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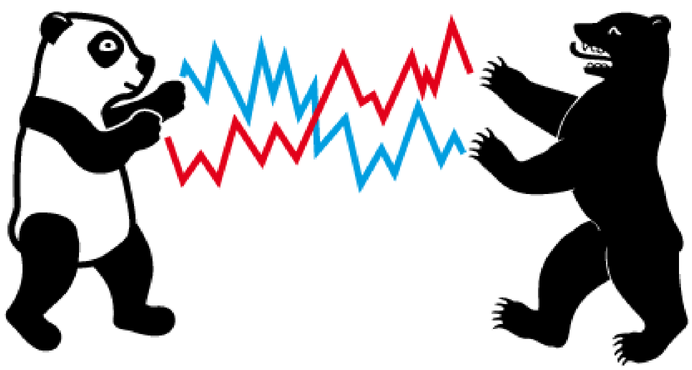

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Estimating low sampling frequency risk measures by high-frequency data

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Abstract

Weekly, quarterly and yearly risk measures are crucial for risk reporting according to Basel III and Solvency II. For the respective data frequencies, the authors show in a simulation and backtest study that available data series are not sufficient in order to estimate Value at Risk and Expected Shortfall sufficiently, given confidence levels of 99.9% and 99.99%. Accordingly, this paper presents a semi-parametric estimation method, rescaling data from high- to low-frequency which allows to obtain significantly more data points for the estimation of the respective risk measures. The presented methodology in the α-stable framework, which is able to mimic multifractal behavior in asset returns, provides tail events which never occurred in the original low-frequency dataset.

Keywords: high-frequency, multifractal, stable distribution, rescaling, risk management, Value at Risk, quantile distribution

JEL Classification: C14, C22, C46, C53, G32

1. Introduction

Value at Risk and Expected Shortfall estimation for high confidence levels, i.e. 99.9% or 99.99% are central parts of risk reporting for banking (Basel III) and insurance (Solvency II) institutions. As the aimed holding periods range from weekly to annual [BIS, 2017], available data-sets for respective sampling frequencies cover a limited amount of data, which is not sufficient in order to model the tails of the loss / return distribution. In this regard, the authors develop a methodology in the α-stable framework which rescales an empirical high-frequency distribution to a lower sampling frequency. Given more data points provided, the semi-parametric method provides an efficient estimation for risk measures with high confidence levels.

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We start to contribute to the literature by formulating a concrete specification of the sample quantile estimation problem for larger confidence levels given a limited number of observations. Within a simulation study, we show by introducing sample quantile bias and overestimation as a function of the number of data points available, that datasets of ten years history and less do not suffice in order to efficiently estimate high confidence quantiles (99.9% / 99.99%) for weekly and lower frequencies. The developed frequency rescaling methodology allows to estimate low-frequency risk measures by rescaling high-frequency data, inducing tail events which never occurred in the history of the lower frequency. Furthermore we show that the multifractal scaling law can be mimicked by the frequency rescaling methodology, which employs long-memory GARCH methods. The backtest evaluates the frequency rescaling method in terms of efficiency and coverage and indicates the outperformance over diverse methods for the weekly sampling frequency.

Basel regulations demand that "assumptions made within the internal model are appropriate and do not underestimate risk. This may include the assumption of the normal distribution" [BIS, 2016]. As leptokurtic distributions with finite variance - such as Student-t for the daily frequency - are governed by the Central Limit Theorem, the relevant annual distribution would be approximately Gaussian. To circumvent this asymptotic inevitability, returns from different sampling frequencies are analyzed in the framework of α-stable distributions, which have their own domain of attraction and limit theorems [Gnedenko and Kolmogorov, 1954]. Deviating from the Gaussian random walk, first applications in finance are due to Mandelbrot [Mandelbrot, 1963, 1967] and his student Fama [Fama, 1965], who rest their work on Lévy stable processes [Lévy, 1925]. In contrast to the theoretical results of stability under addition [Fama and Roll, 1968], Fama [1976], McFarland et al. [1982], Boothe and Glassman [1987], Dacorogna et al. [2001] as well as Grabchak and Samorodnitsky [2010], show that the stability exponent decreases empirically for higher sampling frequencies. This implies that high-frequency returns are heavier tailed than low-frequency returns, still differing significantly from Gaussianity. Modeling the dispersion of price increments as a function of its past absolute or squared returns, Bollerslev and Domowitz [1993], Andersen and Bollerslev [1997], Andersen and Bollerslev [1998] and Bollerslev et al. [2000] show that in terms of temporal dependence, higher frequency returns exhibit a longer memory, which call for different function approximations over sampling frequencies. Allowing for hyperbolic decay of the autocorrelation function, long memory can be modelled accordingly [Taylor, 1986]. In contrast Mandelbrot and van Ness [1968], Mandelbrot [1982], Mantegna and Stanley [1995] and Xu and Gencay [2003] rest their analysis on the empirical scaling law, which deviates empirically from uni- and mesofractal models. Mandelbrot et al. [1997]

and Mandelbrot et al. [1997b] model multifractality by modeling the price increments under a subordinated (fractional) Brownian motion. The subordinator is represented by a multifractal measure, which controls tail behavior and long memory [Calvet and Fisher, 2002]. On the foundation of frequency dependent stochastic properties in terms of heavy tails, time dependence and multifractality, we analyze ten years of Level-1 tick data for Microsoft, which are gathered from Lobster. For computation, Matlab has been utilized. The paper is organized as follows: The first chapter formulates the financial model and introduces the sample quantile estimation problem in a simulation study. The high-frequency data-set of Microsoft is subsequently analyzed over varying sampling frequencies in chapter two. Given the varying behavior over sampling frequencies, a method to rescale the data-set from high- to low-frequency is introduced in chapter three. The last chapter verifies the performance of the respective quantile estimates in an in- and out-of-sample backtest.

2. Sample quantile distribution

Financial stake-holders are interested in estimating the (conditional) quantile of the wealth distribution for large confidence levels in order to report capital at risk, given a fixed probability of ungovernable events. Although it is possible to estimate risk measures for all confidence levels, given a limited amount of data, we will show that respective risk measures are significantly underestimated. From mathematical statistics it is well-known that the asymptotic distribution of the sample quantile is unbiased and Gaussian by CLT, given stationarity and finite second moments [Ruppert, 2010]. Given that empirical time series provide a limited amount of data points, we are going to examine the bootstrapped sample quantile distribution for relevant confidence levels and distribution assumptions.

2.1. Financial model

Let $X_t \in \mathbb{R}^k,\ k \in \mathbb{N}^+$ be multidimensional log-returns from distribution P^t, where t indicates the scale, e.g. days. $\tilde{X}_t$ represents the according discrete returns. For horizon $T \in \mathbb{N}^+$ days, the wealth equation

$$W_T(f_t) = W_0 \prod_{t=1}^{T} \left\{1 + f_t^\top \tilde{X}_t\right\} = W_0 \prod_{t=1}^{T} \left\{f_t^\top \exp(X_t)\right\} \tag{1}$$

can be simplified, given constant investment fractions $f \in \mathbb{R}^k$ over time $f_t = f\ \forall t = 0, \dots, T$.

$$W_T(f) = W_0 \left\{ f^\top \exp\left(\sum_{t=1}^{T} X_t\right)\right\} = W_0 \left\{f^\top \exp(X)\right\},\ X \stackrel{\text{def}}{=} \sum_{t=1}^{T} X_t \tag{2}$$

For respective cdf $F_{W_T}(x)$ the spectral measure with weight function $\phi(x)$ is defined through the quantile function $F^{-1}_{W_T}(x) \stackrel{\text{def}}{=} \{x : \mathrm{P}\left(W_T(f_t) \leq x\right) = \tau\}$, $\tau \in (0,1)$.

$$S_\phi \left\{W_T(f_t)\right\} = \int_0^1 \phi(x) F^{-1}_{W_T}(x) \mathrm{d}x \tag{3}$$

Within the context of risk measures, the spectral measure will be coherent iff the weight function is positive $\phi(x) \geq 0$, increasing $\phi'(x) \geq 0$ and normalized $\int_0^1 \phi(x) = 1$ [Acerbi, 2002]. Two specific risk measures to assess the risk of the portfolio are quantile (VaR) and expected shortfall (ES) constraints:

- S_1 : The quantile constraint (VaR) is a special case of the spectral risk measure from (3)
$$\phi_{\mathrm{Q}_\alpha}(x) = \delta(x = \tau) \tag{4}$$
where $\delta(x = \tau)$ is the Dirac delta function, well-known to be a non-coherent risk measure. Further drawbacks of the quantile constraint are treated in Basak and Shapiro [2001]. However, the quantile restriction allows to indicate a loss, which his not exceeded with probability $1 - \alpha$.

- S_2 : In contrast, Expected Shortfall is a coherent risk measure representing the average loss beyond a given quantile constraint. Being a special case of the spectral measure, the weight function is given as
$$\phi_{\mathrm{ES}_\tau}(x) = \tau^{-1}\mathbf{1}(x < \tau). \tag{5}$$
Avoiding the weaknesses of VaR, the Basel Committee on Banking supervision proposed to shift the quantitative risk measurement from VaR to expected shortfall (ES) [BIS, 2013, 2016]. As ES is shown to be sub-additive and assesses events beyond the quantile, ES is becoming present in the financial industry.

2.2. A Microsoft Investor

Presume a stock-market investor is calculating the weekly VaR of his portfolio by using close to ten years of Microsoft data, representing 471 weekly returns. Accordingly, Gaussian, Student-t ($\nu = 5$) and Stable ($\alpha = 1.7$) distributions are fitted via MLE. The respective quantiles of the discrete wealth return distribution

$$Q(\tau) = \int_0^1 \delta(x = \tau) F^{-1}(x) dx \tag{6}$$

are estimated by the sample quantile

$$\hat{Q}_n(\tau) = \int_0^1 \delta(x = \tau) \hat{F}_n^{-1}(x) dx, \tag{7}$$

given in Table 2. For the confidence level of 99% the maximum losses of the Microsoft stock under standard parametric assumptions are -8.38% (Gaussian), -11.95% (Student-t) and -11.64% (Stable). For confidence intervals larger than 99.9% the tails of the Stable distribution are heavier than the tails of Student-t and Gaussian. In contrast to the asymptotic Gaussianity of the sample quantile

Confidence $1-\tau$	Gaussian	Student-t	Stable
99%	-8.38	-11.95	-11.64
99.9%	-11.01	-20.05	-35.59
99.99%	-13.13	-30.90	-81.31

Table 1: : Quantiles (in %) of discrete weekly Microsoft returns for Gaussian, Student-t ($\nu = 5$) and Stable ($\alpha = 1.7$) for confidence levels 99%, 99.9% and 99.99%

under stationarity and finite second moments we illustrate the effect of limited sample size via bootstrap.

- draw $B = 10^4$ independent bootstrap samples $X^{(1)}, \ldots, X^{(B)}$ of size $n = 471$ from parametric distribution $\hat{F}_{\hat{\theta},n}$, $\hat{\theta} = \{\text{Gaussian, Student-}t\text{, }\alpha\text{−Stable}\}$ and

- calculate $B = 10^4$ quantile estimators $\hat{Q}_{i,n}(\tau) \sim \hat{G}_n,\ i = 1, \ldots, B$ plotted as histograms in Figure 1.

We chose the block-length n to be 471 as it represents the number of observations in the weekly frequency of our data-set. The vertical lines in Figure 1 represent the asymptotic quantiles under their parametric assumptions. The bootstrapped quantile distributions for Gaussian, Student-t and Stable are plotted as histograms. Whereas the bootstrapped quantile distributions are unbiased under Gaussianity, the quantile distribution under stable laws is significantly skewed.

In order to evaluate bias and overestimation of the bootstrapped quantile estimates, two measures are introduced as a function of the sample size $n \in \mathbb{N}^+$. The average bias represents the difference between the asymptotic quantile and the bootstrapped quantile:

$$\bar{b}_n = \frac{1}{B} \sum_{i=1}^{B} \left\{ Q(\tau) - \hat{Q}_{i,n}(\tau) \right\}. \tag{8}$$

The upper part of Figure 2 shows the average bias as function of the sample size. The more leptokurtic the distribution, the more data points are needed in order to obtain an unbiased estimate. Overall, for a confidence level of 99%, 471 observations suffice to provide an unbiased estimate. The quantile overestimation gives the minimum overestimation of the quantile for 5% of the bootstrapped quantiles. In other words, in 5% of the cases the overestimation of the quantile is larger than $O_n(\bar{\alpha})$, where

$$O_n(\bar{\alpha}) = Q(\tau) - \hat{G}_n^{-1}(\bar{\alpha}), \ \bar{\alpha} = 95\%. \tag{9}$$

The lower part of Figure 2 shows the quantile overestimation as function of the sample size. The more leptokurtic the distributions, the more data points are needed in order to reduce the overestimation. Overall, for a confidence level of 99% and 471 observations the quantile is overestimated by more than 1% / 2% / 3% for Gaussian, Student-t and Stable assumption in 5% of the cases. The quantile overestimation implies that the respectively reported VaR estimates are exceeded more often than the confidence level presumes.

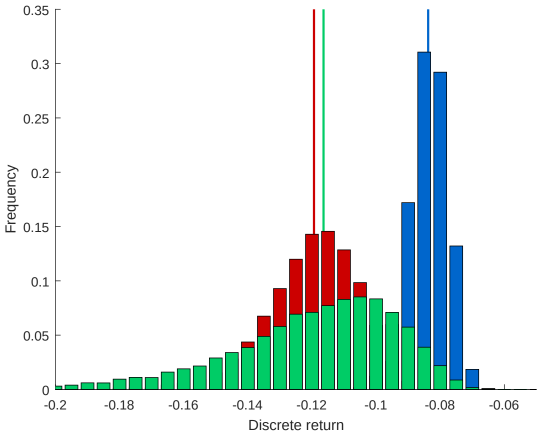

Figure 1: Sample quantile histograms for $n = 471$ observations with asymptotic quantile lines ($\tau = 1\%$)

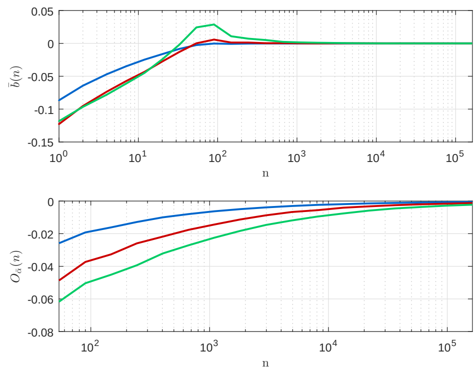

Figure 2: (Upper) Sample quantile bias (Lower) 95% quantile overestimation ($\tau = 1\%$)

For a confidence level of 99.9% 471 observations barely suffice in order to obtain an unbiased estimate (see Figure 3). The quantile is overestimated by more than 2% / 7% / 21% for Gaussian, Student-t and Stable assumption in 5% of the cases (see Figure 4). In order to obtain a reliable estimate in terms of unbiasedness and overestimation more than 10^4 observations are necessary. For a confidence level of 99.99% 471 observations do not suffice in order to obtain an unbiased estimate. In that respect, the quantile is overestimated by more than 4% / 18% / 66% for Gaussian, Student-t and Stable assumption in 5% of the cases. To obtain a reliable estimate in terms of unbiasedness and overestimation more than 10^5 observations are necessary. The clear implication is that although risk measures can always be calculated for high confidence levels, the limited amount of data points leads inevitably to underestimated risk estimates as shown for the special case of the quantile (VaR). For Expected Shortfall the degree of unbiasedness

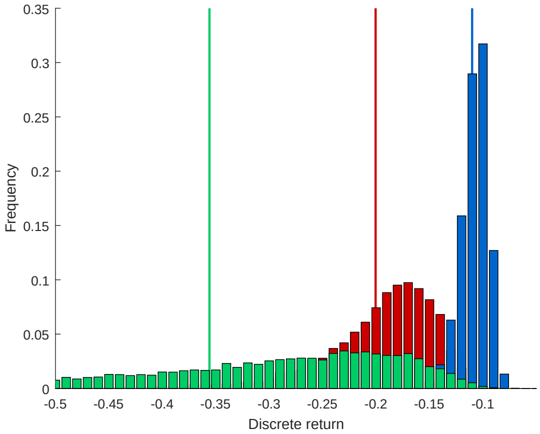

Figure 3: Sample quantile histograms for $n = 471$ observations with asymptotic quantile lines ($\tau = 0.1\%$)

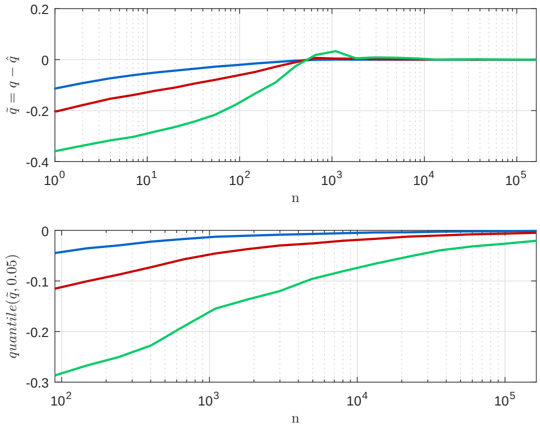

Figure 4: (Upper) Sample quantile bias (Lower) 95% quantile overestimation ($\tau = 0.1\%$)

and overestimation is even aggravated due to the conditional formulation of the quantile. Straight-forward, financial institutions should only report risk measures for those confidence levels, which can be estimated efficiently for available data. Within this paper, we are going to describe how to filter and rescale data from higher frequencies, implying significantly more data points, in order to estimate lower frequency risk measures.

3. Data

3.1. High-frequency data

The data-set represents transaction level data (Level 1) from 2007-06-27 till 2016-11-16 for Microsoft gathered from Lobster. By utilizing the previous-tick method, each day gives 390 trading minutes, representing 6 1/2 hours of trading from 09:30 a.m. till 04:00 p.m., see also Dacorogna et al. [2001]. After transforming the price data to log-returns, the returns are aggregated to their respective frequencies, up to one week, representing 471 weeks (see Table 2). Two excerpts of the data-set are given in Figure 5. The blue line represents minute, the red line hourly, the orange line daily and the violet line weekly prices.

Frequency	1 min	1 hour	1 day	1 week
Data points	921959	15365	2363	471

Table 2: : Number of data points for different sampling frequencies

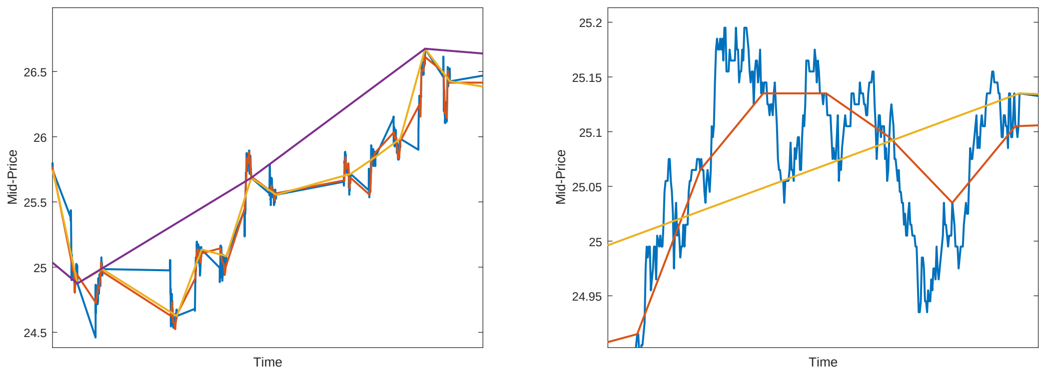

Figure 5: Excerpt of Microsoft mid-price data for minute (blue), hourly (red), daily (orange) and weekly (violet) frequency

3.2. Return characteristics

Data series from various fields of research, i.e. finance, economics, biology and physics, share the same characteristics over time and frequency. Starting with Mandelbrot [1963], log-transformations of cotton prices and later wheat prices, railroad stocks and financial rates [Mandelbrot, 1967] deviated from the Gaussian by exhibiting heavier tails than presumed. An overview over various heavy-tailed models in finance is given in Rachev [2003]. Stable laws specifically are treated in Zolotarev [1986], Samorodnitsky and Taqqu [1994] and Nolan [2017]. For financial returns, the degree of leptokurticity decreases with increasing sampling frequency [Fama, 1976, McFarland et al., 1982, Boothe and Glassman, 1987, Dacorogna et al., 2001, Grabchak and Samorodnitsky, 2010]. For Microsoft log-returns, the sample kurtosis decreases from 1153.2 for 1-minute returns to 6.1 for weekly returns (see Figure 6). Hence, the leptokurtic behavior of Microsoft log-returns decrease over the sampling frequencies, still differing significantly from the Gaussian assumption.

In order to verify if the sample kurtosis for lower frequencies is underestimated due to the lack of data points, we sample 10^4 blocks of 471 minutes, hours and days and plot the resulting sample kurtosis confidence interval (confidence level 95%) in Figure 6. Whereas the sample kurtosis increases exponentially with increasing frequencies, the average bootstrapped kurtosis remains in a range of six to eleven. For frequencies higher than 10-minutes, even the upper 95% confidence bound of the bootstrapped kurtosis lays below the empirically observed kurtosis. The implication is that 471 data points do not suffice to replicate the sample kurtosis of higher-frequency returns, inducing evidence that the kurtosis of the weekly frequency is underestimated. Indeed, the sample kurtosis increases with increasing data points. In Figure 7 the sample kurtosis is plotted as function of number of data points, used for the respective frequencies. The more data points are used,

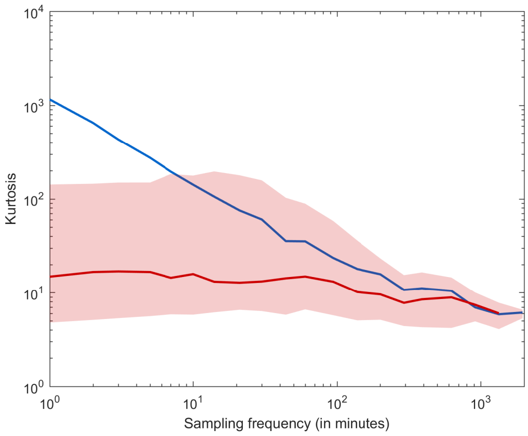

Figure 6: Sample kurtosis over frequency (blue) with bootstrapped kurtosis (red) for 471 data points (weekly frequency)

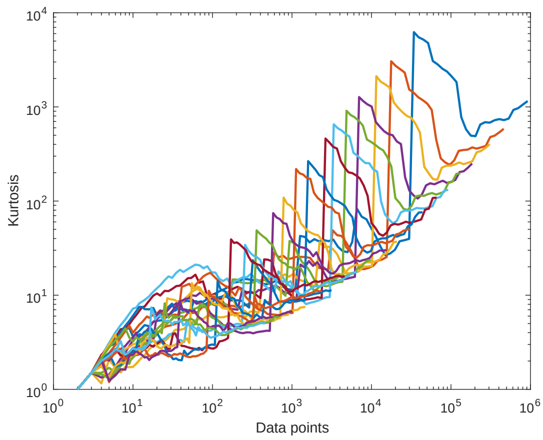

Figure 7: Sample kurtosis for increasing data points for frequencies from 1 minute to 1 week

the higher is the sample kurtosis, arguing against the finiteness of sample kurtosis. Accordingly, we resort to analyze the varying distributions over frequency in the α-stable framework for section 4, as stable distributions belong to their own domain of attraction.

Volatility clustering is the empirical observation that "large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes" [Mandelbrot, 1963]. Subsequently, two branches in the analysis of time-dependence developed: On the one hand the (FI)GARCH approach of Engle [1982], Bollerslev [1986] and Baillie et al. [1996], who aim to model the variance as a linear function of the past squared daily returns and on the other hand, the multifractal approach of Mandelbrot et al. [1997], Fisher et al. [1997], Calvet et al. [1997] and Calvet and Fisher [2002]. For higher sampling frequencies than daily, see for example Bollerslev and Domowitz [1993], Andersen and Bollerslev [1997], Andersen and Bollerslev [1998] and Bollerslev et al. [2000]. For increasing frequency, temporal dependence in absolute and squared returns increases (see Figure 8). Specifically, 5-minute and 1-hour absolute Microsoft returns exhibit strong intraday seasonalities, which are not captured by standard time-series models.

The empirical scaling law states that the mean absolute (squared) returns, as functions of their time intervals, are proportional to a power of the interval size [Mandelbrot, 1982, Mantegna and Stanley, 1995, Mandelbrot et al., 1997b, Calvet and Fisher, 2002, Xu and Gencay, 2003]. Starting from the self-affine process

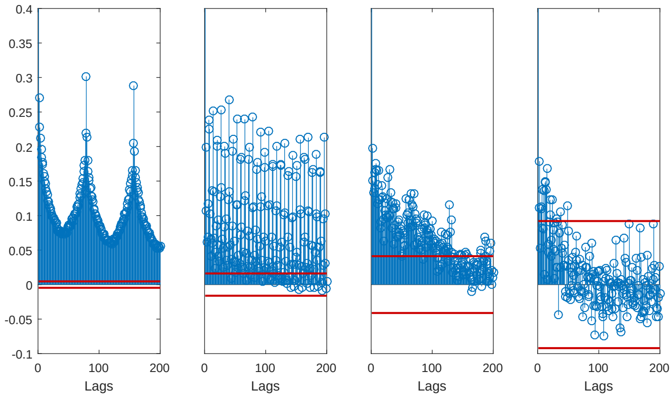

Figure 8: Sample autocorrelation function of absolute returns for 5-minute, hourly, daily and weekly frequencies

$\{X_t\}\,,\ t \geq 0$ with Hurst exponent $H > 0$ and $c > 0$,

$$X_{ct} \stackrel{\mathcal{L}}{=} c^H X_t \tag{10}$$

$$\mathsf{E}\left[|X_{ct}|^p\right] = c^{Hp}\,\mathsf{E}\left[|X_t|^p\right] \tag{11}$$

the scaling relation-ship in moments of order $p \in \mathbb{R}$ is derived. For $c(p) = \mathsf{E}\left[|X_t|^p\right]$ and $D(p) = H(p)p$ Mandelbrot et al. [1997] define a fractal process in terms of its moments as it remains graphically tractable (see Figure 9).

$$\mathsf{E}\left(|X_t|^p\right) = c(p)t^{D(p)} \tag{12}$$

For normalization in p, raise the scaling law of equation (12) to the power of $1/p$, giving

$$\mathsf{E}\left(|X_t|^p\right)^{1/p} = \left[c(p)t^{D(p)}\right]^{1/p} \tag{13}$$

$$\Leftrightarrow \frac{1}{p}\log \mathsf{E}\left(|X_t|^p\right) = \frac{1}{p}\log c(p) + H(p)\log t \tag{14}$$

If the absolute moments would scale with a unique Hurst exponent $0.5 \leq H \leq 1$ for all powers p, the underlying process would come from a Fractional Browian motion (Unifractal). For the Lévy stable motion the stability exponent would imply $H = 1/\alpha$ for $p \leq \alpha$ and $H = 1/q$ for $p > \alpha$, $0 < \alpha \leq 2$ (Mesofractal). But Figure 9 and 10 indicate that the Hölder exponents vary with increasing order of the moment p (Multifractal). Muller et al. [1990] argue that the empirically observed scaling law can only be explained by varying distributions for different time intervals, leading to subordinated (fractional) Brownian motions, see Mandelbrot et al. [1997b]. The Hölder exponents are estimated by (log-log) linear regression, see formula (14) and the generalized Hurst exponent by Matteo et al. [2005].

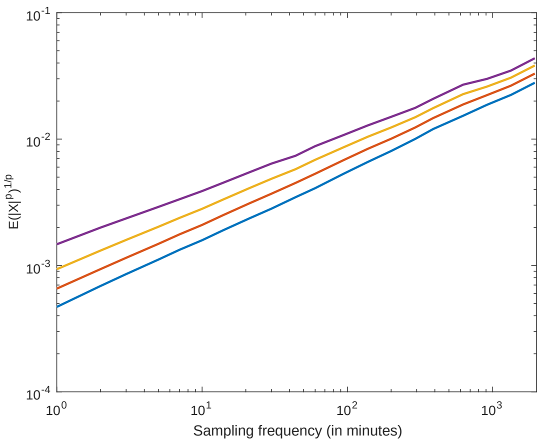

Figure 9: Empirical scaling law for Microsoft returns formulated as powers $p = \{0.5, 1, 1.5, 2\}$ of absolute moments (from the bottom up)

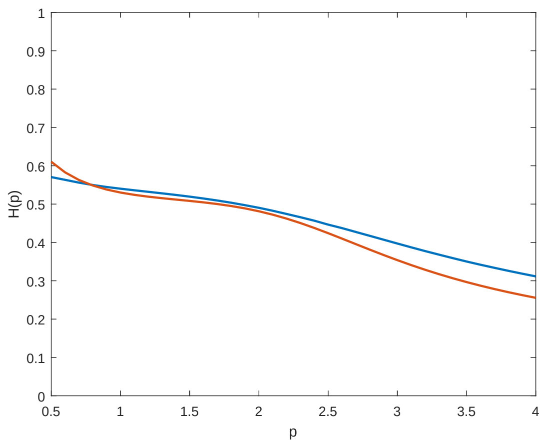

Figure 10: Hölder exponents $H(p)$ of Microsoft returns for absolute moments of order p, estimated by log-log regression (blue) and generalized Hurst exponent (red)

4. Estimation

4.1. Mesofractality

A one-dimensional random variable $X \sim S(\alpha, \beta, \gamma, \delta)$ will be α-stable distributed with parameters $0 < \alpha \leq 2,\ -1 \leq \beta \leq 1,\ \gamma \geq 0$ and $\delta \in \mathbb{R}$ [Nolan, 2017, Cizek et al., 2011], if

$$X \stackrel{\mathcal{L}}{=} \left\{ \begin{array}{r} \gamma Z + \delta,\ \alpha \neq 1 \\ \gamma Z + (\delta + \beta \frac{2}{\pi} \gamma \log \gamma),\ \alpha = 1 \end{array} \right\}, \tag{15}$$

where $S(Z \mid \alpha, \beta, 1, 0)$ represents the standard stable form. As only special cases of stable distributions are available as real-valued densities (Gaussian, Cauchy and Lévy), α-stable distributions are expressed as Fourier transforms of the characteristic function $\varphi_X(u)$.

$$S(X \mid \alpha, \beta, \gamma, \delta) = \frac{1}{2\pi} \int \varphi_X(u) - \exp(-iuX) du \tag{16}$$

The according characteristic function representation is given by

$$\log \varphi_X(u) = \left\{ \begin{array}{r} iu\delta - \gamma^\alpha |u|^\alpha \left\{1 + i\beta \tan\left(\frac{\alpha\pi}{2}\right)(\operatorname{sign} u)\right\},\ \alpha \neq 1 \\ iu\delta - \gamma |u| \left\{1 + i\beta \frac{2}{\pi} (\operatorname{sign} u) \log(|u|)\right\},\ \alpha = 1. \end{array} \right. \tag{17}$$

Scale invariance under addition implies that for the sum of α-stable variables $X_t \sim S(\alpha, \beta, \gamma, \delta)$, $t = 1, \ldots, T$

$$X_1 + X_2 + \ldots + X_T = \sum_{t=1}^{T} X_t = X \sim S(\alpha, \beta, T^{\frac{1}{\alpha}}\gamma, T\delta). \tag{18}$$

The characteristic function of X is consequently given by

$$T \log \varphi_X(u) = \begin{cases} iu(T\delta) - T\left(\gamma^\alpha\right)|u|^\alpha \left\{1 + i\beta \tan\left(\frac{\alpha\pi}{2}\right)(\operatorname{sign} u)\right\}, & \alpha \neq 1 \\ iu(T\delta) - T\gamma|u| \left\{1 + i\beta \frac{2}{\pi}(\operatorname{sign} u)\log(|u|)\right\}, & \alpha = 1. \end{cases} \tag{19}$$

According to Gnedenko and Kolmogorov [1954], the limiting distribution of T i.i.d. α-stable random variables, $0 < \alpha \leq 2$ is

$$a_T \left(\sum_{t=1}^{T} X_t\right) - b_T \xrightarrow{\mathcal{L}} S(\alpha, \beta, 1, 0), \tag{20}$$

where $a_T > 0$ and $b_T \in \mathbb{R}$. The special case of the Generalized Central Limit Theorem (GCLT) is the standard CLT for $\alpha = 2$, $\beta = 0$, $\gamma = \frac{\sigma}{\sqrt{2}}$ and $\delta = \mu$, given $a_T = \frac{1}{\sigma\sqrt{T}}$ and $b_T = \frac{\sqrt{T}\mu}{\sigma}$. In general, for $0 < \alpha \leq 2$,

$$T^{-\frac{1}{\alpha}} \sum_{t=1}^{T}(X_t - \delta) \xrightarrow{\mathcal{L}} S(\alpha, 0, \gamma, 0). \tag{21}$$

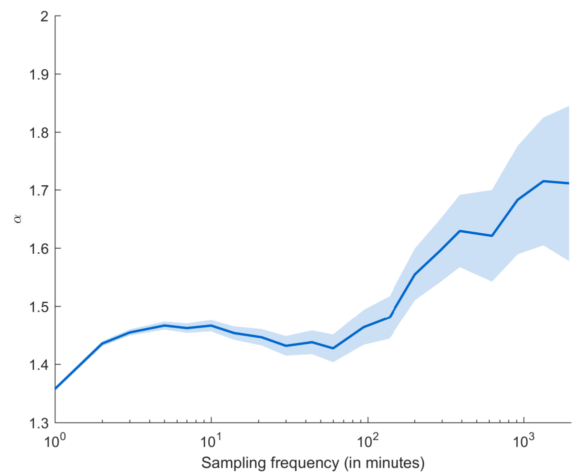

Figure 11: Stability index α over frequency

If the scaling exponent α would be constant over the sampling frequency, the sum of the higher frequency returns under α-stability, see equation (2), could be modelled under one specific stable distribution [Fama and Roll, 1968]. Figure 11 shows the MLE of the stability index α with respect to the sample frequency, including the 95% confidence intervals from the numerical Fisher information [Nolan, 2001]. For all sampling frequencies, the respective distributions are more leptokurtic than under Gaussianity ($\alpha = 2$) and more platykurtic than under the Cauchy assumption ($\alpha = 1$). In contrast to the analysis of higher moments, i.e. kurtosis, this class of Stable Paretian distributions $1 \leq \alpha \leq 2$ provides a well-defined framework in order to assess the tails of the distributions of different sampling frequencies. But, as argued in the context of scaling laws of subsection 3.2, varying distributions over sampling sequences are observed. Consequently, the mesofractal assumption of raw Microsoft returns has to be denied. If the distribution of the higher frequency returns X_t would be modelled under finite variance, such as generalized hyperbolic distributions with normal-inverse Gaussian (NIG) [Hartmann et al., 2010] or Student-t [Chen et al., 2010] as special cases, the horizon distribution, which is heavy tailed by empirical observation (see Figure 6), would be asymptotically Gaussian by the standard Central Limit Theorem.

4.2. Filter for seasonality and time-dependence

In order to examine the intraday seasonalities of Figure 8, we plot the absolute 1-minute returns over the course of the day in Figure 12. As the apparent convex shape, see also Engle and Sokalska [2012], is not covered by economic theory, the literature proposed to estimate these intraday seasonalities by universal function approximators. Whereas Giot [2005] models the intraday patterns with cubic splines, Andersen and Bollerslev [1997, 1998] use flexible fourier forms. We follow Andersen et al. [2003] and Engle and Sokalska [2012] by averaging the absolute returns over each minute $k = 1, \ldots, 390$ of the day.

$$s_k = \frac{1}{T} \sum_{t=1}^{T} |X_{t,k}| \quad k = 1, \ldots, 390 \tag{22}$$

After normalizing the raw one-minute returns by

$$Z_{t,k} = \frac{X_{t,k}}{\sqrt{s_k}} \quad k = 1, \ldots, 390 \tag{23}$$

the according lower frequencies are calculated and the sample autocorrelation functions of absolute returns are plotted in Figure 13. The ACFs of the deseasonalized absolute returns in Figure 14 indicates the hyperbolic decay which can be observed for sampling frequencies up to daily, see also Taylor [1986], Robinson

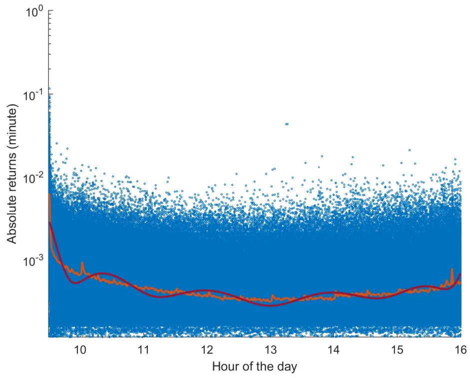

Figure 12: Absolute returns (minute) with average (orange) over hour of the day and polynomial fit (red)

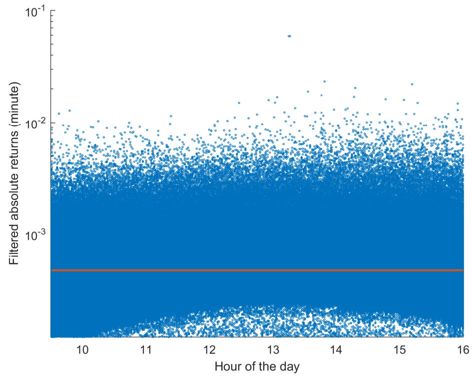

Figure 13: Filtered absolute returns (minute) with average (orange) over hour of the day

[1991], Ding et al. [1993] and Andersen and Bollerslev [1997]. Accordingly we model the observed phenomena of heavy tails, seasonality and long-range dependence by employing the FIGARCH methodology [Baillie et al., 1996] for seasonally filtered returns [Andersen and Bollerslev, 1998, Muller et al., 1990] under stable laws [Paolella et al., 2002] in order to explain the instability in distributions, as observed in subsection 3.2.

The Stable paretian power GARCH process $S^{\delta}_{\alpha,\beta}\text{GARCH}(r,s)$ with seasonality is given by

$$X_t = \delta + s_k \gamma_t \varepsilon_t, \;\; \varepsilon_t \sim S_{\alpha,\beta}(1,0) \tag{24}$$

$$\gamma_t^{\delta} = \theta_0 + \sum_{i=1}^{r} \theta_i |\varepsilon_{t-i}|^{\delta} + \sum_{j=1}^{s} \phi_j \gamma_{t-j}^{\delta} \tag{25}$$

$$= \theta_0 + \theta(L)\varepsilon_t^{\delta} + \phi(L)\gamma_t^{\delta}. \tag{26}$$

The according GARCH equation can be rewritten in lag polynomial form:

$$\{1 - \theta(L) - \phi(L)\}\, \varepsilon_t^{\delta} = \theta_0 + \{1 - \phi(L)\} \left\{\varepsilon_t^{\delta} - \gamma_t^{\delta}\right\}. \tag{27}$$

In order to allow for a slower decay than exponential, the fractional difference operator $(1-L)^d, \; 0 < d < 1$ is introduced, obtaining the fractionally integrated GARCH (FIGARCH) equation:

$$\{1 - \theta(L) - \phi(L)\}\,(1-L)^d \varepsilon_t^{\delta} = \theta_0 + \{1 - \phi(L)\} \left\{\varepsilon_t^{\delta} - \gamma_t^{\delta}\right\}. \tag{28}$$

In contrast to the special cases of GARCH ($d = 0$) and IGARCH ($d = 1$), Davidson [2004] shows that this class of processes is able to reproduce more flexible temporal dependencies, i.e. long memory. As GARCH processes are modelled separately for each sampling frequency Mandelbrot et al. [1997] argues that this family of fractionally integrated models is neither self-affine nor scale consistent. Still, Fisher et al. [1997] find evidence that the class of FIGARCH-models can mimic multifractality.

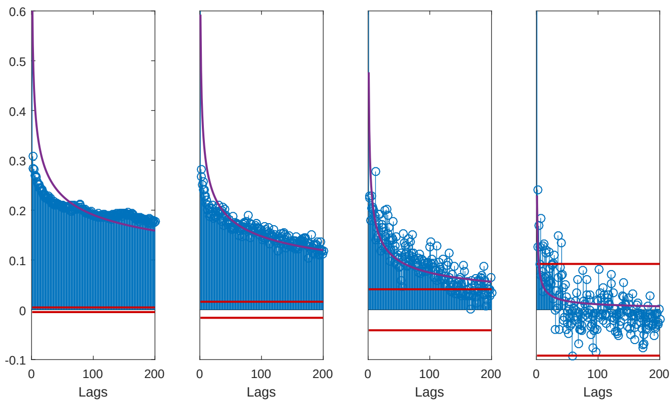

Figure 14: Sample autocorrelation function with hyperbolic fit for 5-minute, hourly, daily and weekly frequency

4.3. Frequency rescaling

Figure 15 shows the ML estimates of the stability parameter α after deseasonalizing with (green) and without (blue) FIGARCH(1,1) filter. Comparable to Figure 11, deseasonalizing price increments without FIGARCH(1,1) filter gives again evidence for different generating distributions for different sampling frequencies. After accounting for temporal dependence via FIGARCH(1,1) filter, the stability parameter remains to be constant for sampling frequencies larger than five minutes. The specific parameters of the FIGARCH(1,1) models for the different sampling frequencies are available on request. Here, the increase of the stability index with decreasing sampling frequency can be explained to a large extent by intraday seasonality and time dependence. For higher frequencies than five minutes, microstructure effects lead to a overestimated deviations and hence a smaller stability index [Zumbach et al., 2002, Chaboud et al., 2010].

The result is supported from the perspective of scaling laws, see equation (14). The observation of multifractality is not evident in the residuals of the FIGARCH(1,1) model. Figure 16 plots Hölder exponents, which remain to be constant over the residuals moments of order p. We have shown that the filtered log-returns give evidence for mesofractality,

$$\varepsilon_t \sim S_{\gamma,\delta}(\alpha, \beta) \tag{29}$$

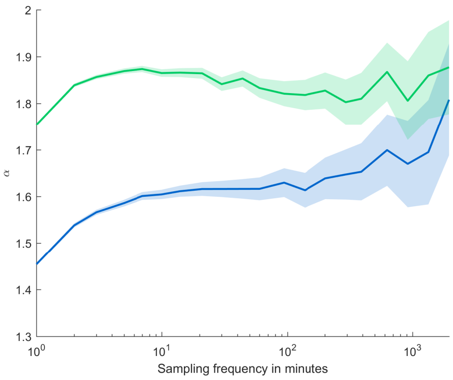

Figure 15: Stability parameter of innovations ε_t over frequency with (blue) and without (green) FIGARCH filter

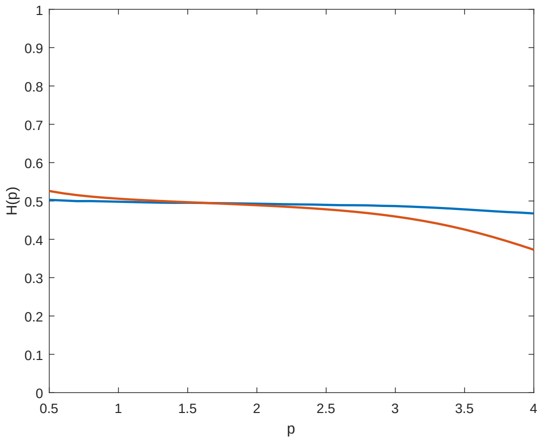

Figure 16: Hölder exponents $H(p)$ of FIGARCH-residuals for absolute moments of order p, estimated by log-log regression (blue) and generalized Hurst exponent (red)

for all sampling frequencies larger than five minutes, which makes it possible to rescale between different frequencies under α-stability, see equation (18). The proposed semiparametric rescaling method rests on the higher frequency data-set itself, but uses the α-stable assumption beneath:

i. Filter higher frequency returns X for intraday seasonality and time dependence.
   - Let higher frequency returns $X \sim S(\alpha_X, \beta_X, \gamma_X, \delta_X)$
   - Let higher frequency residuals $\varepsilon_X \sim S(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, \delta_\varepsilon)$
   - Let lower frequency returns $Y \sim S(\alpha_Y, \beta_Y, \gamma_Y, \delta_Y)$

ii. Evaluate if $\alpha_\varepsilon = \alpha_Y \stackrel{\text{def}}{=} \alpha$, $\beta_\varepsilon = \beta_Y \stackrel{\text{def}}{=} \beta$

iii. Normalize higher frequency residuals (for auxiliary models)

$$Z = \frac{\varepsilon_X - \delta_\varepsilon}{\gamma_\varepsilon}, \ Z \sim S(\alpha, \beta, 1, 0) \tag{30}$$

iv. Rescale normalized residuals for lower frequency with drift $\delta_{\tilde{Y}} = T\delta_X$ and scale $\gamma_{\tilde{Y}} = T^{\frac{1}{\alpha_X}} \gamma_X$.

$$\tilde{Y} = \delta_{\tilde{Y}} + \gamma_{\tilde{Y}} Z, \ \tilde{Y} \sim S(\alpha, \beta, \gamma_{\tilde{Y}}, \delta_{\tilde{Y}}) \tag{31}$$

v. Estimate the risk measure from the nonparametric, rescaled distribution $F_{\tilde{Y}}$, see equation (3)

$$S_\phi(\tilde{Y}) = \int_0^1 \phi(x) F_{\tilde{Y}}^{-1}(x) dx \tag{32}$$

By obtaining a lower-frequency distribution from high-frequency data, the problem of insufficient data points from high-confidence risk measures of section 2.2 is addressed. For the aimed weekly frequency, we rescale 5-minute returns to the weekly frequency. For aimed frequencies higher than weekly, the time-dependence structure of the lower frequency would have to be included. Figure 17 compares the empirical and the frequency-rescaled distributions representing 471 weeks and 184391 rescaled weeks respectively. The bootstrapped confidence intervals for the kernel density estimates are smaller under frequency rescaling as increasingly more data points are being used. Albeit coming from the same data-set, the semi-parametric method allows for sampling positive and negative events, which never occurred in the original weekly data history.

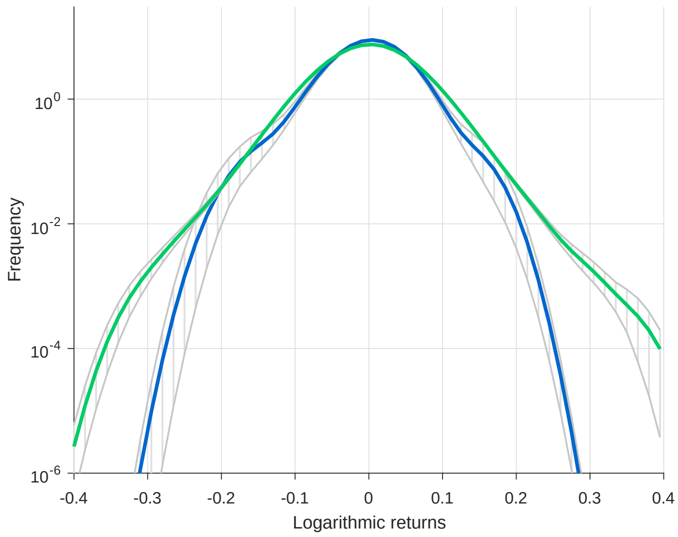

Figure 17: Empirical (blue) and frequency rescaled (green) kernel density estimates of weekly return distribution with bootstrapped 95% confidence intervals

5. Implementation: VaR

5.1. Backtest methodology

Due to transaction costs investors and other stakeholders are not rebalancing their portfolios at high-frequencies. Consequently institutional investors and also regulators are naturally interested in low frequency risk measures at weekly, quarterly or annual time scale. Here we focus on the 5-day "minimum liquidity horizon" to guarantee a sufficient amount of data, e.g. for the backtest [BIS, 2017]. With the spectral risk measure (3) the exceedance process $I_t,\ t = 1, \ldots, T$ is defined as

$$I_t = \begin{cases} 0, & X_t > S_{\phi_\alpha}() \\ 1, & X_t \leq S_{\phi_\alpha}(). \end{cases} \tag{33}$$

Given i.i.d. data X_t, the unconditional coverage is given by

$$\mathsf{E}\left(I_t\right) = \tau. \tag{34}$$

Statistical coverage tests by Kupiec [1995] and Christoffersen [1998] utilize that under such an i.i.d. assumption $I_t \sim \text{Bern}(\tau)$ and accordingly $\sum_{t=1}^{T} I_t \sim \text{Bin}(T+1, 1-\tau)$. A test statistics to evaluate

$$H_0 : \hat{\tau} = \tau \text{ vs.} \tag{35}$$

$$H_1 : \hat{\tau} \neq \tau. \tag{36}$$

is easily constructed. Alternatively, BIS [1996] propose to use a traffic light approach, which we will follow here. We are going to compare the proposed frequency rescaling method (FRM) of section 4.3 with the VaR forecasts of the following models for the weekly sampling frequency: Under the independence assumption Gaussian, Student-t, Stable and nonparametric VaR estimates are directly calculated from the according ML fits given weekly data. Accounting for time-dependence in the data (see Figure 8), GARCH(1,1) and FIGARCH(1,1) risk estimates are obtained for weekly and 10-minute data. Additionally, the VaR forecasts for the realized GARCH(1,1) and realized FIGARCH(1,1) are included.

As argued in section 2, 471 observations from the weekly frequency do not suffice to construct reliable VaR estimates, especially for large confidence levels. This holds for the backtest of the weekly VaR. Given $471-h,\ h \in \mathbb{N}^+$ testing weeks, where $h = 41$ is the number of periods used to initially estimate the VaR levels, the exceedance probability cannot be sufficiently estimated for large confidence levels. But as empirical data remain the only viable foundation for the backtest, the in-sample and out-of-sample exceedances (in%) are given in Table 3 and 4 for confidence levels 95%, 99% and 99.9%. The unconditional coverage holds for all respective confidence levels, in- and out-of-sample, for the 3) Stable, 6) Weekly FIGARCH and 11) Frequency Rescaling models.

Among the models satisfying the theoretically presumed confidence levels, a smaller level of VaR is beneficial for banks, insurances and other financial institutions. Accordingly Figure 18 and 19 show in- and out-of-sample median VaR over time with 95% confidence bounds for confidence levels 95% (green), 99% (red)

Model	$\tau = 5\%$		$\tau = 1\%$		$\tau = 0.1\%$	
	# Exc	$\hat{\tau}$ in %	# Exc	$\hat{\tau}$ in %	# Exc	$\hat{\tau}$ in %
1) Gaussian	23	5.42	6	1.42	4	0.94
2) Student-t	24	5.66	6	1.42	0	0
3) Stable	23	5.42	3	0.71	0	0
4) Nonparametric	22	5.19	4	0.94	0	0
5) Weekly GARCH(1,1)	12	2.83	3	0.71	0	0
6) Weekly FIGARCH(1,1)	11	2.59	5	1.18	0	0
7) 10-min GARCH(1,1)	18	4.25	4	0.94	1	0.24
8) 10-min FIGARCH(1,1)	21	4.95	5	1.18	1	0.24
9) Realized GARCH(1,1)	20	4.72	4	0.94	0	0
10) Realized FIGARCH(1,1)	34	8.02	6	1.42	0	0
11) FRM	10	2.36	4	0.94	0	0

Table 3: : In-sample weekly Value at Risk exceedances with according probabilities for given confidence levels 95%, 99% and 99.9%

and 99.9% (blue). Whereas the Stable models holds the unconditional coverage by exhibiting the largest VaR of all tested models, the weekly FIGARCH models ensures coverage by time-varying VaR forecasts, resulting, on average, in smaller VaR forecasts. Although VaR forecasts from the frequency rescaling method (FRM) are comparable to the weekly FIGARCH models in terms of size, the FRM has the advantage of producing time-constant VaR in-sample forecasts. Narrower confidence bounds for the VaR forecasts are supported out-of-sample. Out of the VaR models presented, the FRM is presented to be highly beneficial for institutional investors dealing with lower sampling frequencies such as weekly, quarterly and annual. Holding the unconditional coverage at low values of VaR, in contrast to the Lévy stable motion, the VaR forecasts are not time-dependent as in the GARCH models. Moreover portfolio balance sheets should not only report VaR levels, but also hedge respective risks, which reduces portfolio turnovers and transaction costs. In comparison to the nonparametric VaR estimation from weekly data, the FRM utilizes a semiparametric estimation procedure to scale high-frequency data to the lower frequency, inducing tail events, which never happened in the original data history.

6. Conclusion

Estimating risk in a low-frequency context and limited data is difficult for large confidence levels as relevant weekly, quarterly or annual data capture an insufficient history or are not relevant for the risk profile of the financial institution. Utilizing high-frequency data for the estimation of low-frequency risk measures

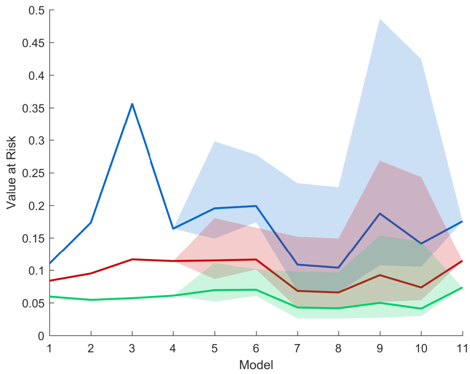

Figure 18: In-sample median Value at Risk over time with 95% confidence bounds for confidence levels 95% (green), 99% (red) and 99.9% (blue)

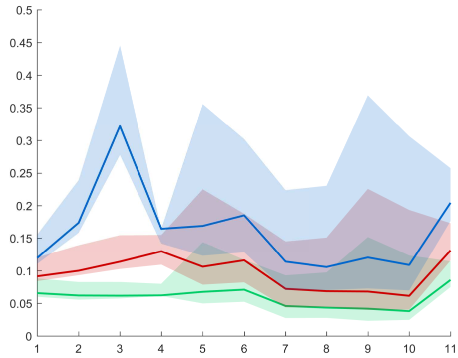

Figure 19: Out-of-sample median Value at Risk over time with 95% confidence bounds for confidence levels 95% (green), 99% (red) and 99.9% (blue)

Model	$\tau = 5\%$		$\tau = 1\%$		$\tau = 0.1\%$	
	# Exc	$\hat{\tau}$ in %	# Exc	$\hat{\tau}$ in %	# Exc	$\hat{\tau}$ in %
1) Gaussian	11	2.59	6	1.42	4	0.94
2) Student-t	13	3.07	6	1.42	0	0
3) Stable	14	3.30	4	0.94	0	0
4) Nonparametric	15	3.54	4	0.94	2	0.47
5) Weekly GARCH(1,1)	10	2.36	4	0.94	1	0.24
6) Weekly FIGARCH(1,1)	13	3.07	4	0.94	0	0
7) 10-min GARCH(1,1)	28	6.60	7	1.65	3	0.71
8) 10-min FIGARCH(1,1)	30	7.08	8	1.89	4	0.94
9) Realized GARCH(1,1)	36	8.49	16	3.77	4	0.94
10) Realized FIGARCH(1,1)	37	8.73	14	3.30	4	0.94
11) FRM	7	1.65	3	0.71	0	0

Table 4: : Out-of-sample weekly Value at Risk exceedances with according probabilities for given confidence levels 95%, 99% and 99.9%

can provide significantly more data points, given that a fractal structure could be verified among different sampling frequencies. The proposed method incorporates a specific form of rescaling, involving filters for seasonality and time dependence in the α-stable framework. In contrast, given finite variance and the classic CLT, Gaussianity cannot be circumvented for long holding periods, which is not supported empirically.

In an i.i.d. simulation study we indicate that ten years of data barely suffice in order to efficiently estimate weekly VaR of a stock market portfolio for a confidence level of 99.9% and larger. For the relevant confidence levels, the authors recommend to use at least 10^3 observations for the 99% confidence, 10^4 observations for the 99.9% confidence and 10^5 observations for the 99.99% confidence level. As deficiencies through underestimation of VaR hold for relevant i.i.d. distributions such as Student-t or Stable, the predicament unfolds in aggravated form for Expected Shortfall under time-dependent processes.

The high-frequency data-set of Microsoft gives evidence for different generating distributions over the sampling frequencies. In order to obtain events for the lower sampling frequencies, we show that by filtering high-frequency returns for seasonality and long-range dependence, the mesofractal assumption (Lévy stable motion) cannot be denied for the residuals of sampling frequencies larger than five minutes. Within the α-stable framework and the underlying fractal structure, 5-min filtered log-returns are scaled to the weekly minimum liquidity horizon. The obtained weekly data points cover tail events which never existed in the original data history, which improve the estimation of the respective risk measures.

Empirically, the backtest reveals that the frequency rescaling method holds the unconditional coverage or all confidence levels reported, in- and out-of sample. Given all models which hold the unconditional coverage, the VaR from the frequency rescaling method are the smallest over time, involving no large deviations given the underlying iid assumptions for the weekly frequency.

In subsequent research we are going to extent the frequency rescaling method (FRM) to the multidimensional case. The open question remains to be if the data series we observe are just a preasymptotic snapshot of a fractional Lévy stable motion [Samorodnitsky and Taqqu, 1994], in which the Hölder exponents have converged. In that case the only relevant filter for the higher frequencies remains to be microstructure noise and seasonality, which are generated due to market makers and cyclical business components. If so, the diverging behavior within the sampling frequencies of the process is merely a relic of lacking data points in the lower sampling frequencies.

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