., Tsay (1987), Lu (1998), Ling (2004, 2007), Zhu and Ling (2013), Nielsen and Rahbek (2014), Li, Ling and Zakoian (2015), Li, Ling and Zhang (2016), Li et al. (2017), Zhu, Zheng and Li (2018).

The strict stationarity condition for model (1) has been well formulated and is closely related to the top Lyapunov exponent

$$\gamma_0 = E \log |\phi_0 + \eta \sqrt{\alpha_0}|,$$

where η is a generic random variable with the same distribution as η_t . Borkovec and Klüppelberg (2001) proved that $\gamma_0 < 0$ is sufficient for strict stationarity of model (1), while Chen, Li and Ling (2014) proved that it is (almost) necessary. Thus, testing strict stationarity is equivalent to testing the following hypothesis:

$$H_0 : \gamma_0 < 0 \quad \text{v.s.} \quad H_1 : \gamma_0 \geq 0. \quad (2)$$

To test (2), a challenge is to obtain a good estimator of γ_0 under the null and alternative hypotheses. Surprisingly, the majority of literature is focused on the inference of $(\phi_0, \alpha_0, \omega_0)$ rather than γ_0 . Chan and Peng (2005) discussed the estimation of γ_0 in the stationary case via simulation studies so as to check the stationarity condition. Nevertheless, they do not consider its asymptotic properties. This may be partly due to the fact that the estimation of γ_0 is nonstandard even in the stationary situation and the related asymptotic theory is hard to pursue. Fig. 1 shows the strictly stationary region of model (1) for three different distributions of η , indicating that γ_0 depends not only on the intrinsic parameters but also on the distribution of the underlying innovations.

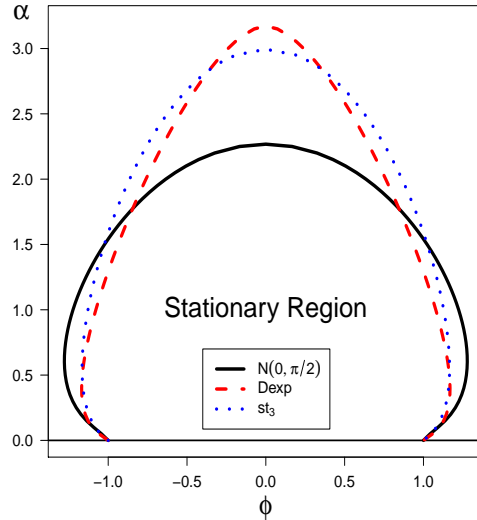


Figure 1: The (strictly) stationary region determined by $\{(\phi, \alpha) : E \log |\phi + \eta\sqrt{\alpha}| < 0\}$ for $\eta \sim N(0, \pi/2)$, the double exponential distribution with density $f(x) = 0.5 \exp(-|x|)$, and the standardized Student's t_3 (st_3) with density $f(x) = 4\pi^2/(\pi^2 + 4x^2)^2$.

It is notable that [Ling \(2004\)](#) considered the following testing problem for model (1):

$$H_0 : (\phi_0, \alpha_0) = (\pm 1, 0) \quad \text{v.s.} \quad H_1 : (\phi_0, \alpha_0) \neq (\pm 1, 0). \quad (3)$$

Under H_0 in (3), $\{y_t\}$ is a standard unit root process. However, we cannot claim that $\{y_t\}$ is stationary even if H_0 in (3) is rejected. [Ling \(2004\)](#) further points out that testing $\gamma_0 < 0$ in (2) remains an interesting issue. One aim of this paper is to develop a tractable procedure for testing (2) within the DAR framework.

This paper has two major contributions. First, we propose a data-driven procedure for testing (2). The basic steps are as follows. We first provide a natural estimator of γ_0 and demonstrate that it possesses nice asymptotic properties without any stationarity assumption. We then develop a random weighting method to approximate its asymptotic covariance. A feature of our procedure is that it does not rely on the expression of asymptotic variance and turns out to capture the sampling uncertainty adaptively. Based on these estimators, we propose a t -type statistic for testing strict stationarity and construct a consistent critical region. It is remarkable that our basic idea is similar to that in [Francq and Zakoïan \(2012\)](#) but the underlying large-sample theories own considerably more mathematical gaps. In other words, the proof techniques in [Francq and Zakoïan \(2012\)](#) fail to apply here. To solve this problem, we rely on modern empirical process theory. A new

theoretical insight on this issue is provided in Section 3.

In the literature, numerous methods have been developed for inference on model (1). Ling (2004, 2007) considered the quasi-maximum likelihood estimation (QMLE) of $(\phi_0, \alpha_0, \omega_0)$ and proved its asymptotics under the conditions $\gamma_0 < 0$ and $E\eta^4 < \infty$. On the other hand, in the explosive case (i.e., $\gamma_0 > 0$), Ling and Li (2008) investigated a constrained QMLE of (ϕ_0, α_0) in the sense that ω_0 is fixed and obtained asymptotic normality when η is standard normal. Recently, Chen, Li and Ling (2014) studied an unconstrained QMLE of (ϕ_0, α_0) when η is symmetric with $E\eta^4 < \infty$. One disadvantage of the QMLE is that the assumption of $E\eta^4 < \infty$ is indispensable for valid inference but too restrictive in practice. To tackle this challenge, a robust estimation procedure is provided, see Chan and Peng (2005) and Zhu and Ling (2013) for the stationary case. In the presence of nonstationarity or heavy-tailed noises, the inference becomes more challenging and no results are available so far.

The second contribution of this article is to offer a unified framework for parameter estimation of model (1) in both stationary and explosive cases. This framework does not require the fourth moment of the innovation η_t to be finite and hence can cover heavy-tailed cases. Specifically, we propose an unconstrained global least absolute deviation estimation (GLADE) for $(\phi_0, \alpha_0, \omega_0)$ when ω_0 is not fixed. Here, ‘global’ means that the convergence rate of the estimator is first obtained and the limiting distribution is then derived, see Zhu and Ling (2011, 2013). Under mild conditions, the proposed estimator of (ϕ_0, α_0) are always strongly consistent and asymptotically normal when the data mechanism is stationary or explosive. To measure the accuracy of the estimator, we again propose a random weighting method to estimate its asymptotic covariance matrix. It is worth stressing that even if the GLADE of (ϕ_0, α_0) is consistent in every situation, the intercept term ω_0 is only consistent in the stationary case. This further demonstrates the importance of testing the sign of γ_0 .

The remainder of the paper is organized as follows. Section 2 presents the global LADE with asymptotic properties and discusses how to estimate the asymptotic variance matrix via the random weighting approach. In Section 3, we address the estimation of γ_0 and tests of stationarity and nonstationarity. The relevant asymptotics are also established. Section 4 reports numerical results on the finite-sample performance of the proposed methodology and analyzes a real dataset of interest rates. A concluding remark is in Section 5 and all the technical proofs can be found in the Appendix.

2 Global Least Absolute Deviation Estimation

2.1 Asymptotic Properties

Suppose that the observations $\{y_0, y_1, \dots, y_n\}$ are from model (1). When η is double exponential, the log-likelihood function (ignoring a constant) can be written as

$$L_n(\theta) = \sum_{t=1}^n \left\{ \frac{1}{2} \log(\omega + \alpha y_{t-1}^2) + \frac{|y_t - \phi y_{t-1}|}{\sqrt{\omega + \alpha y_{t-1}^2}} \right\},$$

where $\theta = (\phi, \alpha, \omega)^T$ is the parameter. The proposed estimator is defined as

$$\hat{\theta}_n = (\hat{\phi}_n, \hat{\alpha}_n, \hat{\omega}_n)^T = \arg \min_{\theta \in \Theta} L_n(\theta), \quad (4)$$

where Θ is a compact subset of $R \times R_+^2$ containing the true value $\theta_0 = (\phi_0, \alpha_0, \omega_0)^T$. Here $R_+ = (0, \infty)$. Since we do not assume that η is double exponential, the estimator $\hat{\theta}_n$ is often called the quasi-maximum exponential likelihood estimator as in [Zhu and Ling \(2013\)](#) or the least absolute deviation estimator (LADE) as in [Chan and Peng \(2005\)](#). Throughout the paper, following a traditional naming convention in literature, we refer to $\hat{\theta}_n$ as the (global) LADE of θ_0 .

Theorem 2.1. *Suppose that $\{\eta_t\}$ is i.i.d. and symmetric with $E|\eta_t| = 1$ and Θ is compact. Then, for the DAR(1) model (1), the LADE defined in (4) satisfies the following properties.*

- (i). *If $\gamma_0 < 0$, then $\hat{\phi}_n \rightarrow \phi_0$, $\hat{\alpha}_n \rightarrow \alpha_0$, and $\hat{\omega}_n \rightarrow \omega_0$ a.s. as $n \rightarrow \infty$.*
- (ii). *If $\gamma_0 > 0$, then $\hat{\phi}_n \rightarrow \phi_0$ and $\hat{\alpha}_n \rightarrow \alpha_0$ a.s. as $n \rightarrow \infty$.*

To further discuss asymptotic distribution of $\hat{\theta}_n$, we need three assumptions.

Assumption 1. *$\{\eta_t\}$'s are i.i.d. with $E|\eta_1| = 1$ and $\kappa_\eta = E\eta_1^2 < \infty$.*

Assumption 2. *The density $f(x)$ of η_1 is symmetric and bounded continuous on R with $f(0) > 0$.*

Assumption 3. *Θ is compact and θ_0 is an interior point of Θ .*

Theorem 2.2. *Suppose that Assumptions 1-3 hold.*

- (i). *If $\gamma_0 < 0$, then*

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = O_p(1) \quad \text{and} \quad \sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} N(0, \mathcal{J}_S)$$

as $n \rightarrow \infty$, where ' $\xrightarrow{\mathcal{L}}$ ' stands for the convergence in distribution,

$$\mathcal{J}_S = \text{diag} \left\{ \frac{1}{4\sigma_{11}f^2(0)}, 4(\kappa_\eta - 1)\Sigma^{-1} \right\}, \quad (5)$$

and both Σ and σ_{11} are defined as

$$\Sigma = \begin{pmatrix} \sigma_{22} & \sigma_{12} \\ \sigma_{12} & \sigma_{02} \end{pmatrix} \quad \text{with} \quad \sigma_{ij} = E \left\{ \frac{y_t^{2i}}{(\omega_0 + \alpha_0 y_t^2)^j} \right\}, \quad i, j = 0, 1, 2. \quad (6)$$

(ii). If $\gamma_0 > 0$, then

$$\sqrt{n}(\hat{\phi}_n - \phi_0, \hat{\alpha}_n - \alpha_0)^T = O_p(1) \quad \text{and} \quad \sqrt{n}(\hat{\phi}_n - \phi_0, \hat{\alpha}_n - \alpha_0)^T \xrightarrow{\mathcal{L}} N(0, \mathcal{J}_N),$$

as $n \rightarrow \infty$, where

$$\mathcal{J}_N = \text{diag} \left\{ \frac{\alpha_0}{4f^2(0)}, 4(\kappa_\eta - 1)\alpha_0^2 \right\}. \quad (7)$$

Remark 1. From Theorem 2.2, we can see that our LADE is global, i.e., the convergence rate is first obtained and the limiting distribution then derived, see [Zhu and Ling \(2011, 2013\)](#). This is totally different from the local LADE of regression or time series models. On the other hand, it is worth noting that $\hat{\omega}_n$ is inconsistent when $\gamma_0 > 0$. A similar phenomenon was observed by [Chen, Li and Ling \(2014\)](#), who studied an unconstrained QMLE of an explosive DAR(1) model, and by [Francq and Zakoïan \(2012\)](#) who studied the QMLE of nonstationary GARCH(1,1) models, see also [Jensen and Rahbek \(2004a,b\)](#).

Remark 2. We explore the hidden relationship between expressions (5) and (7), which reveals why several results in the stationary situation can still be applicable in the explosive one. Note that all σ_{ij} 's in (6) are finite constants when $\gamma_0 < 0$ since $\{y_t\}$ is strictly stationary and ergodic. In the explosive situation, i.e. $\gamma_0 > 0$, however, $|y_t|$ diverges to infinity at an exponential rate as $t \rightarrow \infty$, see Theorem 1 in [Chen, Li and Ling \(2014\)](#) or Theorem 2.1 in [Liu, Li and Kang \(2018\)](#). Thus, $\sigma_{11} \rightarrow 1/\alpha_0$, $\sigma_{22} \rightarrow 1/\alpha_0^2$ and other σ_{ij} 's go to zero a.s. as $t \rightarrow \infty$, which implies that \mathcal{J}_S can be reduced to $\text{diag}(\mathcal{J}_N, 0)$.

Remark 3. Theorem 2.2 rules out the boundary case $\alpha_0 = 0$ or $\omega_0 = 0$. Clearly, if $\alpha_0 = 0$, the GLADE $\hat{\alpha}_n$ is not asymptotically normal at all since $\hat{\alpha}_n \geq 0$. A similar phenomenon can be found in Example 5 for an ARCH(1) model in [Francq and Zakoïan \(2007\)](#). When $\omega_0 = 0$, model (1) reduces to a double AR(1) model without intercept, which has been studied by [Li, Guo and Zhu \(2018\)](#).

Remark 4. Unlike the GARCH(1, 1) model in [Francq and Zakoïan \(2012\)](#), we don't establish asymptotic properties of the GLADE for the case $\gamma_0 = 0$ since the asymptotic behavior of $|y_t|$ is unknown in this scenario. Although we may pay some extra cost, such as more unverifiably restrictive conditions on η_t and y_t , to obtain asymptotics of the GLADE, we don't pursue this at this moment and left it for future work.

Remark 5. *The GLADE proposed here can be easily extended to a strictly stationary DAR(p) ($p \geq 1$) model with an intercept in the conditional mean. For a nonstationary DAR(1) model with an intercept in the conditional mean, it can be shown that the GLADE of this intercept is not consistent, just like ω_0 in this case. However, for nonstationary DAR(p) model with $p \geq 2$, we cannot establish its asymptotics following the routine of nonstationary high-order GARCH models as they pose considerably huge gap between these two different processes. We left this for future work.*

Remark 6. *In standard linear time series analysis, nonstationarity leads to completely different asymptotic inferences, such as the Dickey-Fuller statistics. However, this is not so in DAR models, where a unit root in the conditional mean does not necessarily imply nonstationarity as lagged observations of the process enter the conditional variance. This point can be seen from Fig.1. See also [Borkovec and Klüppelberg \(2001\)](#), [Ling \(2004\)](#) and [Nielsen and Rahbek \(2014\)](#).*

2.2 Variance Estimation

Theorem 2.2 demonstrates that the proposed global robust estimator shares nice asymptotic normality in both cases. To evaluate the accuracy of the estimator, we need a consistent estimator of their asymptotic covariance matrix. Unlike the QMLE, however, their asymptotic covariance matrix involves the density function $f(\cdot)$ and cannot be precisely estimated using the plug-in rules. One choice is to use the nonparametric kernel method to estimate the density, see [Zhu and Ling \(2013\)](#). However, this method involves the choice of the bandwidth, which is another problem. To avoid density estimation, we propose a random weighting approach for covariance matrix estimation. This approach, as a variant of the traditional wild bootstrap in [Wu \(1986\)](#), was initially proposed by [Jin, Ying and Wei \(2001\)](#). So far, it has been widely used for statistical inference in regression and time series models when the asymptotic covariance matrix of certain estimator can not be directly estimated; see [Koul and Ossiander \(1994\)](#), [Koul \(2002\)](#), [Chen, et al. \(2008, 2010\)](#), [Li, Leng and Tsai \(2014\)](#) and [Zhu \(2016\)](#), etc.

To be specific, let $\varpi_1, \dots, \varpi_n$ be a sequence of i.i.d. nonnegative random variables, with both mean and variance equal to one. For example, the standard exponential distribution satisfies this requirement. Define

$$L_n^*(\theta) = \sum_{t=1}^n \varpi_t \left\{ \frac{1}{2} \log(\omega + \alpha y_{t-1}^2) + \frac{|y_t - \phi y_{t-1}|}{\sqrt{\omega + \alpha y_{t-1}^2}} \right\}$$

and $\hat{\theta}_n^* = \arg \min_{\theta \in \Theta} L_n^*(\theta)$. In the stationary case, the distribution of $\sqrt{n}(\hat{\theta}_n - \theta_0)$ can be approximated by the resampling distribution of $\sqrt{n}(\hat{\theta}_n^* - \hat{\theta}_n)$. It turns out that, in the

explosive case, the distribution of $\sqrt{n}(\hat{\phi}_n - \phi_0, \hat{\alpha}_n - \alpha_0)^T$ can still be approximated by that of $\sqrt{n}(\hat{\phi}_n^* - \hat{\phi}_n, \hat{\alpha}_n^* - \hat{\alpha}_n)^T$.

Theorem 2.3. *Suppose Assumptions 1–3 hold.*

(i). *If $\gamma_0 < 0$, then, conditionally on the data $\{y_0, y_1, \dots, y_n\}$, in probability,*

$$\sqrt{n}(\hat{\theta}_n^* - \hat{\theta}_n) \xrightarrow{\mathcal{L}} N(0, \mathcal{J}_S), \quad \text{as } n \rightarrow \infty;$$

(ii). *If $\gamma_0 > 0$, then, conditionally on the data $\{y_0, y_1, \dots, y_n\}$, in probability,*

$$\sqrt{n}(\hat{\phi}_n^* - \hat{\phi}_n, \hat{\alpha}_n^* - \hat{\alpha}_n)^T \xrightarrow{\mathcal{L}} N(0, \mathcal{J}_N), \quad \text{as } n \rightarrow \infty.$$

The proof of Theorem 2.3 is provided in the Appendix. The inference procedure via random weighting is done as follows. First, i.i.d. $\{\varpi_1, \dots, \varpi_n\}$ are generated B times from standard exponential distribution, where B is a large number. Each time, the minimizer $\hat{\theta}_n^*$ is computed. Denote them as $\hat{\theta}_n^{*1}, \dots, \hat{\theta}_n^{*B}$. Then, the asymptotic variances of $\sqrt{n}(\hat{\phi}_n - \phi_0)$ and $\sqrt{n}(\hat{\alpha}_n - \alpha_0)$ can be approximated by, respectively,

$$\hat{\sigma}_\phi^2 = \frac{1}{B-1} \sum_{b=1}^B (\hat{\phi}_n^{*b} - \bar{\phi}_n^*)^2 \quad \text{and} \quad \hat{\sigma}_\alpha^2 = \frac{1}{B-1} \sum_{b=1}^B (\hat{\alpha}_n^{*b} - \bar{\alpha}_n^*)^2,$$

with $\bar{\phi}_n^* = B^{-1} \sum_{b=1}^B \hat{\phi}_n^{*b}$ and $\bar{\alpha}_n^* = B^{-1} \sum_{b=1}^B \hat{\alpha}_n^{*b}$. In the stationary case, the asymptotic variance of $\sqrt{n}(\hat{\omega}_n - \omega_0)$ is estimated analogously.

3 Strict Stationarity Testing

As described in the Introduction, the parameter γ_0 plays a key role in characterizing the strict stationarity of DAR(1) model. It also determines the consistency of $\hat{\omega}_n$. In this section, we propose a consistent estimator $\hat{\gamma}_n$ of γ_0 and then construct t -type tests based on $\hat{\gamma}_n$ for whether $\{y_t\}_{t=1}^n$ is strictly stationary or not. The asymptotic normality of $\hat{\gamma}_n$ is established and a random weighting approach is correspondingly proposed to estimate asymptotic variance of $\hat{\gamma}_n$ in Section 3.1. Section 3.2 shows that the t -type stationarity tests are consistent.

3.1 Consistent Estimator of γ_0

Define the rescaled residuals as

$$\hat{\eta}_t = \eta_t(\hat{\theta}_n), \quad \eta_t(\theta) = \frac{y_t - \phi y_{t-1}}{\sqrt{\omega + \alpha y_{t-1}^2}}. \quad (8)$$

By the definition of γ_0 and the symmetry of η_t , a natural estimator of γ_0 is

$$\tilde{\gamma}_n = \frac{1}{2n} \sum_{t=1}^n (\log |\hat{\phi}_n + \hat{\eta}_t \sqrt{\hat{\alpha}_n}| + \log |\hat{\phi}_n - \hat{\eta}_t \sqrt{\hat{\alpha}_n}|). \quad (9)$$

To facilitate our proofs, we propose a truncated estimator for γ_0 :

$$\hat{\gamma}_n = \frac{1}{2n} \left(\sum_{t \in \mathcal{A}_1} \log |\hat{\phi}_n + \hat{\eta}_t \sqrt{\hat{\alpha}_n}| + \sum_{t \in \mathcal{A}_2} \log |\hat{\phi}_n - \hat{\eta}_t \sqrt{\hat{\alpha}_n}| \right), \quad (10)$$

where $\mathcal{A}_1 = \{t : \hat{\phi}_n + \hat{\eta}_t \sqrt{\hat{\alpha}_n} \in \mathcal{I}_n, 1 \leq t \leq n\}$ and $\mathcal{A}_2 = \{t : \hat{\phi}_n - \hat{\eta}_t \sqrt{\hat{\alpha}_n} \in \mathcal{I}_n, 1 \leq t \leq n\}$ with $\mathcal{I}_n = [-n^2, -n^{-2}] \cup [n^{-2}, n^2]$. Indeed, in practice, both estimates are almost identical unless $|\hat{\phi}_n + \hat{\eta}_t \sqrt{\hat{\alpha}_n}|$ or $|\hat{\phi}_n - \hat{\eta}_t \sqrt{\hat{\alpha}_n}|$ is extremely small for some t .

To prove the asymptotics of $\hat{\gamma}_n$, we impose the following assumption.

Assumption 4. *The density $f(\cdot)$ of η_1 is positive and differentiable a.s. on R with $\sup_{x \in R} f(x) < \infty$, and $\int (\log |\phi_0 + x\sqrt{\alpha_0}|) \{f(x) + |xf'(x)|\} dx$ exists and is finite.*

Remark 7. *The most commonly used distributions, like normal and Student's t distributions, satisfy this condition. The double exponential (i.e., Laplace) distribution is also allowed since it is differentiable except one point.*

To state the asymptotic normality of $\hat{\gamma}_n$, we first introduce some notation. Denote

$$\begin{aligned} \mu_1 &= -\alpha_0^{-1/2} \int \log |\phi_0 + x\sqrt{\alpha_0}| f'(x) dx, \\ \mu_2 &= -\alpha_0^{-1/2} \int \log |\phi_0 + x\sqrt{\alpha_0}| \{f(x) + xf'(x)\} dx. \end{aligned}$$

For each t , define

$$\begin{aligned} \zeta_t &= (\zeta_{1t} + \zeta_{2t})/2, \\ \zeta_{1t} &= \log |\phi_0 + \eta_t \sqrt{\alpha_0}| - E(\log |\phi_0 + \eta_t \sqrt{\alpha_0}|), \\ \zeta_{2t} &= \log |\phi_0 - \eta_t \sqrt{\alpha_0}| - E(\log |\phi_0 - \eta_t \sqrt{\alpha_0}|). \end{aligned}$$

For $\gamma_0 < 0$, denote $\xi_t = (\xi_{1t}, \xi_{2t}^T)^T$, where

$$\xi_{1t} = \frac{1}{2\sigma_{11}f(0)} \frac{y_{t-1} \text{sign}(\eta_t)}{\sqrt{\omega_0 + \alpha_0 y_{t-1}^2}}, \quad \xi_{2t} = \Sigma^{-1} \left(\frac{2y_{t-1}^2}{\omega_0 + \alpha_0 y_{t-1}^2}, 2 \right)^T (|\eta_t| - 1),$$

and $\nu = (\nu_1, \nu_2)^T$, where

$$\nu_1 = E \frac{\omega_0}{2\sqrt{\alpha_0}(\omega_0 + \alpha_0 y_{t-1}^2)}, \quad \nu_2 = -E \frac{\sqrt{\alpha_0}}{2(\omega_0 + \alpha_0 y_{t-1}^2)}.$$

In the case of $\gamma_0 > 0$, we denote

$$\tilde{\xi}_{1t} = \frac{\sqrt{\alpha_0}}{2f(0)} \text{sign}(\eta_t).$$

The following theorem gives the asymptotic properties of $\hat{\gamma}_n$.

Theorem 3.1. *Suppose that Assumptions 1–4 hold.*

(i). *If $\gamma_0 < 0$, then*

$$\begin{aligned} \sqrt{n}(\hat{\gamma}_n - \gamma_0) &= \frac{1}{\sqrt{n}} \sum_{t=1}^n (\zeta_t + \mu_1 \xi_{1t} + \mu_2 \nu^T \xi_{2t}) + o_p(1) \\ &\xrightarrow{\mathcal{L}} N(0, E(\zeta_t + \mu_1 \xi_{1t} + \mu_2 \nu^T \xi_{2t})^2). \end{aligned}$$

(ii). *If $\gamma_0 > 0$, then*

$$\begin{aligned} \sqrt{n}(\hat{\gamma}_n - \gamma_0) &= \frac{1}{\sqrt{n}} \sum_{t=1}^n (\zeta_t + \mu_1 \tilde{\xi}_{1t}) + o_p(1) \\ &\xrightarrow{\mathcal{L}} N(0, E(\zeta_t + \mu_1 \tilde{\xi}_{1t})^2). \end{aligned}$$

Here we provide a heuristic sketch so as to understand why its associated theory involves $f(\cdot)$ and $f'(\cdot)$ and is totally different from that in [Francq and Zakoïan \(2012\)](#). In the Appendix, we will give detailed proofs. For illustration, we consider the following toy problem. Suppose that we observe i.i.d. data $\{(X_i, Z_i)^T\}_{i=1}^n$ sampled from a bivariate normal variable $(X, Z)^T$ with mean $(0, a)^T$ and an identity covariance matrix, where a is an unknown parameter and $a \neq 0$. We aim to estimate the quantity $\gamma_1 = E \log |X + a|$, similar to our γ_0 in a DAR(1) model. Clearly, a natural estimator is

$$\hat{\gamma}_1 = \frac{1}{n} \sum_{i=1}^n \log |X_i + \hat{a}_n|,$$

where $\hat{a}_n = n^{-1} \sum_{i=1}^n Z_i$. Decompose $\hat{\gamma}_1 - \gamma_1$ as

$$\hat{\gamma}_1 - \gamma_1 = \frac{1}{n} \sum_{i=1}^n (\log |X_i + \hat{a}_n| - \log |X_i + a|) + \frac{1}{n} \sum_{i=1}^n (\log |X_i + a| - \gamma_1) := I_1 + I_2.$$

A challenge is from I_1 . Noting that $\log |x + a| = \frac{1}{2} \log(x + a)^2$, by Taylor's expansion, the

term I_1 becomes

$$I_1 = \frac{1}{n} \sum_{i=1}^n \frac{1}{X_i + a^*} (\hat{a}_n - a).$$

where a^* is some value between \hat{a}_n and a . However, for any constant $c > 0$, $n^{-1} \sum_{i=1}^n (X_i + c)^{-1}$ does not converge as $n \rightarrow \infty$. In other words, the Taylor's expansion approach fails.

To solve the mathematical problem for I_1 , a possible remedy is to apply modern empirical process theory. Denote empirical and true distributions of X by $P_n(x) = n^{-1} \sum_{i=1}^n I(X_i \leq x)$ and $P(x) = P(X \leq x)$, where $I(A)$ is an indicator function of a set A . Then $I_1 = \int_{x \in R} \log |x| d(P_n(x - \hat{a}_n) - P_n(x - a))$. Empirical process approximations enable us to obtain that

$$\sup_{x \in R, |a_1 - a| \leq \delta_n} |P_n(x - a_1) - P(x - a_1) - P_n(x - a) + P(x - a)| = o_p(n^{-1/2-\epsilon}),$$

provided by $\delta_n \downarrow 0$ at some rate with some $\epsilon > 0$, and hence, intuitively,

$$I_1 = \int_{x \in R} \log |x| d(P(x - \hat{a}_n) - P(x - a)) + o_p(n^{-1/2}). \quad (11)$$

(Of course, this assertion is not easy to prove and needs to be analyzed carefully since $\log |x|$ is unbounded on R . For instance, we added a truncation into our estimator. One reason is to prove (11) in a much easier way.) Through this approximation, I_1 reduces to

$$I_1 = - \int \log |x| f'(x - a) dx (\hat{a}_n - a) + o_p(n^{-1/2}),$$

where $f(\cdot)$ is the density of X . Define $c = - \int \log |x + a| f'(x) dx$. Then,

$$\sqrt{n}(\hat{\gamma}_1 - \gamma_1) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \{c(Z_i - a) + \log |X_i + a| - \gamma_1\} + o_p(1). \quad (12)$$

This explains why asymptotic variance of $\hat{\gamma}_1$ involves the derivative function $f'(\cdot)$.

We now discuss how to estimate the asymptotic variance of $\sqrt{n}(\hat{\gamma}_n - \gamma_0)$. Since it involves the density function $f(\cdot)$ and its derivative $f'(\cdot)$, the asymptotic expression would not be applicable directly. Similar to Section 2, we propose a random weighting approach for variance estimation. In particular, let $\varpi_1, \dots, \varpi_n$ be a sequence of i.i.d. nonnegative random variables, with both mean and variance equal to one. Define

$$L_n^*(\theta) = \sum_{t=1}^n \varpi_t \left\{ \frac{1}{2} \log(\omega + \alpha y_{t-1}^2) + \frac{|y_t - \phi y_{t-1}|}{\sqrt{\omega + \alpha y_{t-1}^2}} \right\}$$

and $\hat{\theta}_n^* = \arg \min_{\theta \in \Theta} L_n^*(\theta)$. For $t = 1, \dots, n$ and $k = 1, 2$, denote

$$\begin{aligned}\hat{\eta}_t^* &= \eta_t(\hat{\theta}_n^*), \quad \mathcal{A}_k^* = \{t : \hat{\eta}_t^* \sqrt{\hat{\alpha}_n^*} - (-1)^k \hat{\phi}_n^* \in \mathcal{I}_n\}, \\ \hat{\gamma}_{nk}^* &= \frac{1}{\sum_{t \in \mathcal{A}_k^*} \varpi_t} \sum_{t \in \mathcal{A}_k^*} \varpi_t \log |\hat{\eta}_t^* \sqrt{\hat{\alpha}_n^*} - (-1)^k \hat{\phi}_n^*|.\end{aligned}\tag{13}$$

The final resampling estimator of $\hat{\gamma}_n$ is defined as

$$\hat{\gamma}_n^* = (\hat{\gamma}_{n1}^* + \hat{\gamma}_{n2}^*)/2.$$

To construct the resampling estimator, we insert the random weights into two locations in (13). They correspond to two different parts in (12). The following theorem shows that the distribution of $\sqrt{n}(\hat{\gamma}_n - \gamma_0)$ can be approximated by the resampling distribution of $\sqrt{n}(\hat{\gamma}_n^* - \hat{\gamma}_n)$.

Theorem 3.2. *Suppose that Assumptions 1–4 hold and $\gamma_0 \neq 0$. Then, conditionally on the data $\{y_0, y_1, \dots, y_n\}$, in probability, $\sqrt{n}(\hat{\gamma}_n^* - \hat{\gamma}_n)$ is asymptotically normal as $n \rightarrow \infty$, and its asymptotic variance is the same as that of $\sqrt{n}(\hat{\gamma}_n - \gamma_0)$.*

In applications, the asymptotic variance of $\sqrt{n}(\hat{\gamma}_n - \gamma_0)$ can be estimated by using random weighting as follows. First, i.i.d. $\{\varpi_1, \dots, \varpi_n\}$ are generated B times from standard exponential distribution, where B is a large number. Each time, minimize $L_n^*(\theta)$ to obtain $\hat{\theta}_n^*$ and then $\hat{\gamma}_n^*$ is accordingly computed. Denote them as $\hat{\gamma}_n^{*1}, \dots, \hat{\gamma}_n^{*B}$. The asymptotic variance σ_γ^2 of $\sqrt{n}(\hat{\gamma}_n - \gamma_0)$ is approximated by

$$\hat{\sigma}_\gamma^2 = \frac{1}{B-1} \sum_{b=1}^B (\hat{\gamma}_n^{*b} - \bar{\gamma}_n^*)^2 \quad \text{with} \quad \bar{\gamma}_n^* = \frac{1}{B} \sum_{b=1}^B \hat{\gamma}_n^{*b}.$$

Once variance estimation is given, confidence intervals can be constructed, as stated in the following corollary.

Corollary 3.1. *Under the conditions in Theorem 3.1 and $B = O(n)$, $\hat{\sigma}_\gamma^2 \rightarrow \sigma_\gamma^2$ in probability as $n \rightarrow \infty$. Therefore, at the significance level $\underline{\alpha} \in (0, 1)$, a confidence interval for γ_0 is*

$$\left[\hat{\gamma}_n - \frac{\hat{\sigma}_\gamma}{\sqrt{n}} \Phi^{-1} \left(1 - \frac{\underline{\alpha}}{2} \right), \hat{\gamma}_n + \frac{\hat{\sigma}_\gamma}{\sqrt{n}} \Phi^{-1} \left(1 - \frac{\underline{\alpha}}{2} \right) \right],$$

where $\Phi(\cdot)$ is the standard normal distribution.

3.2 Strict Stationarity Testing

Consider the strict stationarity testing

$$H_0 : \gamma_0 < 0 \quad \text{v.s.} \quad H_1 : \gamma_0 \geq 0, \quad (14)$$

and

$$H_0 : \gamma_0 > 0 \quad \text{v.s.} \quad H_1 : \gamma_0 \leq 0. \quad (15)$$

With the aid of estimation of γ_0 and its asymptotics, the above tests are tractable. We give asymptotic critical regions for both testing problems as follows.

Theorem 3.3. *Suppose that Assumptions 1–4 hold. Let*

$$T_n = \sqrt{n} \frac{\hat{\gamma}_n}{\hat{\sigma}_\gamma} \quad (16)$$

be the test statistics for (14) and (15), where $\hat{\sigma}_\gamma$ is estimated with the resampling size $B = O(n)$.

(i). *For the test (14), the test defined by the stationary (ST) critical region*

$$C^{\text{ST}} = \{T_n > \Phi^{-1}(1 - \underline{\alpha})\}$$

has its asymptotic significance level $\underline{\alpha}$ and is consistent for all $\gamma_0 > 0$.

(ii). *For the test (15), the test defined by the nonstationary (NS) critical region*

$$C^{\text{NS}} = \{T_n < \Phi^{-1}(\underline{\alpha})\}$$

has its asymptotic significance level $\underline{\alpha}$ and is consistent for all $\gamma_0 < 0$.

Here we use the normal approximation to construct critical regions for (14) and (15). An alternative approach is to use the resampling distribution of $\hat{\gamma}_n$. If such an approach is applied, the resampling size would need to be larger.

4 Numerical Studies

4.1 Simulation Studies

In this subsection, we conduct numerical studies to assess the finite-sample performance of our GLADE and strict stationarity tests. In particular, we compare our proposed estimation method with the QMLE studied in [Ling \(2004\)](#) and [Chen, Li and Ling \(2014\)](#).

First, we are interested in the performance of the estimators $(\hat{\phi}_n, \hat{\alpha}_n)$ and $\hat{\gamma}_n$ in finite samples. We generate the innovations $\{\eta_t\}$ in the following three scenarios: (a) *the normal distribution* $N(0, \pi/2)$; (b) *the Laplace distribution with density* $f(x) = 0.5 \exp(-|x|)$; and (c) *the standardized Student's t_3 (st_3) distribution with density* $f(x) = 4\pi^2/(\pi^2 + 4x^2)^2$. The true parameters are set to be $(\phi_0, \alpha_0, \omega_0) = (0.7, 0.4, 0.5)$ and $(1.0, 3.0, 0.5)$, corresponding to the stationary and explosive cases, respectively. The sample size n is 200 and 400, and 1000 replications are used for each configuration. We take the resampling size $B = 500$ for variance estimation for each configuration. For all cases, the true top Lyapunov exponents have been calculated in Table 1.

Table 1: Summary statistics for our proposed procedure under various scenarios based on 1000 replications

η	Parameters	$n = 200$				$n = 400$			
		Bias	SE	SEE	CP	Bias	SE	SEE	CP
Normal	$\phi_0 = 0.700$	-0.003	0.102	0.106	0.944	-0.004	0.072	0.074	0.948
	$\alpha_0 = 0.400$	-0.014	0.085	0.082	0.925	-0.007	0.056	0.058	0.951
	$\omega_0 = 0.500$	0.010	0.103	0.097	0.927	0.006	0.068	0.067	0.948
	$\gamma_0 = -0.523$	-0.005	0.109	0.113	0.958	-0.005	0.074	0.076	0.950
Laplace	$\phi_0 = 0.700$	-0.002	0.068	0.073	0.963	-0.001	0.048	0.049	0.954
	$\alpha_0 = 0.400$	-0.009	0.107	0.102	0.915	-0.005	0.072	0.072	0.941
	$\omega_0 = 0.500$	0.019	0.135	0.129	0.930	0.013	0.092	0.090	0.947
	$\gamma_0 = -0.440$	-0.005	0.086	0.093	0.969	-0.002	0.060	0.062	0.960
st_3	$\phi_0 = 0.700$	-0.002	0.082	0.085	0.943	0.001	0.057	0.058	0.940
	$\alpha_0 = 0.400$	-0.008	0.140	0.124	0.895	-0.009	0.087	0.084	0.903
	$\omega_0 = 0.500$	0.012	0.147	0.139	0.906	0.016	0.111	0.103	0.919
	$\gamma_0 = -0.473$	-0.003	0.106	0.110	0.952	-0.001	0.071	0.075	0.954
Normal	$\phi_0 = 1.000$	0.011	0.193	0.200	0.939	0.003	0.139	0.139	0.942
	$\alpha_0 = 3.000$	-0.009	0.338	0.326	0.932	-0.006	0.223	0.228	0.952
	$\omega_0 = 0.500$	0.586	4.352	8.903	0.895	0.537	1.962	4.050	0.885
	$\gamma_0 = 0.242$	0.004	0.068	0.070	0.951	0.003	0.045	0.048	0.953
Laplace	$\phi_0 = 1.000$	0.002	0.128	0.137	0.953	-0.001	0.090	0.094	0.959
	$\alpha_0 = 3.000$	-0.001	0.438	0.430	0.923	-0.002	0.292	0.302	0.946
	$\omega_0 = 0.500$	1.401	7.593	80.96	0.863	2.207	25.28	28.69	0.865
	$\gamma_0 = 0.227$	0.001	0.073	0.075	0.948	-0.001	0.049	0.052	0.958
st_3	$\phi_0 = 1.000$	-0.006	0.158	0.159	0.942	0.001	0.109	0.110	0.949
	$\alpha_0 = 3.000$	0.022	0.536	0.521	0.921	0.001	0.375	0.356	0.926
	$\omega_0 = 0.500$	2.004	15.60	252.8	0.859	8.122	180.7	68.95	0.845
	$\gamma_0 = 0.183$	0.005	0.075	0.079	0.954	0.001	0.054	0.055	0.947

Note: Bias and SE are the finite sample bias and standard error of the parameter estimator, SEE means the mean of standard error estimator, and CP is the empirical coverage probabilities of the 95% confidence intervals.

In Table 1, we report the finite-sample biases (Bias), the standard errors (SE), the sample mean of the standard error estimators (SEE) and the empirical coverage probabilities (CP) of the 95% confidence intervals via normal approximation. Except for $\hat{\omega}_n$ in nonstationary cases, we observe that biases of all other estimators are very small, the variance estimators accurately reflect the true variations and the coverage probabilities agree with the nominal level 95% for almost all cases. These findings confirm that the stationarity assumption is not necessary for estimation of these parameters. They also illustrate that the resampling

approach works well for variance estimation. We would like to point out that in some cases, the coverage rates of α_0 is slightly lower than the nominal level 95%. One possible reason is that $\sqrt{n}(\hat{\alpha}_n - \alpha_0)$ is skewed relative to the normal distribution, especially for small positive α_0 , since all $\hat{\alpha}_n$ are positive. The basic idea for improvement is to incorporate the skewness information into confidence interval construction. This may be done, for example, by better confidence intervals proposed by Efron (1987). From Table 1, we can also see that $\hat{\omega}_n$ is not estimable in nonstationary cases.

Next, we evaluate the finite-sample performance of our estimators $(\hat{\phi}_n, \hat{\alpha}_n)$ by comparing with the QMLE. We consider $n = 200$ and generate 1000 replications for each configuration. Since both estimation procedures require different conditions on η for identification in model (1), it is not a good idea to compare them directly. To make fair comparison, we use the average absolute errors (AAE), defined as

$$\text{AAE} = \frac{1}{2} \left(\left| \hat{\phi} - \phi_0 \right| + \left| \frac{\hat{\alpha}}{\alpha_0} - 1 \right| \right),$$

where $(\hat{\phi}, \hat{\alpha})$ are the GLADE or QMLE of (ϕ_0, α_0) , respectively. Unlike Zhu and Ling (2013), the advantage of this AAE is that it is not necessary to rescale the estimators for both procedures.

Fig. 2 shows the box-plots of the AAE for the GLADE and QMLE. It is observed

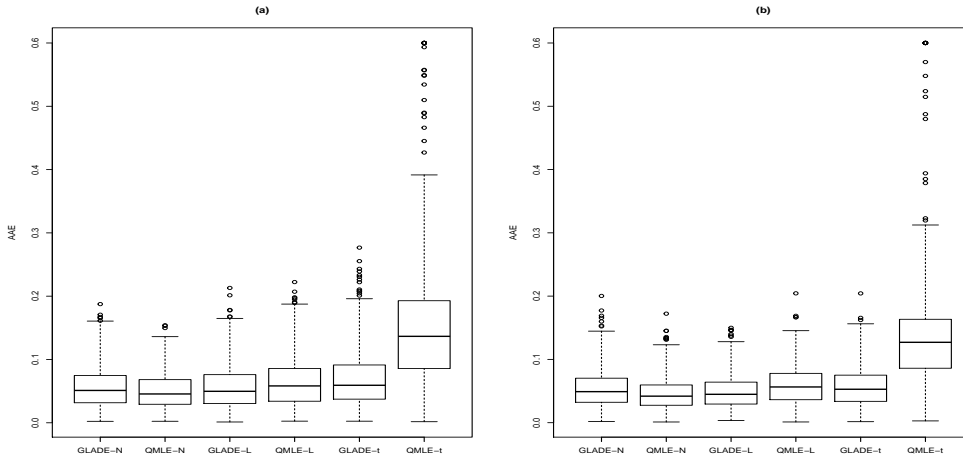


Figure 2: Boxplots of the AAE for the GLADE and the QMLE based on 1000 replications. (a) $(\phi_0, \alpha_0, \omega_0) = (0.7, 0.4, 0.5)$ and (b) $(\phi_0, \alpha_0, \omega_0) = (1.0, 3.0, 0.5)$. ‘GLADE-N’, ‘GLADE-L’ and ‘GLADE-t’ mean ‘GLADE’s when errors are normal, Laplace and st_3 , respectively. ‘QMLE-N’, ‘QMLE-L’ and ‘QMLE-t’ are defined similarly.

from Fig. 2 that the GLADE outperforms the QMLE when the innovation η follows st_3

and Laplace distribution, thereby supporting the robust properties of our GLADE, either stationary or explosive. In particular, the QMLE performs much worse when errors have infinite fourth moments. We also observe that the QMLE with normal errors behaves better. It is not surprising since in this case, the QMLE is efficient, while the GLADE is not.

Finally, we illustrate the performance of strict stationarity tests developed in Section 3. We keep the standardized Student's t_3 distribution for η_t , $\omega_0 = 0.5$ but ϕ_0 varies from 0.6 to 1.3 and $\alpha_0 = 2\phi_0$. In this scenario, we have $\gamma_0 = 0$ for $(\phi_0, \alpha_0) = (0.922, 1.844)$ and hence, $\gamma_0 > 0$ or < 0 if $\phi_0 > 0.922$ or < 0.922 , respectively. We take the sample size $n = 200, 400, 800$, and 1000 replications for each configuration.

Tables 2 and 3 summarize empirical frequencies of rejection for (14) and (15) for various values of ϕ_0 , respectively. It is observed that when $\gamma_0 = 0$, the rejection frequencies of the two tests agree with the nominal level 5% as n increases. As expected, the frequency of rejection of the C^{ST} test increases with γ_0 , while that of the C^{NS} test decreases. Overall, the power of the two tests is significant, as shown in theory.

Table 2: Relative frequency of rejection of the test (14): $H_0 : \gamma_0 < 0$ with $(\omega_0, \alpha_0) = (0.5, 2\phi_0)$ based on 1000 replications.

	ϕ_0						
	0.600	0.700	0.800	0.922	1.000	1.100	1.300
$n = 200$	0.000	0.000	0.001	0.053	0.173	0.528	0.943
$n = 400$	0.000	0.000	0.000	0.062	0.252	0.754	1.000
$n = 800$	0.000	0.000	0.000	0.057	0.452	0.957	1.000

Table 3: Relative frequency of rejection of the test (15): $H_0 : \gamma_0 > 0$ with $(\omega_0, \alpha_0) = (0.5, 2\phi_0)$ based on 1000 replications.

	ϕ_0						
	0.600	0.700	0.800	0.922	1.000	1.100	1.300
$n = 200$	0.954	0.713	0.313	0.038	0.009	0.000	0.000
$n = 400$	0.996	0.933	0.501	0.048	0.003	0.000	0.000
$n = 800$	1.000	1.000	0.796	0.054	0.002	0.000	0.000

4.2 An Empirical Study

To illustrate the use of our proposed inference procedure, we consider a data set of the monthly 3-Month London Interbank Offered Rate (LIBOR) $\{y_t\}$ (based on Japanese Yen) over the period from January 1986 to September 2016. The observed level series of LIBOR is plotted in Fig. 3. Here we aim to check whether this level series over the given period

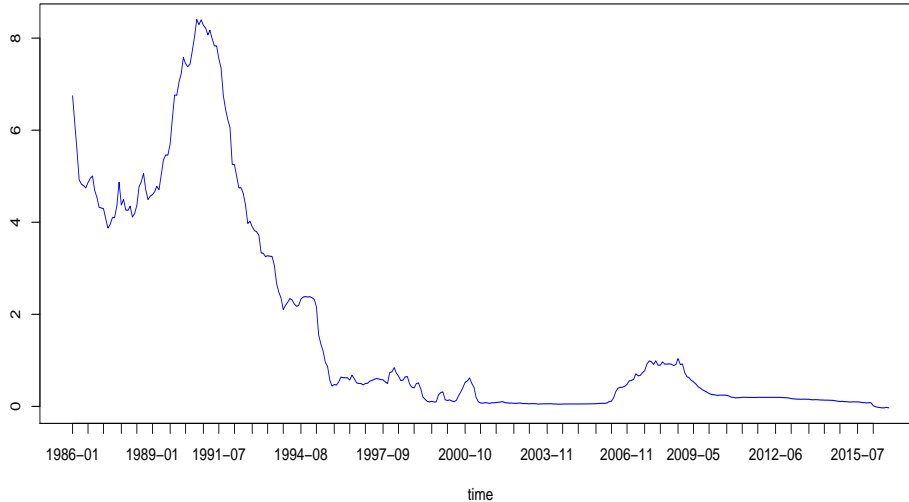


Figure 3: The monthly 3-Month LIBOR (in percent), based on Japanese Yen, from Jan 1986 to Sep 2016.

is strictly stationary by fitting a DAR(1) model. If the significance level is set to be 5%, then the stationarity hypothesis of test (14) can be tested by comparing the t -type statistic T_n with 1.64 and rejecting the null hypothesis if $T_n > 1.64$. To calculate T_n , we set the resampling size to be 1000 for variance estimation of $\hat{\gamma}_n$.

We fit the level series directly instead of its log-return series since this data set contains negative observations around year 2016. Table 4 reports estimates of θ , the top Lyapunov exponent γ and the value of test statistics T_n for (14).

Table 4: Parameter estimates and test statistic T_n of the stationarity test in (14).

Index	$\hat{\phi}$	$\hat{\alpha}$	$\hat{\omega}$	$\hat{\gamma}$	T_n
LIBOR	0.9940	0.0086	1.5454×10^{-5}	-0.0237	-4.7760
	(0.0036)	0.0026	(0.7976×10^{-5})	(0.0050)	(1.0000)

Note: the values in the first four parentheses are the standard deviations of parameter estimates and the value in the last parenthesis is the p -value of T_n .

From Fig. 3, the level LIBOR series seems to be nonstationary intuitively. Meanwhile, the value of $\hat{\phi}$ in Table 4 is insignificant from 1. However, we should do the judgement with more caution in the DAR model since a unit root in the conditional mean does not necessarily imply nonstationarity as lagged values of the process enter into the conditional variance. As a matter of fact, the test statistic T_n is -3.4209 , which is less than 1.64,

meaning that we do not have significant evidence to reject the null hypothesis of strict stationarity (p -value=0.999). This further demonstrates that the series constructed from the DAR(1) model is allowed to be persistent but remain strictly stationary, see [Nielsen and Rahbek \(2014\)](#).

The final estimated model is presented as following:

$$y_t = 0.994y_{t-1} + \eta_t \sqrt{1.5454 \times 10^{-5} + 0.0086y_{t-1}^2}, \quad t = 2, \dots, 369. \quad (17)$$

The standardized residuals and conditional variance from model (17) are plotted in Fig. 4.

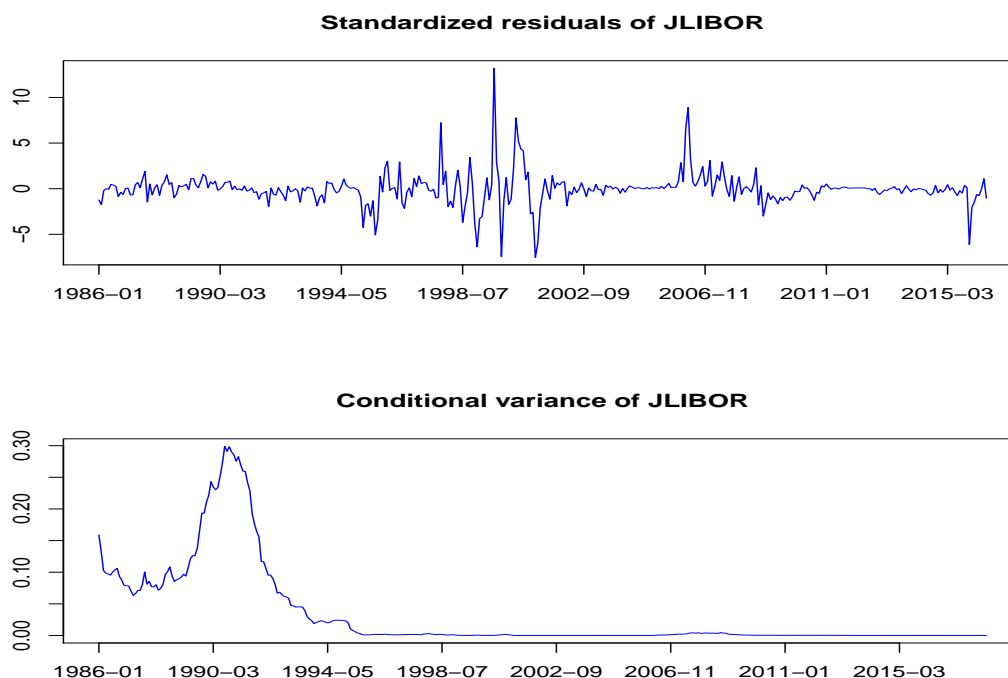


Figure 4: The standardized residuals and conditional variance from model (17).

To assess the performance of the fitting DAR(1) model, we simulate a sample path from the estimated DAR(1) model (17) with an initial value y_1 , where $\{\eta_t\}$ is i.i.d. Student's t_3 . Here, the Student's t distribution is chosen based on the histogram of the residuals in Fig. 5 and the degree of freedom 3 is approximately obtained from the standard deviation (i.e., 1.932) of the residuals. Clearly, the model (17) is strictly stationary according to the stationarity region, see, e.g., [Borkovec and Klüppelberg \(2001\)](#) and [Chen, Li and Ling \(2014\)](#). A simulated path is plotted in Fig. 6, from which we can see that the simulated

series behaves like that of the true LIBOR level series. This further provides an evidence for our findings.

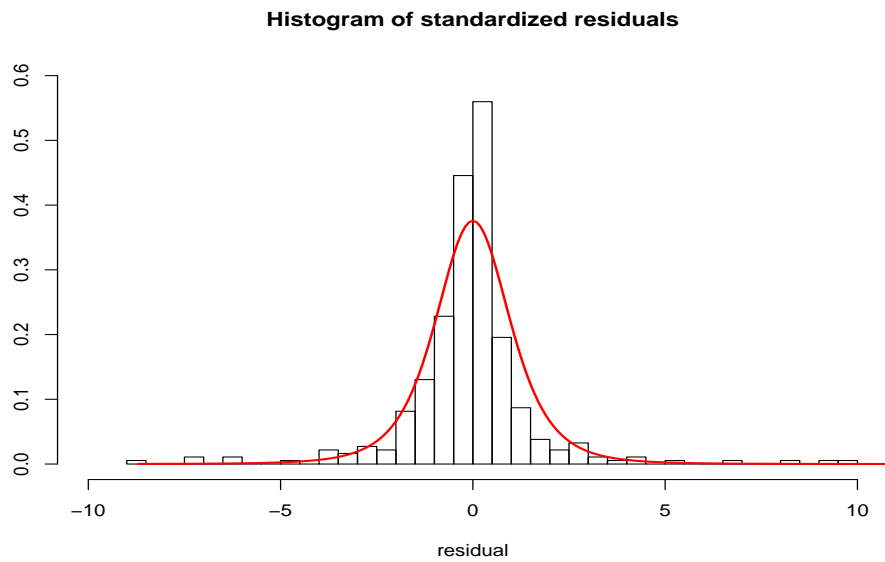


Figure 5: The histogram of the residuals $\{\hat{\eta}_t\}$ from model (17).

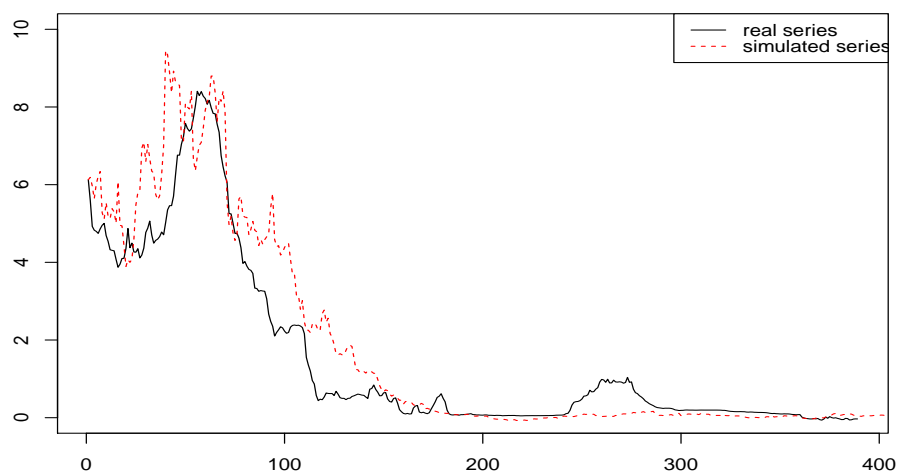


Figure 6: The simulated series with $\hat{\phi}$, $\hat{\alpha}$ and $\hat{\omega}$ and t_3 distributed innovations.

5 Conclusion

Testing strict stationarity is important in the context of nonlinear time series analysis. This paper addresses the strict stationarity testing on DAR models and develops a unified framework for the inference of both stationary and explosive DAR(1) processes. We also propose a global LADE of (ϕ_0, α_0) in the DAR(1) model and establish its asymptotic theory without strict stationarity assumption. If one is interested in the inference on ϕ_0 or α_0 , strict stationarity testing is unnecessary.

It is worth noting that one of assumptions is η is symmetric, which is met by most commonly used errors. When this assumption is violated, the statistical inference of DAR models is absent since the dynamic behavior of $\{y_t\}$ is unknown in the nonstationary case. We leave it to future studies.

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Appendix

A Proofs of Theorems

Denote by M any positive constant whose value is unimportant and can be different throughout the proofs. Recall the fact that

$$|y_t|/\rho^t \rightarrow \infty \quad \text{a.s. as } t \rightarrow \infty \quad (18)$$

for any $\rho \in (1, e^{\gamma_0})$ by Theorem 1 in [Chen, Li and Ling \(2014\)](#).

A.1 Proof of Theorem 2.1

The result stated in (i) is standard, see [Zhu and Ling \(2013\)](#). Consider the case (ii). Clearly, $(\hat{\phi}_n, \hat{\alpha}_n, \hat{\omega}_n)^T = \arg \min_{\theta \in \Theta} Q_n(\theta)$, where $Q_n(\theta) = \{L_n(\theta) - L_n(\theta_0)\}/n$. We have

$$Q_n(\theta) = O_n(\phi, \alpha) + R_{1n}(\theta) + R_{2n}(\theta),$$

where

$$O_n(\phi, \alpha) = \frac{1}{n} \sum_{t=1}^n \left\{ \frac{1}{2} \log \frac{\alpha}{\alpha_0} + \frac{|\eta_t \text{sign}(y_{t-1}) \sqrt{\alpha_0} - (\phi - \phi_0)|}{\sqrt{\alpha}} - |\eta_t| \right\}$$

and

$$R_{1n}(\theta) = \frac{1}{2n} \sum_{t=1}^n \log \frac{\alpha_0(\omega + \alpha y_{t-1}^2)}{\alpha(\omega_0 + \alpha_0 y_{t-1}^2)},$$

$$R_{2n}(\theta) = \frac{1}{2n} \sum_{t=1}^n \left\{ \frac{|\eta_t \sqrt{\omega_0 + \alpha_0 y_{t-1}^2} - (\phi - \phi_0) y_{t-1}|}{\sqrt{\omega + \alpha y_{t-1}^2}} - \frac{|\eta_t \text{sign}(y_{t-1}) \sqrt{\alpha_0} - (\phi - \phi_0)|}{\sqrt{\alpha}} \right\}.$$

Note that $(\eta_1 \text{sign}(y_0), \dots, \eta_n \text{sign}(y_{n-1}))$ and (η_1, \dots, η_n) have the same distribution. Thus,

$$\begin{aligned} \lim_{n \rightarrow \infty} O_n(\phi, \alpha) &= \frac{1}{2} \log \frac{\alpha}{\alpha_0} + \frac{E|\eta_t \sqrt{\alpha_0} - (\phi - \phi_0)|}{\sqrt{\alpha}} - 1 \quad \text{a.s.} \\ &\geq \frac{1}{2} \log(\alpha/\alpha_0) + \sqrt{\alpha_0/\alpha} - 1 \geq 0 \end{aligned}$$

because $E|\eta_t - c| \geq E|\eta_t|$ for any $c \in R$ and the inequality $\log x \leq x - 1$ for $x > 0$. The equality holds if and only if $\phi = \phi_0$ and $\alpha = \alpha_0$.

For $R_{1n}(\theta)$, by the mean value theorem, the compactness of Θ and (18), we have

$$\lim_{n \rightarrow \infty} \sup_{\theta \in \Theta} |R_{1n}(\theta)| \leq M \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \frac{1}{1 + y_{t-1}^2} \rightarrow 0 \quad \text{a.s.}$$

For $R_{2n}(\theta)$, by the compactness of Θ and (18), a simple calculus yields

$$\lim_{n \rightarrow \infty} \sup_{\theta \in \Theta} |R_{2n}(\theta)| \leq M \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \frac{|\eta_t|}{|y_{t-1}|} \rightarrow 0 \quad \text{a.s.}$$

Thus, the proof can be completed by standard arguments, invoking the compactness of Θ .

A.2 Proof of Theorem 2.2

(i). When $\gamma < 0$, $\{y_t\}$ is strictly stationary and ergodic, and there exists some $\iota > 0$ such that $E|y_t|^\iota < \infty$, see [Borkovec and Klüppelberg \(2001\)](#). Thus, Assumption 2 in [Zhu and Ling \(2013\)](#) is satisfied and in turn the result holds.

(ii). We first reparameterize the objective function as

$$H_n(u) = L_n(\theta_0 + u) - L_n(\theta_0), \quad (19)$$

where $u \in \Lambda := \{u = (u_1, u_2, u_3)^T : u + \theta_0 \in \Theta\}$.

Let $\hat{u}_n = \hat{\theta}_n - \theta_0 := (\hat{u}_{1n}, \hat{u}_{2n}, \hat{u}_{3n})^T = (\hat{v}_n^T, \hat{u}_{3n})^T$. We can see that \hat{u}_n is the minimizer of $H_n(u)$ in Λ . By Theorem 2.1(ii), $\hat{v}_n = o_p(1)$. Using the similar arguments as in [Zhu and Ling \(2011, 2013\)](#), by the fact (18), we have

$$H_n(\hat{u}_n) = (\sqrt{n}\hat{v}_n)^T T_n + (\sqrt{n}\hat{v}_n)^T \Omega (\sqrt{n}\hat{v}_n) + o_p(\sqrt{n}\|\hat{v}_n\| + n\|\hat{v}_n\|^2),$$

where

$$T_n = -\frac{1}{\sqrt{n}} \sum_{t=1}^n \left(\frac{\text{sign}(\eta_t)}{\sqrt{\alpha_0}}, \frac{|\eta_t| - 1}{2\alpha_0} \right)^T \text{ and } \Omega = \text{diag}(f(0)/\alpha_0, 1/(8\alpha_0^2)).$$

By the central limit theorem, we have $T_n \xrightarrow{\mathcal{L}} N(0, \text{diag}(1/\alpha_0, (\kappa_\eta - 1)/(4\alpha_0^2)))$. Let $\lambda_{\min} = \min\{f(0)/\alpha_0, 1/(8\alpha_0^2)\} > 0$. Then

$$H_n(\hat{u}_n) \geq -\|\sqrt{n}\hat{v}_n\| \{\|T_n\| + o_p(1)\} + \|\sqrt{n}\hat{v}_n\|^2 \{\lambda_{\min} + o_p(1)\}.$$

Note that $H_n(\hat{u}_n) \leq 0$ by the definition of $\hat{\theta}_n$. Thus,

$$\|\sqrt{n}\hat{v}_n\| \leq \{\lambda_{\min} + o_p(1)\}^{-1} \{\|T_n\| + o_p(1)\} = O_p(1).$$

Next, let $v_n^* = -\Omega^{-1}T_n/(2\sqrt{n})$. Then,

$$\sqrt{n}v_n^* = -\Omega^{-1}T_n/2 \xrightarrow{\mathcal{L}} N(0, \mathcal{J}_N).$$

Using the previous facts, a simple calculus gives that

$$\begin{aligned} H_n(\hat{u}_n) - H_n(u_n^*) &= (\sqrt{n}\hat{v}_n - \sqrt{n}v_n^*)^T \Omega (\sqrt{n}\hat{v}_n - \sqrt{n}v_n^*) + o_p(1) \\ &\geq \lambda_{\min} \|\sqrt{n}\hat{v}_n - \sqrt{n}v_n^*\|^2 + o_p(1), \end{aligned}$$

where $u_n^* = (v_n^{*T}, 0)^T$. Note that $H_n(\hat{u}_n) - H_n(u_n^*) = L_n(\theta_0 + \hat{u}_n) - L_n(\theta_0 + u_n^*) \leq 0$ a.s. Thus, we have $\|\sqrt{n}\hat{v}_n - \sqrt{n}v_n^*\| = o_p(1)$.

Finally, we have $\sqrt{n}(\hat{\phi}_n - \phi_0, \hat{\alpha}_n - \alpha_0)^T = \sqrt{n}\hat{v}_n = \sqrt{n}v_n^* + o_p(1)$ and then (ii) holds.

A.3 Proof of Theorem 2.3

We first prove (ii). Let P^* be the joint probability of $(\varpi_1, \dots, \varpi_n)$ and (y_1, \dots, y_n) . Replacing $L_n(\theta)$ in (19) by $L_n^*(\theta)$ and repeating the proof of Theorem 2.2, we have that

$$\begin{aligned}\sqrt{n}(\hat{\phi}_n^* - \phi_0) &= \frac{\sqrt{\alpha_0}}{2f(0)} \frac{1}{\sqrt{n}} \sum_{t=1}^n \varpi_t \text{sign}(\eta_t) + o_{P^*}(1), \\ \sqrt{n}(\hat{\alpha}_n^* - \alpha_0) &= \frac{2\alpha_0}{\sqrt{n}} \sum_{t=1}^n \varpi_t \{|\eta_t| - 1\} + o_{P^*}(1).\end{aligned}$$

Note that $(\hat{\phi}_n^* - \hat{\phi}_n) = (\hat{\phi}_n^* - \phi_0) - (\hat{\phi}_n - \phi_0)$ and so does for $(\hat{\alpha}_n^* - \hat{\alpha}_n)$. Hence, together with asymptotic representations of $\sqrt{n}(\hat{\phi}_n - \phi_0)$ and $\sqrt{n}(\hat{\alpha}_n - \alpha_0)$ in the proof of Theorem 2.2, we immediately obtain that

$$\begin{aligned}\sqrt{n}(\hat{\phi}_n^* - \hat{\phi}_n) &= \frac{\sqrt{\alpha_0}}{2f(0)} \frac{1}{\sqrt{n}} \sum_{t=1}^n (\varpi_t - 1) \text{sign}(\eta_t) + o_{P^*}(1), \\ \sqrt{n}(\hat{\alpha}_n^* - \hat{\alpha}_n) &= \frac{2\alpha_0}{\sqrt{n}} \sum_{t=1}^n (\varpi_t - 1) \{|\eta_t| - 1\} + o_{P^*}(1),\end{aligned}$$

and therefore, conditional on the data $\{y_0, y_1, \dots, y_n\}$, the part (ii) follows. The proof of (i) is proved analogously.

A.4 Proof of Theorem 3.1

First, we decompose $\hat{\phi}_n + \hat{\eta}_t \sqrt{\hat{\alpha}_n}$ for each $t = 1, \dots, n$. In fact, $\hat{\phi}_n + \hat{\eta}_t \sqrt{\hat{\alpha}_n}$ is rewritten as

$$\begin{aligned}\hat{\phi}_n + \hat{\eta}_t \sqrt{\hat{\alpha}_n} &= \hat{\phi}_n + \frac{y_t - \hat{\phi}_n y_{t-1}}{\sqrt{\hat{\omega}_n + \hat{\alpha}_n y_{t-1}^2}} \sqrt{\hat{\alpha}_n} \\ &= \hat{\phi}_n + \frac{(\phi_0 - \hat{\phi}_n) y_{t-1}}{\sqrt{\hat{\omega}_n + \hat{\alpha}_n y_{t-1}^2}} \sqrt{\hat{\alpha}_n} + \sqrt{\frac{\omega_0 + \alpha_0 y_{t-1}^2}{\hat{\omega}_n + \hat{\alpha}_n y_{t-1}^2}} \eta_t \sqrt{\hat{\alpha}_n} \\ &= \phi_0 + \eta_t \sqrt{\alpha_0} + v_t^{(1)}(\hat{\theta}_n) + \eta_t u_t(\hat{\theta}_n).\end{aligned}$$

where $\hat{\theta}_n = (\hat{\phi}_n, \hat{\alpha}_n, \hat{\omega}_n)^T$,

$$v_t^{(1)}(\theta) = (\phi - \phi_0) \left(1 - \frac{y_{t-1} \sqrt{\alpha}}{\sqrt{\omega + \alpha y_{t-1}^2}}\right), \quad u_t(\theta) = \sqrt{\frac{\omega_0 + \alpha_0 y_{t-1}^2}{\omega + \alpha y_{t-1}^2}} \sqrt{\alpha} - \sqrt{\alpha_0}.$$

Similarly,

$$\hat{\eta}_t \sqrt{\hat{\alpha}_n} - \hat{\phi}_n = \eta_t \sqrt{\alpha_0} - \phi_0 + v_t^{(2)}(\hat{\theta}_n) + \eta_t u_t(\hat{\theta}_n),$$

where $v_t^{(2)}(\theta) = -(\phi - \phi_0) \left(1 + \frac{y_{t-1} \sqrt{\alpha}}{\sqrt{\omega + \alpha y_{t-1}^2}}\right)$.

(I) Consider the stationary case, i.e. $\gamma_0 < 0$. Let $v_{nt}^{(k)}(\vartheta) = n^{1/2} v_t^{(k)}(\theta_0 + n^{-1/2} \vartheta)$, $k = 1, 2$, and $u_{nt}(\vartheta) = n^{1/2} u_t(\theta_0 + n^{-1/2} \vartheta)$. For $k = 1, 2$, define

$$\begin{aligned} H_{nk}(x; \vartheta) &= \frac{1}{n} \sum_{t=1}^n I \left(\eta_t \leq \frac{x + (-1)^k \phi_0 - n^{-1/2} v_{nt}^{(k)}(\vartheta)}{\sqrt{\alpha_0} + n^{-1/2} u_{nt}(\vartheta)} \right), \\ H_k(x; \vartheta) &= \frac{1}{n} \sum_{t=1}^n F \left(\frac{x + (-1)^k \phi_0 - n^{-1/2} v_{nt}^{(k)}(\vartheta)}{\sqrt{\alpha_0} + n^{-1/2} u_{nt}(\vartheta)} \right). \end{aligned}$$

If we denote $\hat{\vartheta}_n = n^{1/2}(\hat{\theta}_n - \theta_0)$ and $\mathcal{I}_n = [-n^2, -n^{-2}] \cup [n^{-2}, n^2]$, then

$$2\hat{\gamma}_n = \int_{\mathcal{I}_n} \log |x| dH_{n1}(x; \hat{\vartheta}_n) + \int_{\mathcal{I}_n} \log |x| dH_{n2}(x; \hat{\vartheta}_n) = \hat{\gamma}_{n1} + \hat{\gamma}_{n2}.$$

We decompose $\hat{\gamma}_{nk} - \gamma_0$ into

$$\begin{aligned} \hat{\gamma}_{nk} - \gamma_0 &= \int_{\mathcal{I}_n} \log |x| d(H_{nk}(x; \hat{\vartheta}_n) - H_{nk}(x; 0) - H_k(x; \hat{\vartheta}_n) + H_k(x; 0)) \\ &\quad + \int_{\mathcal{I}_n} \log |x| d(H_{nk}(x; 0) - H_k(x; 0)) + \int_{R/\mathcal{I}_n} \log |x| dH_k(x; 0) \\ &\quad + \int_{\mathcal{I}_n} \log |x| d(H_k(x; \hat{\vartheta}_n) - H_k(x; 0)) \\ &= I_{n1}^{(k)} + I_{n2}^{(k)} + I_{n3}^{(k)} + I_{n4}^{(k)}. \end{aligned}$$

In the following, we will show that

- (a) $\sqrt{n} I_{n1}^{(k)} = o_p(1)$, $k = 1, 2$;
- (b) $\sqrt{n} (I_{n2}^{(1)} + I_{n2}^{(2)}) = \frac{1}{\sqrt{n}} \sum_{t=1}^n (\zeta_{1t} + \zeta_{2t}) + o_p(1)$;
- (c) $\sqrt{n} I_{n3}^{(k)} = o_p(1)$, $k = 1, 2$;
- (d) $\sqrt{n} (I_{n4}^{(1)} + I_{n4}^{(2)}) = \frac{1}{\sqrt{n}} \sum_{t=1}^n (2\mu_1 \xi_{1t} + 2\mu_2 \nu^T \xi_{2t}) + o_p(1)$.

Suppose that (a)-(d) are proved. It immediately follows that

$$\sqrt{n}(\hat{\gamma}_n - \gamma_0) = \frac{1}{\sqrt{n}} \sum_{t=1}^n (\zeta_t + \mu_1 \xi_{1t} + \mu_2 \nu^T \xi_{2t}) + o_p(1),$$

and hence $\sqrt{n}(\hat{\gamma}_n - \gamma_0)$ is asymptotically normal.

(a) Consider the term $I_{n1}^{(k)}$ ($k = 1, 2$). It follows from Lemma B.1 below that, for any fixed $0 < b < \infty$, there exists a positive constant $\epsilon > 0$ such that, for $k = 1, 2$,

$$\sup_{x \in R, \|\vartheta\| \leq b} \sqrt{n} |H_{nk}(x; \vartheta) - H_{nk}(x; 0) - H_k(x; \vartheta) + H_k(x; 0)| = O_p(n^{-\epsilon}),$$

and therefore

$$\sup_{x \in R} |H_{nk}(x; \hat{\vartheta}_n) - H_{nk}(x; 0) - H_k(x; \hat{\vartheta}_n) + H_k(x; 0)| = O_p(n^{-1/2-\epsilon}).$$

Note that, through integration by parts, for a function $G(x)$,

$$\int_{n^{-2}}^{n^2} \log |x| dG(x) = \log(x)G(x)|_{n^{-2}}^{n^2} - \int_{n^{-2}}^{n^2} G(x) d \log(x).$$

Hence,

$$\left| \int_{n^{-2}}^{n^2} \log |x| dG(x) \right| \leq 8 \log(n) \sup_{x \in [n^{-2}, n^2]} |G(x)|.$$

Similarly, $\left| \int_{-n^2}^{-n^{-2}} \log |x| dG(x) \right| \leq 8(\log n) \sup_{x \in [-n^2, -n^{-2}]} |G(x)|$. Applying them to $|I_{n1}^{(k)}|$, we immediately have that, for $k = 1, 2$,

$$\begin{aligned} |I_{n1}^{(k)}| &\leq 16 \log(n) \sup_{x \in R} |H_{nk}(x; \hat{\vartheta}_n) - H_{nk}(x; 0) - H_k(x; \hat{\vartheta}_n) + H_k(x; 0)| \\ &= o_p(n^{-1/2}). \end{aligned}$$

(b) Consider $I_{n2}^{(k)}$ ($k = 1, 2$). Observe that $\sqrt{n}(1 - P(|\eta_t \sqrt{\alpha_0} + \phi_0| \in \mathcal{I}_n)) = o(1)$. Thus,

$$\begin{aligned} \sqrt{n} I_{n1}^{(k)} &= \frac{1}{n} \sum_{t=1}^n (I(|\eta_t \sqrt{\alpha_0} - (-1)^k \phi_0| \in \mathcal{I}_n) \log |\eta_t \sqrt{\alpha_0} - (-1)^k \phi_0| - \int_{\mathcal{I}_n} \log |x| dH_k(x; 0)) \\ &= \frac{1}{\sqrt{n}} \sum_{t=1}^n \zeta_{kt} + o_p(1), \end{aligned}$$

and consequently,

$$\sqrt{n}(I_{n1}^{(1)} + I_{n2}^{(2)}) = \frac{1}{\sqrt{n}} \sum_{t=1}^n (\zeta_{1t} + \zeta_{2t}) + o_p(1).$$

(c) Consider the term $I_{n3}^{(k)}$ ($k = 1, 2$). Since $E\eta^2 < \infty$ and $f(\cdot)$ is bounded,

$$|I_{n3}^{(k)}|^2 \leq E(\log |(-1)^{k-1}\phi_0 + \eta\sqrt{\alpha_0}|)^2 P(|(-1)^{k-1}\phi_0 + \eta\sqrt{\alpha_0}| \notin \mathcal{I}_n) = o(n^{-1}).$$

(d) Consider the term $I_{n4}^{(k)}$ ($k = 1, 2$). Define

$$\tilde{v}_t^{(k)} = (-1)^{k-1} \left(1 + \frac{(-1)^k y_{t-1} \sqrt{\alpha_0}}{\sqrt{\omega_0 + \alpha_0 y_{t-1}^2}} \right), \quad \tilde{u}_{1t} = \frac{1}{\sqrt{\alpha_0}} \tilde{u}_{2t}, \quad \tilde{u}_{2t} = \frac{-\sqrt{\alpha_0}}{2(\omega_0 + \alpha_0 y_{t-1}^2)}.$$

By Taylor's expansion, $v_t^{(k)}(\hat{\theta}_n)$ ($k = 1, 2$) and $u_t(\hat{\theta}_n)$ can be approximated by

$$\begin{aligned} v_t^{(k)}(\hat{\theta}_n) &= \tilde{v}_t^{(k)}(\hat{\phi}_n - \phi_0) + o_p(n^{-1/2}), \\ u_t(\hat{\theta}_n) &= \tilde{u}_{1t}(\hat{\alpha}_n - \alpha_0) + \tilde{u}_{2t}(\hat{\omega}_n - \omega_0) + o_p(n^{-1/2}), \end{aligned}$$

respectively. It follows from the laws of large numbers that

$$\frac{1}{n} \sum_{t=1}^n \tilde{u}_{kt} \rightarrow \nu_k, \text{ a.s.}, \quad k = 1, 2.$$

Hence, it follows from Lemma B.2 and Theorem 2.2 that

$$\begin{aligned} \sqrt{n}(I_{n4}^{(1)} + I_{n4}^{(2)}) &= 2\mu_1 \sqrt{n}(\hat{\phi}_n - \phi_0) + 2\mu_2 \nu_1 \sqrt{n}(\hat{\alpha}_n - \alpha_0) + 2\mu_2 \nu_2 \sqrt{n}(\hat{\omega}_n - \omega_0) + o_p(1) \\ &= \frac{1}{\sqrt{n}} \sum_{t=1}^n (2\mu_1 \xi_{1t} + 2\mu_2 \nu^T \xi_{2t}) + o_p(1). \end{aligned}$$

The part (d) follows.

(II) Consider the nonstationary case, i.e., $\gamma_0 > 0$. Let $v_{nt}^{(k)}(\vartheta) = n^{1/2} v_t^{(k)}((\phi_0 + n^{-1/2}\vartheta_1, \alpha_0 + n^{-1/2}\vartheta_2, \omega_0 + \vartheta_3)^T)$ and $u_{nt}(\vartheta) = n^{1/2} u_t((\phi_0 + n^{-1/2}\vartheta_1, \alpha_0 + n^{-1/2}\vartheta_2, \omega_0 + \vartheta_3)^T)$. Define

$$\begin{aligned} \tilde{H}_{nk}(x; \vartheta) &= \frac{1}{n} \sum_{t=1}^n I\left(\eta_t \leq \frac{x + (-1)^k \phi_0 - n^{-1/2} v_{nt}^{(k)}(\vartheta)}{\sqrt{\alpha_0 + n^{-1/2} u_{nt}(\vartheta)}}\right), \\ \tilde{H}_k(x; \vartheta) &= \frac{1}{n} \sum_{t=1}^n F\left(\frac{x + (-1)^k \phi_0 - n^{-1/2} v_{nt}^{(k)}(\vartheta)}{\sqrt{\alpha_0 + n^{-1/2} u_{nt}(\vartheta)}}\right). \end{aligned}$$

Denote $\hat{\vartheta}_n = (n^{1/2}(\hat{\phi}_n - \phi_0), n^{1/2}(\hat{\alpha}_n - \alpha_0), \hat{\omega}_n - \omega_0)^T$, then

$$2\hat{\gamma}_n = \int_{\mathcal{I}_n} \log |x| d\tilde{H}_{n1}(x; \hat{\vartheta}_n) + \int_{\mathcal{I}_n} \log |x| d\tilde{H}_{n2}(x; \hat{\vartheta}_n) = \hat{\gamma}_{n1} + \hat{\gamma}_{n2}.$$

We decompose the $\hat{\gamma}_{nk} - \gamma_0$ ($k = 1, 2$), respectively, as

$$\begin{aligned} \hat{\gamma}_{nk} - \gamma_0 &= \int_{\mathcal{I}_n} \log |x| d \left(\tilde{H}_{nk}(x; \hat{\vartheta}_n) - \tilde{H}_{nk}(x; 0) - \tilde{H}_k(x; \hat{\vartheta}_n) + \tilde{H}_k(x; 0) \right) \\ &\quad + \int_{\mathcal{I}_n} \log |x| d(\tilde{H}_{nk}(x; 0) - \tilde{H}_k(x; 0)) + \int_{R/\mathcal{I}_n} \log |x| d\tilde{H}_k(x; 0) \\ &\quad + \int_{\mathcal{I}_n} \log |x| d(\tilde{H}_k(x; \hat{\vartheta}_n) - \tilde{H}_k(x; 0)) \\ &:= I_{n1}^{(k)} + I_{n2}^{(k)} + I_{n3}^{(k)} + I_{n4}^{(k)}. \end{aligned}$$

To complete the proof, it suffices to show that

- (a) $\sqrt{n}I_{n1}^{(k)} = o_p(1)$, $k = 1, 2$;
- (b) $\sqrt{n}(I_{n2}^{(1)} + I_{n2}^{(2)}) = \frac{1}{\sqrt{n}} \sum_{t=1}^n (\zeta_{1t} + \zeta_{2t}) + o_p(1)$;
- (c) $\sqrt{n}I_{n3}^{(k)} = o_p(1)$, $k = 1, 2$;
- (d) $\sqrt{n}(I_{n4}^{(1)} + I_{n4}^{(2)}) = \frac{1}{\sqrt{n}} \sum_{t=1}^n (2\mu_1 \xi_{1t}) + o_p(1)$.

Combining (a)-(d), we immediately obtain that

$$\sqrt{n}(\hat{\gamma}_n - \gamma_0) = \frac{1}{\sqrt{n}} \sum_{t=1}^n (\zeta_t + \mu_1 \tilde{\xi}_{1t}) + o_p(1).$$

In fact, (a), (b) and (c) can be obtained in a similar fashion as those in the stationary case. Now we only need to prove (d). First, it follows from Lemma B.1 below and Taylor's expansion that

$$\begin{aligned} \sqrt{n}(I_{n4}^{(1)} + I_{n4}^{(2)}) &= \frac{\mu_1}{\sqrt{n}} \sum_{t=1}^n \{v_t^{(1)}(\hat{\theta}_n) + v_t^{(2)}(\hat{\theta}_n)\} + \frac{2\mu_2}{\sqrt{n}} \sum_{t=1}^n u_t(\hat{\theta}_n) + o_p(1) \\ &= \frac{1}{\sqrt{n}} \sum_{t=1}^n (2\mu_1 \tilde{\xi}_{1t}) + \frac{2\mu_2}{\sqrt{n}} \sum_{t=1}^n u_t(\hat{\theta}_n) + o_p(1). \end{aligned}$$

By the compactness of Θ and the fact (18), we have

$$\sum_{t=1}^n |u_t(\hat{\theta}_n)| \leq M \sum_{t=1}^n \frac{1}{1 + y_{t-1}^2} < \infty.$$

Thus, $\sum_{t=1}^n |u_t(\hat{\theta}_n)| = O_p(1)$ and

$$\sqrt{n}(I_{n4}^{(1)} + I_{n4}^{(2)}) = \frac{1}{\sqrt{n}} \sum_{t=1}^n (2\mu_1 \tilde{\xi}_{1t}) + o_p(1).$$

The proof is complete.

A.5 Proof of Theorem 3.2

Consider the case of $\gamma_0 < 0$. Using similar arguments in the proof of Theorem 3.1, we obtain that

$$\sqrt{n}(\hat{\gamma}_n^* - \gamma_0) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \varpi_t (\zeta_t + \mu_1 \xi_{1t} + \mu_2 \nu^T \xi_{2t}) + o_{P^*}(1).$$

Then, with the asymptotic representation of $\sqrt{n}(\hat{\gamma}_n - \gamma_0)$, it follows that

$$\sqrt{n}(\hat{\gamma}_n^* - \hat{\gamma}_n) = \frac{1}{\sqrt{n}} \sum_{t=1}^n (\varpi_t - 1) (\zeta_t + \mu_1 \xi_{1t} + \mu_2 \nu^T \xi_{2t}) + o_{P^*}(1).$$

Thus, conditionally on $\{y_0, y_1, \dots, y_n\}$, in probability, as $n \rightarrow \infty$,

$$\sqrt{n}(\hat{\gamma}_n^* - \hat{\gamma}_n) \xrightarrow{\mathcal{L}} N(0, E(\zeta_t + \mu_1 \xi_{1t} + \mu_2 \nu^T \xi_{2t})^2).$$

The results for $\gamma_0 > 0$ can be proved analogously. The proof is complete.

A.6 Proof of Theorem 3.3

The proof is immediate from Theorem 3.2, and hence it is omitted.

B Lemmas with proofs

In this subsection, two lemmas are given for the proofs of Theorems 3.1 and 3.2.

Lemma B.1. *Let $\{\eta_t\}$ be i.i.d. with the distribution $F(\cdot)$, which has an a.e. positive bounded density $f(\cdot)$ with $\int_{x \in \mathbb{R}} |x| f(x) dx < \infty$. Let \mathcal{A}_{ni} be an array of sub σ -fields such that $\mathcal{A}_{nt} \subset \mathcal{A}_{n(t+1)}$, $1 \leq t \leq n$, $n \geq 1$; for each θ , $(v_{n1}(\theta), u_{n1}(\theta))$ is \mathcal{A}_{n1} -measurable and*

$(\eta_1, \dots, \eta_{t-1}, v_{nt}(\theta), u_{nt}(\theta))$, $t \leq j$ are \mathcal{A}_{nj} measurable, $2 \leq j \leq n$; η_t is independent of \mathcal{A}_{nt} , $1 \leq t \leq n$. For $v_{nt}(\theta)$ and $u_{nt}(\theta)$, assume that

$$\frac{1}{n} \sum_{t=1}^n \sup_{\|\theta_1 - \theta_2\| \leq \delta} |v_{nt}(\theta_1) - v_{nt}(\theta_2)| \leq C_1 \delta, \quad a.s.,$$

and

$$\frac{1}{n} \sum_{t=1}^n \sup_{\|\theta_1 - \theta_2\| \leq \delta} |u_{nt}(\theta_1) - u_{nt}(\theta_2)| \leq C_2 \delta, \quad a.s.$$

for some universal positive constants C_1 and C_2 . For each $x \in R$, define

$$H_n(x; \theta) = \frac{1}{n} \sum_{t=1}^n I\left(\eta_t \leq \frac{x - n^{-1/2}v_{nt}(\theta)}{\sqrt{\alpha_0} + n^{-1/2}u_{nt}(\theta)}\right) \quad \text{and} \quad H(x; \theta) = \frac{1}{n} \sum_{t=1}^n F\left(\frac{x - n^{-1/2}v_{nt}(\theta)}{\sqrt{\alpha_0} + n^{-1/2}u_{nt}(\theta)}\right),$$

where α_0 are a positive constant. Then, for any constant b with $0 < b < \infty$, there exists a positive constant ϵ such that

$$\sup_{x \in R, \|\theta\| \leq b} \sqrt{n} |H_n(x; \theta) - H_n(x; 0) - H(x; \theta) + H(x; 0)| = O_p(n^{-\epsilon}).$$

PROOF. This lemma is similar to that of Lemma 8.3.2 in [Koul \(2002\)](#) but the rate of convergence is strengthened. It should be noted that we use the pseudometric

$$d_b(x, y) = \sup_{|z| \leq b} \left| F\left(\frac{x - \phi_0 - n^{-1/2}z_1}{\sqrt{\alpha_0} + n^{-1/2}z_2}\right) - F\left(\frac{y - \phi_0 - n^{-1/2}z_1}{\sqrt{\alpha_0} + n^{-1/2}z_2}\right) \right|$$

for this local and scale setup, where $z = (z_1, z_2) \in R^2$, $|z| = |z_1| \vee |z_2|$, $x, y \in R$, $b > 0$. If we let $\mathfrak{N}(\delta, b)$ be the cardinality of the minimal δ -net of (R, d_b) , we can show that $\mathfrak{N}(\delta, b) \leq C_b \delta^{-4}$ with $C_b < \infty$ for any $0 < n^{-1/2}b < 1$. We can repeat the similar arguments as in Lemma 8.3.2 in [Koul \(2002\)](#) using the metric d_b . The convergence rate is strengthened since we make stronger conditions of the terms $u_{nt}(\theta)$ and $v_{nt}(\theta)$. The details are omitted here.

Lemma B.2. *Suppose that η is a random variable with the density $f(\cdot)$ and the derivative $f'(x)$ of $f(x)$ exists for each x . The constants ϕ_0 and α_0 are finite and $\alpha_0 > 0$. Assume $b_1 = -\alpha_0^{-1/2} \int \log |\phi_0 + \eta\sqrt{\alpha_0}| f'(\eta) d\eta < \infty$ and $b_2 = -\alpha_0^{-1/2} \int \log |\phi_0 + \eta\sqrt{\alpha_0}| (f(\eta) + \eta f'(\eta)) d\eta < \infty$. Denote $\delta = (\delta_1, \delta_2)^T$ and $h(\eta; \delta) = \log |\phi_0 + \eta\sqrt{\alpha_0} + \delta_1 + \eta\delta_2|$. Then, for small constants δ_1 and δ_2 with $\delta_1 \rightarrow 0$ and $\delta_2 \rightarrow 0$, the following approximation holds:*

$$E(h(\eta; \delta) - h(\eta; 0)) = b_1 \delta_1 + b_2 \delta_2 + o(|\delta_1| + |\delta_2|).$$

PROOF. $Eh(\eta; \delta)$ is expressed as

$$Eh(\eta; \delta) = \int f\left(\frac{\eta - \phi_0 - \delta_1}{\sqrt{\alpha_0} + \delta_2}\right) \frac{\log |\eta|}{\sqrt{\alpha_0} + \delta_2} d\eta.$$

Hence,

$$\begin{aligned} E(h(\eta; \delta) - h(\eta; 0)) &= \int \log |\eta| \left\{ f\left(\frac{\eta - \phi_0 - \delta_1}{\sqrt{\alpha_0} + \delta_2}\right) - f\left(\frac{\eta - \phi_0}{\sqrt{\alpha_0}}\right) \right\} d\eta \frac{1}{\sqrt{\alpha_0} + \delta_2} \\ &\quad + \left(\frac{1}{\sqrt{\alpha_0} + \delta_2} - \frac{1}{\sqrt{\alpha_0}} \right) \int \log |\eta| f\left(\frac{\eta - \phi_0}{\sqrt{\alpha_0}}\right) d\eta \\ &= I_1 + I_2. \end{aligned}$$

Let us handle the first term I_1 . By Taylor's expansion for $f(\cdot)$ at the point $(\eta - \phi_0)/\sqrt{\alpha_0}$, we obtain that

$$\begin{aligned} f\left(\frac{\eta - \phi_0 - \delta_1}{\sqrt{\alpha_0} + \delta_2}\right) - f\left(\frac{\eta - \phi_0}{\sqrt{\alpha_0}}\right) &= f'\left(\frac{\eta - \phi_0}{\sqrt{\alpha_0}}\right) \left(\frac{\eta - \phi_0 - \delta_1}{\sqrt{\alpha_0} + \delta_2} - \frac{\eta - \phi_0}{\sqrt{\alpha_0}} \right) + o(|\delta_1| + |\delta_2|) \\ &= f'\left(\frac{\eta - \phi_0}{\sqrt{\alpha_0}}\right) \left(\frac{-\delta_1\sqrt{\alpha_0} - \delta_2\eta + \phi_0\delta_2}{\alpha_0} \right) + o(|\delta_1| + |\delta_2|). \end{aligned}$$

Thus,

$$\begin{aligned} I_1 &= \int \log |\eta| f'\left(\frac{\eta - \phi_0}{\sqrt{\alpha_0}}\right) \left(\frac{-\delta_1\sqrt{\alpha_0} - \delta_2\eta + \phi_0\delta_2}{\alpha_0} \right) d\eta \frac{1}{\sqrt{\alpha_0}} + o(|\delta_1| + |\delta_2|) \\ &= -\alpha_0^{-1/2} \delta_1 \int \log |\phi_0 + \eta\sqrt{\alpha_0}| f'(\eta) d\eta \\ &\quad - \alpha_0^{-1/2} \delta_2 \int \log |\phi_0 + \eta\sqrt{\alpha_0}| f'(\eta) \eta d\eta + o(|\delta_1| + |\delta_2|). \end{aligned}$$

Similarly, we have that

$$I_2 = -\alpha_0^{-1/2} \delta_2 \int \log |\phi_0 + \eta\sqrt{\alpha_0}| f(\eta) d\eta + o(\delta_2).$$

Combining the above results, the proof follows.

References

Bollerslev, T., 1986. Generalized autoregressive conditional heteroskedasticity. *J. Econometrics* 31, 307-327.

- Borkovec, M., Klüppelberg, C., 2001. The tail of the stationary distribution of an autoregressive process with ARCH(1) errors. *Ann. Appl. Prob.* 11, 1220-1241.
- Chan, N.H., Peng, L., 2005. Weighted least absolute deviation estimation for an AR(1) process with ARCH(1) errors. *Biometrika* 92, 477-484.
- Chen, K., Ying, Z., Zhang, H., Zhao, L., 2008. Analysis of least absolute deviation. *Biometrika* 95, 107-122.
- Chen, K., Guo, S., Lin, Y., Ying, Z., 2010. Least absolute relative error estimation. *J. Amer. Statist. Assoc.* 105, 1104-1112.
- Chen, M., Li, D., Ling, S., 2014. Non-stationarity and quasi-maximum likelihood estimation on a double autoregressive model. *J. Time Series Anal.* 35, 189-202.
- Cline, D.B.H., Pu, H.H., 2004. Stability and the Lyapounov exponent of threshold AR-ARCH models. *Ann. Appl. Prob.* 14, 1920-1949.
- Efron, B., 1987. Better bootstrap confidence intervals. *J. Amer. Statist. Assoc.* 82, 171-185.
- Engle, R.F., 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* 50, 987-1007.
- Francq, C., Zakoïan, J.-M., 2007. Quasi-maximum likelihood estimation in GARCH processes when some coefficients are equal to zero. *Stoch. Proc. Appl.* 117, 1265-1284.
- Francq, C., Zakoïan, J.-M., 2012. Strict stationarity testing and estimation of explosive and stationary generalized autoregressive conditional heteroscedasticity models. *Econometrica* 80, 821-861.
- Francq, C., Zakoïan, J.-M., 2013. Inference in nonstationary asymmetric GARCH models. *Ann. Stat.* 41, 1970-1998.
- Jensen, S.T., Rahbek, A., 2004a. Asymptotic normality of the QMLE estimator of ARCH in the nonstationary case. *Econometrica* 72, 641-646.
- Jensen, S.T., Rahbek, A., 2004b. Asymptotic inference for nonstationary GARCH. *Econometric Theory* 20, 1203-1226.
- Jin, Z., Ying, Z., Wei, L.J., 2001. A simple resampling method by perturbing the minimand. *Biometrika* 88, 381-390.
- Koul, H.L., 2002. *Weighted Empirical Processes in Dynamic Nonlinear Models*. Springer-Verlag, New York.

- Koul, H.L., Ossiander, M., 1994. Weak convergence of randomly weighted dependent residual empiricals with applications to autoregression. *Ann. Stat.* 22, 540-562.
- Li, D., Guo, S., Zhu, K., 2018. Double AR model without intercept: An alternative to modeling nonstationarity and heteroscedasticity. *Econometric Rev.* <https://doi.org/10.1080/07474938.2017.1310080>.
- Li, D., Ling, S., Zakoian, J.-M., 2015. Asymptotic inference in multiple-threshold double autoregressive models. *J. Econometrics* 189, 415-427.
- Li, D., Ling, S., Zhang, R.M., 2016. On a threshold double autoregressive model. *J. Bus. Econom. Statist.* 34, 68-80.
- Li, G., Leng, C., Tsai, C.L., 2014. A hybrid bootstrap approach to unit root tests. *J. Time Ser. Anal.* 35, 299-321.
- Li, G., Zhu, Q., Liu, Z., Li, W.K., 2017. On mixture double autoregressive time series models. *J. Bus. Econom. Statist.* 35, 306-317.
- Ling, S., 2004. Estimation and testing stationarity for double autoregressive models. *J. Roy. Statist. Soc. Ser. B* 66, 63-78.
- Ling, S., 2007. A double AR(p) model: structure and estimation. *Statist. Sinica.* 17, 161-175.
- Ling, S., Li, D., 2008. Asymptotic inference for a nonstationary double AR(1) model. *Biometrika* 95, 257-263.
- Liu, F., Li, D., Kang, X.M., 2018. Sample path properties of an explosive double autoregressive model. *Econometric Rev.* 37, 484-490.
- Lu, Z., 1998. On the geometric ergodicity of a non-linear autoregressive model with an autoregressive conditional heteroscedastic term. *Statist. Sinica.* 8, 1205-1217.
- Nielsen, H.B., Rahbek, A., 2014. Unit root vector autoregression with volatility induced stationarity. *Journal of Empirical Finance* 29, 144-167.
- Tsay, R.S., 1987. Conditional heteroscedastic time series models, *J. Amer. Statist. Assoc.* 82, 590-604.
- Weiss, A.A., 1984. ARMA models with ARCH errors. *J. Time Ser. Anal.* 5, 129-143.
- Wu, C.F.J., 1986. Jackknife, bootstrap and other resampling methods in regression analysis (with discussion). *Ann. Statist.* 14, 1261-1350.

- Zhu, K., 2016. Bootstrapping the portmanteau tests in weak auto-regressive moving average models. *J. Roy. Statist. Soc. Ser. B* 78, 463-485.
- Zhu, K., Ling, S., 2011. Global self-weighted and local quasi-maximum exponential likelihood estimators for ARMA-GARCH/IGARCH models. *Ann. Statist.* 39, 2131-2163.
- Zhu, K., Ling, S., 2013. Quasi-maximum exponential likelihood estimators for a double AR(p) model. *Statist. Sinica.* 23, 251-270.
- Zhu, Q., Zheng, Y., Li, G. (2018). Linear double autoregression. *J. Econometrics.* (in press)

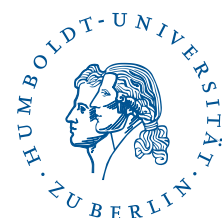
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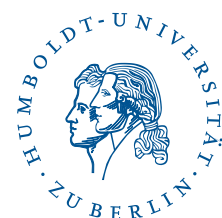
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