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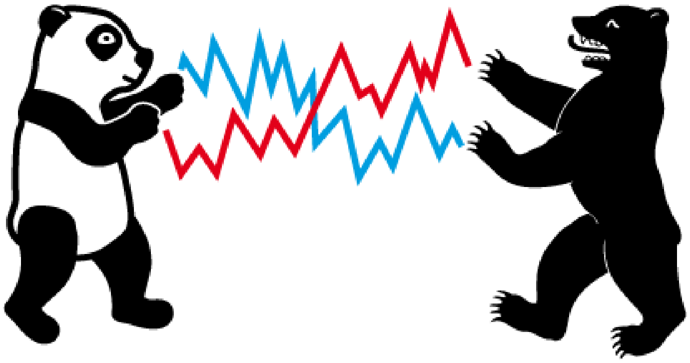

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Abstract This paper provides a detailed framework for modeling portfolios, achieving the highest growth rate under subjective risk constraints such as Value at Risk (VaR) in the presence of stable laws. Although the maximization of the expected logarithm of wealth induces outperforming any other significantly different strategy, the Kelly Criterion implies larger bets than a risk-averse investor would accept. Restricting the Kelly optimization by spectral risk measures, the authors provide a generalized mapping for different measures of growth and security. Analyzing over 30 years of S&P 500 returns for different sampling frequencies, the authors find evidence for leptokurtic behavior for all respective sampling frequencies. Given that lower sampling frequencies imply a smaller number of data points, this paper argues in favor of α-stable laws and its scaling behavior to model financial market returns for a given horizon in an i.i.d. world. Instead of simulating from the class of elliptically stable distributions, a nonparametric scaling approximation, based on the data-set itself, is proposed. Our paper also uncovers that including long put options into the portfolio optimization, improves the growth criterion for a given security level, leading to a new Kelly portfolio providing the highest geometric mean.

Keywords: growth-optimal, Kelly criterion, protective put, portfolio optimization, stable distribution, Value at Risk
JEL Classification: C13, C46, C61, C73, G11

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Given a set of investment opportunities, how should the investment weights be chosen in order to have more wealth than any other investor at the end of the investment period? The Kelly growth-optimum strategy is a betting scheme for an investor, who seeks to asymptotically maximize his growth rate of capital. This strategy outperforms any other significantly different strategy, given knowledge of the true underlying process (Breiman, 1961). But, the sole use of the Kelly Criterion implies larger bets than a representative, risk-averse investor would accept in terms of risk (Clark and Ziemba, 1987; Hausch and Ziemba, 1985). Thus, the Kelly optimization needs to be restricted by a security measure. We use stable laws and its scaling behavior in order to model the underlying financial market returns. Upon the Generalized Central Limit Theorem (GCLT), the horizon distribution is modelled in an discrete i.i.d. framework.

The aim is to maximize the geometric portfolio return, i.e. Kelly Criterion and restrict the objective to a subjective risk constraint, formulated as spectral risk measure, including quantile (VaR) or Expected Shortfall as special cases. The formulated trade-off introduces a mapping over growth and security in order to evaluate the investment decision. The contribution of this paper is three-fold: The first contribution represents the application of multidimensional stable laws, in the form of elliptically stable distributions, to the constrained Kelly portfolio. Second, instead of simulating from the class of elliptically stable distributions, a nonparametric scaling approximation, based on the data set itself, is proposed. Third, assets with non-linear payoff structure, long put-options, are incorporated into the nonlinear optimization to allow for asymmetric payoffs, which lead to a higher growth criterion, given a fixed security constraint.

The Kelly Criterion originates from Kelly (1956), dealing with, from the point of information theory, an optimal investment strategy in a binary channel. Breiman (1961) formally proves the asymptotic outperformance of the Kelly strategy for arbitrary distributions in an i.i.d. world. For arbitrarily distributed, possibly non-stationary processes, those results have been extended by Algeot and Cover (1988). Incorporating security measures into the Kelly optimization, MacLean, Ziemba, and Blazenko (1992) discuss the growth-security trade-off in terms of efficiency. Roll (1973) compares the Markowitz arithmetic mean maximization with the Kelly geometric mean maximization. In contrast to Constant Proportion Portfolio Insurance (CPPI), the investment strategy remains fixed fraction, given stationarity.

The distribution of financial market returns for a chosen horizon is modelled as the sum of daily random variables. As the distribution in some horizon is presumed to be non-Gaussian, the classical Central Limit Theorem (CLT) does not apply as second and higher moments may not exist. Thus, the generalized central limit theorem (GCLT) of Gnedenko and Kolmogorov (1954) is applied for the sum of random variables, whose second and higher moments may not be bounded. For the financial application this implies the use of stable laws (Fama, 1965; Lévy, 1925; Mandelbrot, 1963). As multidimensional stable random variables are difficult to evaluate for larger dimensions, elliptical sta-

ble distributions are employed, allowing for efficient portfolio estimation for dimensions $k \leq 40$ (Nolan, 2013) in the presence of linear dependence.
Price data, both for assets with linear and non-linear payoff structure, were gathered from Bloomberg. For computation, Matlab 2016a was utilized. In order to solve the formulated nonlinear optimization problem the sequential quadratic algorithm in *fmincon* was employed. If a figure containts a link below, the Matlab-code can be obtained in the form of a quantlet.
The paper is organized as follows: In chapter one the portfolio allocation problem is stated. The financial model is formulated by using generalized measures for growth and security. Chapter two, the estimation, starts with a case for non-Gaussianity of financial log-returns of different sampling frequencies, reasoning the utilization of stable laws. For the multidimensional case, elliptically stable distributions are introduced in order have an analytically tractable class of distributions. As the nonparametric scaling approximation is introduced, the estimation of location and scale is illustrated. An application is given in chapter three, the implementation. For a representative investor with a horizon of one year and a quantile-constraint for his wealth return distribution, the optimally constrained Kelly portfolio is found, benefitting from the protective put strategy.

1 Model

1.1 Portfolio allocation problem

Given initial wealth of the investor $W_0 \in \mathbb{R}^+$, there are $j = 1, ..., k$ investment opportunities with fractions $f_t = [f_{1,t}, \dots, f_{k,t}]^\top \in \mathbb{R}^k$ in period $t = 1, 2, \dots, T$. $T \in \mathbb{N}^+$ represents the planning horizon. Assessing solely self-financing strategies, the budget constraint is given by $\sum_{j=1}^k f_{j,t} \leq 1$. As continuous returns $X_t \in \mathbb{R}^k$ are time additive, but not additive over assets, discrete returns are calculated by

$$\tilde{X}_t = \exp\{X_t\} - 1. \tag{1}$$

Given outcomes in $t-1, \dots, T-1$ the wealth in T is given by

$$\begin{aligned} W_T(f_t) &= W_0 \prod_{t=1}^{T} \left\{ 1 + \sum_{j=1}^{k} f_{j,t} \tilde{X}_{j,t} \right\} \\ &= W_0 \prod_{t=1}^{T} \left\{ 1 + f_t^\top \tilde{X}_t \right\}. \end{aligned} \tag{2}$$

Given the stochastic wealth process, measures for growth and security are formulated in order to choose investment fractions f_t, which suit investor preferences.

For a cdf $F_{W_T}(x)$ the spectral risk/growth measure with weight function $\phi(x)$ is defined through the quantile function $F_{W_T}^{-1}(x) \stackrel{\text{def}}{=} \{x : \mathrm{P}\left(W_T(f_t) \leq x\right) = \alpha\}$, $\alpha \in (0,1)$.

$$M_\phi \left\{W_T(f_t)\right\} = \int_0^1 \phi(x) F_{W_T}^{-1}(x) \mathrm{d}x \tag{3}$$

Within the context of spectral risk measures, the measure will be coherent iff the weight function is positive $\phi(x) \geq 0$, increasing $\phi'(x) \geq 0$ and normalized $\int_0^1 \phi(x) = 1$ (Acerbi, 2002). For the discrete framework (2) with $n \in \mathbb{N}^+$ wealth trajectories, the measure is defined as

$$M_\phi \left\{W_T(f_t)\right\} = \sum_{i=1}^{n} \phi_i W_{T,i}(f_t), \tag{4}$$

where $W_{T,i}$ denotes element i out of n wealth paths with according weight ϕ_i.

Growth measures

Following Roll (1973), there are two main strands dealing with the accumulation of wealth and thus, the allocation of wealth into a portfolio. On the one hand, the Markowitz optimization aims to maximize the expected portfolio return (Lintner, 1965; Markowitz, 1952; Sharpe, 1964; Tobin, 1958). On the other hand, the Kelly growth-optimum approach by Kelly (1956), Breiman (1961) and Thorp (1971), aims to maximize the expected logarithm of wealth, which is equivalent to maximizing the geometric portfolio return. Within the framework of spectral growth/risk measures, the growth measures for the Markowitz and the Kelly optimization are evaluated:

– G_1 : For the expected wealth, the growth criterion from the Markowitz optimization, the weight function is

$$\phi_E(x) = 1,$$

giving

$$G_{\phi_E} \left\{W_T(f_t)\right\} = \int_0^1 F_{W_T}^{-1}(x) \mathrm{d}x = \mathsf{E}\left\{W_T(f_t)\right\}. \tag{5}$$

– G_2 : The expected logarithm of wealth, representing the optimization criterion for the Kelly strategy, is obtained for the weight function

$$\phi_{\mathrm{Elog}}(x) = \log(x),$$

giving

$$G_{\phi_{\mathrm{Elog}}} \left\{W_T(f_t)\right\} = \int_0^1 \log F_{W_T}^{-1}(x) \mathrm{d}x = \mathsf{E}\left\{\log W_T(f_t)\right\}. \tag{6}$$

The growth measure will be denoted by $G_\phi\left\{W_T(f_t)\right\}$ and the optimization for horizon T without risk constraints is formulated as

$$\max_{f_t\in\mathbb{R}^k}\left[G_\phi\left\{W_T(f_t)\right\}\,\middle|\,\sum_{j=1}^{k} f_{j,t}\leq 1\right]. \tag{7}$$

This paper focusses on the Kelly growth criterion as it represents a betting scheme for an investor, who seeks to asymptotically maximize his growth rate of capital. The betting strategy outperforms any other significantly different strategy asymptotically and minimizes the expected time to reach a goal (Algeot and Cover, 1988; Breiman, 1961). For a comprehensive treatment of the Kelly Criterion, see MacLean, Thorp, and Ziemba (2011). Whereas the maximization of the expected wealth in the Markowitz optimization, given favorable investment possibilities, always implies betting the entire fortune, the maximization of the expected logarithm of wealth leads to one growth-optimal portfolio, which is not necessarily optimal in terms of the Markowitz portfolio (Thorp, 1971). The log-optimal strategy is fixed fraction, independent of time (MacLean, Ziemba, and Blazenko, 1992).

Security measures

The sole use of the Kelly Criterion implies larger bets than a representative, risk-averse investor would accept in terms of risk (Clark and Ziemba, 1987; Hausch and Ziemba, 1985). In order to formulate individual security measures for different investors, the spectral risk measure from (3), denoted by $S_\phi\left\{W_T(f_t)\right\}$, will be used. Two specific risk measures to include the degree of risk-aversion into the portfolio optimization are quantile (VaR) and expected shortfall (ES) constraints:

- S_1 : The quantile constraint (VaR) is a special case of the spectral risk measure from (3)

$$\phi_{\mathrm{Q}_\alpha}(x)=\delta(x=\alpha),\;\;\alpha\in(0,1), \tag{8}$$

  where $\delta(x=\alpha)$ is the Dirac delta function, well known to be a non-coherent risk measure. Further drawbacks of the quantile constraint are treated in Basak and Shapiro (2001). However, the quantile restriction allows to ask the investor specifically to name a fraction of his wealth he can accept to lose with probability $1-\alpha$.
- S_2 : In contrast, Expected Shortfall is a coherent risk measure representing the average loss beyond a given quantile constraint. Being a special case of the spectral measure, the weight function is given as

$$\phi_{\mathrm{ES}_\alpha}(x)=\alpha^{-1}\mathbf{1}(x<\alpha). \tag{9}$$

Growth-security frontier

Following MacLean, Ziemba, and Blazenko (1992), the possible combinations of growth and security measures are given by the set

$$U = \left[G_\phi \left\{W_T(f_t)\right\}, S_\phi \left\{W_T(f_t)\right\}\right], f_t \text{ feasible.} \tag{10}$$

The growth-security frontier is accordingly formulated as

$$U_t^* = \left[G_\phi \left\{W_T(f_t^*)\right\}, S_\phi \left\{W_T(f_t^*)\right\}\right], f_t^* \text{ feasible,} \tag{11}$$

where the $f_t^* \in \mathbb{R}^k$ is the investment fraction maximizing the growth measure under security restriction.

$$\begin{aligned} f_t^* = \underset{f_t^* \in \mathbb{R}^k}{\arg\max}\ & G_\phi \left\{W_T(f_t)\right\} \\ \text{s.t. } & S_\phi \left\{W_T(f_t)\right\} \leq b,\ b \in \mathbb{R}, \\ & \sum_{j=1}^{k} f_{j,t} \leq 1 \end{aligned} \tag{12}$$

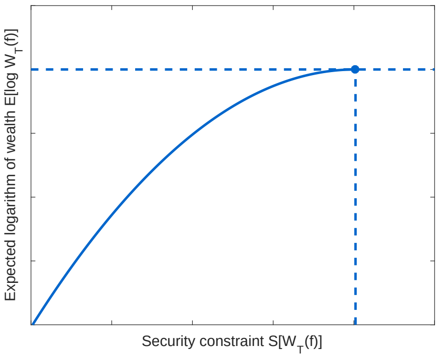

Fig. 1: Kelly-security frontier

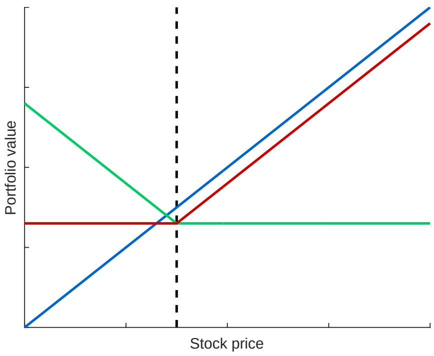

Fig. 2: Protective put strategy

For the Kelly Criterion with a security constraint as proposed, the frontier is illustratively visualized in figure 1. In contrast to the Markowitz maximization, implying a steady tradeoff between mean and security, the geometric mean maximization implies one specific portfolio - the Kelly portfolio - exhibiting the highest geometric mean possible (horizontal dotted line). From this viewpoint, portfolios exhibiting a larger security constraint than the Kelly portfolio (to the right of the vertical dotted line) are not efficient. If the investor prefers a smaller security constraint than the full Kelly investor, restricted Kelly portfolios (solid line) constitute the Kelly-security frontier. These are portfolio strategies with the highest growth criterion given security constraint.

1.2 Tail constraints and non-linear instruments

The introduction of assets as nonlinear functions of the underlyings, derivatives, allows for controlling the asymmetry of the wealth distribution in such a way, that it will be skewed to the left. Albeit the distribution of the risk measure, the loss of the portfolio is limited by construction for high confidence levels. The main instrument utilized to achieve this asymmetric payoff profile are long put options. By construction, corridor options, as argued in the context of quantile constraints, are circumvented (Basak and Shapiro, 2001). A simplified representation of the protective put strategy is given in figure 2, consisting of one stock (blue) and one long put option (green) with chosen strike (dotted black). The result is the protective put strategy (red). The difference in payoff above the strike level is due to the put price, which the option holder has to pay. For k linear assets with multiple put options each, given a pre-specified horizon, the choice of the fraction of linear and nonlinear assets is not obvious.

2 Estimation

2.1 A case for non-Gaussianity

Although Fama (1965) finds evidence for α-stable characteristics for all returns of the Dow Jones Index, it can be observed that financial (log-)returns tend to the Gaussian distribution as the sampling frequency decreases, see also McFarland, Pettit, and Sung (1982), Boothe and Glassman (1987), and Dacorogna, Gencay, Muller, Olsen, and Pictet (2001). The subsequent textbook example for the Standard and Poor's 500 reads as table 1. Due to the 2009 financial crisis, an outlier week of -60% increases (decreases) the sample kurtosis (skewness) for the weekly frequency significantly from 13.67 (-1.27) to 131.09 (-6.7). If the outlier week is omitted, see column S&P (weekly*) of table 1, the general observation of decreasing kurtosis and increasing negative skewness is supported for *different* sample sizes. Still, letting the data speak, erratic behavior of sample moments definitely appears even for this reference data series.

Descriptives	S&P (daily)	S&P (weekly)	S&P (weekly*)	S&P (monthly)	S&P (yearly)
Data points	7564	1513	1512	360	30
Mean (in %)	0.03	0.17	0.21	0.70	8.37
Std (in %)	1.16	2.96	2.51	5.03	16.60
Skewness	−1.29	−6.70	−1.27	−1.87	−1.78
Kurtosis	31.26	131.09	13.67	12.26	7.13

Table 1: Log-return descriptives for the different sampling frequencies, S&P 500 1985-2015 Frequency weekly* omits one week in the financial crisis 2009

The empirical observation of Gaussian convergence for lowering sampling frequencies cannot be shown explicitly by existing data, as data-records capture only 7564 trading days, representing 30 years of data. The empirical verification would require an appropriately large number of weeks, months and years. In order to emphasize the lack of data points for the annual sampling frequency, we randomly sample 10^5 30 subsequent daily returns from the S&P 500 and calculate second, third and fourth moments in order to evaluate dispersion, skewness and leptokurtic behavior. Hence, for the three moments, 10^5 estimators are plotted as histogram in figure 3. The vertical red lines represent the moment estimators for the whole daily data series. In essence, dispersion, skewness and especially leptokurtic behavior of the subsets are significantly biased, compared to the estimator of the whole series, as fewer sampled data-points imply less probability of sampling data in the tails of the return distribution. There was not one out of 10^5 30 day sub-samples, which resulted in a comparable kurtosis of the complete data-series. The result holds for sampling 30 separate days randomly under the i.i.d. assumption. Moments of order larger than one behave erratically over an increasing data sample, as first analyzed for commodity prices in Mandelbrot (1963). Figure 4 plots standard deviation (in %), skewness and kurtosis as function of the used data-points of the series. The red lines represents the empirical moment behavior with increasing daily data points. The blue lines represent 100 trajectories of Gaussian moments with increasing data points. The observation of erratic moment behavior stands in contrast to Gaussian behavior. The observation holds over sampling frequencies daily, weekly, monthly and annually. This specific sample-size problem is crucial in risk management, especially for estimating the quantile of the wealth distribution as in the constrained portfolio optimization in (12). As the confidence level tends to one, having only a limited amount of data, the quantile estimate is systematically biased as the quantile is overestimated. The portfolio analyst has to evaluate if the estimated quantile given the chosen confidence level still has an acceptable distribution. Consequently, for investors with longer investment horizons, such as a year, the sum of daily random variables, constituting the yearly distribution, should not converge to the Gaussian, but to a heavy-tailed distribution, which will turn out to be the class of α-stable distributions. For financial markets, this assumption will imply infinite variance, skewness and kurtosis, leading to non-converging moments, i.e. the observed erratic behavior. The model of chapter 1 will be estimated within a stationary framework for elliptically stable distributions, striving for scale invariance. Although daily and higher frequency returns exhibit non-stationary characteristics, the horizon distribution, i.e. yearly, cannot be shown to exhibit significant volatility clustering.

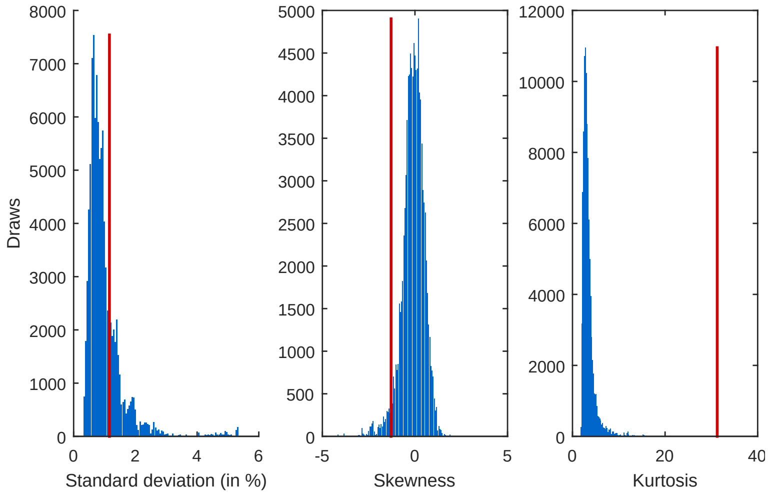

Fig. 3: Whole sample (red) and bootstrapped standard deviations (p.a.), skewness and kurtosis for 10^5 draws of 30 subsequent daily returns (blue) from the S&P 500, 1985 to 2015

Stable_Kelly_MomentDistribution

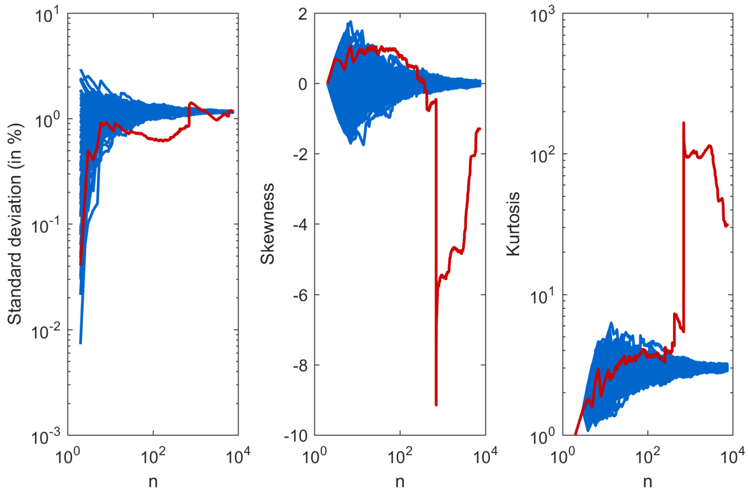

Fig. 4: (Log-)Log plots of standard deviation (in %), skewness and kurtosis with increasing data points, S&P 500 from 1985 to 2015 (red) and 100 Gaussian simulations with S&P 500 moments (blue)

Stable_Kelly_MomentIncrease

2.2 Scale invariance

Let $X_t \in \mathbb{R}^k$ be a multidimensional, i.i.d. random variable from distribution P^t, where t indicates the scale e.g. days. Given the investment horizon of the investor, T days, the wealth equation of (2)

$$W_T(f_t) = W_0 \prod_{t=1}^{T} \left\{1 + f_t^\top \tilde{X}_t\right\} = W_0 \prod_{t=1}^{T} \left\{f_t^\top \exp(X_t)\right\} \tag{13}$$

can be simplified, given $f_t = f \ \forall t = 0, \ldots, T$.

$$W_T(f) = W_0 \left\{f^\top \exp\left(\sum_{t=1}^{T} X_t\right)\right\} = W_0 \left\{f^\top \exp(X)\right\}, \ X \stackrel{\text{def}}{=} \sum_{t=1}^{T} X_t \tag{14}$$

As the horizon T grows, the sum of the random variables X_t tends to the Gaussian as long as the first two moments of the underlying distribution are finite. Formally, let random variable X_t have expectation vector $\mu_t = \mathsf{E}(X_t)$ and covariance matrix $\Sigma_t = \mathsf{E}\left[\{X_t - \mathsf{E}(X_t)\}\{X_t - \mathsf{E}(X_t)\}^\top\right]$. Then

$$\begin{aligned} \sum_{t=1}^{T} X_t &\xrightarrow{\mathcal{L}} \mathrm{N}\left(\sum_{t=1}^{T} \mu_t, \sum_{t=1}^{T} \Sigma_t\right) \\ X &\xrightarrow{\mathcal{L}} \mathrm{N}(\mu, \Sigma) \\ T^{-\frac{1}{2}} \sum_{t=1}^{T} (X_t - \mu_t) &\xrightarrow{\mathcal{L}} \mathrm{N}(0, \Sigma). \end{aligned} \tag{15}$$

If the distribution in horizon T is modelled as the sum of higher frequency distributions, the multidimensional process of returns, which may not be Gaussian, but of finite variance, converges to the Gaussian. In contrast, as argued in section 2.1, returns of horizons beyond the sampled frequency, are presumed to be heavy-tailed. Hence, the standard Central Limit Theorem (CLT) does not apply.

Except for the Gaussian itself, finite variance distributions change their shape under aggregation. In contrast, the class of stable distributions is scale invariant (Mandelbrot, 1963). Scale invariance of distribution P is defined via a continuous function g, such that for all x

$$g(\lambda)P(x) = P(\lambda x), \tag{16}$$

with $\lambda x \geq x_0$ and $x_0 > 0$. Equivalently, distribution P has a power-law tail, implying that for $x \geq x_0 \geq 0, c \geq 0$ and $\alpha > 0$

$$P(x) = cx^{-\alpha}. \tag{17}$$

In that respect, a one-dimensional random variable $X \sim S(\alpha, \beta, \gamma, \delta)$ will be α-stable distributed with parameters $0 < \alpha \leq 2,\ -1 \leq \beta \leq 1,\ \gamma \geq 0$ and $\delta \in \mathbb{R}$ (Cizek, Härdle, and Weron, 2011; Nolan, 2017), if

$$X \stackrel{\mathcal{L}}{=} \begin{cases} \gamma Z + \delta, \ \alpha \neq 1 \\ \gamma Z + (\delta + \beta \frac{2}{\pi} \gamma \log \gamma), \ \alpha = 1. \end{cases} \tag{18}$$

$S(Z \mid \alpha, \beta, 1, 0)$ represents the standard stable form. As only special cases of stable distributions are available as real-valued densities (e.g. Gaussian, Cauchy and Lévy), α-stable distributions are expressed as Fourier transforms of the characteristic function $\varphi_X(u)$.

$$S(X \mid \alpha, \beta, \gamma, \delta) = \frac{1}{2\pi} \int \varphi_X(u) - \exp(-iuX) du \tag{19}$$

The according characteristic function representation is given by

$$\log \varphi_X(u) = \begin{cases} iu\delta - \gamma^\alpha |u|^\alpha \left\{1 + i\beta \tan\left(\frac{\alpha\pi}{2}\right)(\operatorname{sign} u)\right\}, \ \alpha \neq 1 \\ iu\delta - \gamma|u| \left\{1 + i\beta \frac{2}{\pi} (\operatorname{sign} u) \log(|u|)\right\}, \ \alpha = 1. \end{cases} \tag{20}$$

Scale invariance under addition implies that for the sum of α-stable variables $X_t \sim S(\alpha, \beta, \gamma, \delta_t),\ t = 1, \ldots, T$

$$X_1 + X_2 + \ldots + X_T = \sum_{t=1}^{T} X_t = X \sim S(\alpha, \beta, T^{\frac{1}{\alpha}} \gamma, \delta), \tag{21}$$

where $\delta = T\delta_t$. The characteristic function of X is consequently given by

$$T \log \varphi_X(u) = \begin{cases} iu(T\delta_t) - T\left(\gamma^\alpha\right)|u|^\alpha \left\{1 + i\beta \tan\left(\frac{\alpha\pi}{2}\right)(\operatorname{sign} u)\right\}, \ \alpha \neq 1 \\ iu(T\delta_t) - T\gamma|u| \left\{1 + i\beta \frac{2}{\pi} (\operatorname{sign} u) \log(|u|)\right\}, \ \alpha = 1. \end{cases} \tag{22}$$

According to Gnedenko and Kolmogorov (1954), the limiting distribution of T i.i.d. α-stable random variables, $0 < \alpha \leq 2$ is

$$a_T \left(\sum_{t=1}^{T} X_t \right) - b_T \xrightarrow{\mathcal{L}} S(\alpha, \beta, 1, 0), \tag{23}$$

where $a_T > 0$ and $b_T \in \mathbb{R}$. The special case of the Generalized Central Limit Theorem (GCLT) is the CLT of equation (15) for $\alpha = 2,\ \beta = 0,\ \gamma = \frac{\sigma}{\sqrt{2}}$ and $\delta_t = \mu_t$, given $a_T = \frac{1}{\sigma\sqrt{T}}$ and $b_T = \frac{\sqrt{T}\mu}{\sigma}$. In general, for $0 < \alpha \leq 2$,

$$T^{-\frac{1}{\alpha}} \sum_{t=1}^{T} (X_t - \delta_t) \xrightarrow{\mathcal{L}} S(\alpha, 0, \gamma, 0). \tag{24}$$

2.3 Elliptically contoured stable distributions

For the multidimensional estimation, α-stable laws are not extensively accessible as closed-form densities are only available for special cases. One computationally tractable exception are elliptically contoured α-stable laws, which can be efficiently estimated for dimensions $k \leq 40$ (Nolan, 2013). This class of distributions enables the modeling of heavy tails while preserving its shape under aggregation in the presence of linear dependence.
Random vector $Y = [Y_1, \ldots, Y_k]^\top$ has a spherical distribution iff the characteristic function $\varphi_Y(u)$ satisfies for all $u \in \mathbb{R}^k$

$$\varphi_Y(u) = \mathsf{E}\left\{\exp\left(iu^\top Y\right)\right\} = \psi(u^\top u) = \psi(u_1^2 + \ldots + u_k^2), \tag{25}$$

where ψ is the characteristic generator of the spherical distribution.
Random vector $X \sim E_k(\delta, \Gamma, \psi)$ is elliptically distributed with positive definite scaling matrix $\Gamma = AA^\top,\ A \in \mathbb{R}^{k\times k}$ and location vector $\delta \in \mathbb{R}^k$ when

$$X \stackrel{\mathcal{L}}{=} \delta + AY, \tag{26}$$

where Y is spherical with characteristic generator ψ. The characteristic function is given by

$$\varphi_X(u) = \mathsf{E}\left\{\exp\left(iu^\top X\right)\right\} = \exp\left(iu^\top \delta\right)\psi\left(u^\top \Gamma u\right). \tag{27}$$

A subclass of elliptical distributions are normal variance mixtures $X = [X_1, \ldots, X_k]^\top$ for

$$X \stackrel{\mathcal{L}}{=} W^{1/2}AZ + \delta, \tag{28}$$

with $Z \sim \mathrm{N}(0, I_k)$ and $W \geq 0$ being a non-negative one-dimensional random variable, independent of Z (Kring, Rachev, Höchstötter, and Fabozzi, 2009).
A further subclass of normal variance mixtures are α-stable sub-Gaussian $X = [X_1, \ldots, X_k]^\top$ for $W \sim S(\alpha/2, (\cos \pi\alpha/4)^{2/\alpha}, 1, 0),\ 0 < \alpha < 2$, being one-dimensionally α-stable distributed, parameterized following Nolan (2017). $G \sim \mathrm{N}(0, \Gamma)$ is multidimensional Gaussian with scaling matrix $\Gamma = AA^\top$. Then $X \sim E_k(\alpha, \beta, \Gamma, \delta, \psi),\ \beta = 0$ is α-stable sub-Gaussian if

$$\begin{aligned} X &\stackrel{\mathcal{L}}{=} W^{1/2}G + \delta \\ &\stackrel{\mathcal{L}}{=} W^{1/2}AZ + \delta,\ Z \sim \mathrm{N}(0, I_k) \\ &\stackrel{\text{def}}{=} AY + \delta, \end{aligned} \tag{29}$$

while $Y \sim E_k(\alpha, 0, I_k, 0)$ is radially symmetric α-stable. The according characteristic function of X is

$$\begin{aligned} \varphi_X(u) &= \int_{-\infty}^{\infty} f_X(x)\exp(iu^\top X)\mathrm{d}x = \mathsf{E}\left(iu^\top X\right) \\ &= \exp\left\{-\left(\frac{1}{2}u^\top \Gamma u\right)^{\alpha/2} + iu^\top \delta\right\}, \end{aligned} \tag{30}$$

$f_X(x)$ as probability density function. $\Gamma \in \mathbb{R}^{k\times k}$ is the positive definite scale matrix and $\delta \in \mathbb{R}^k$ the location vector. The characteristic generator is therefore given by

$$\psi(s,\alpha) = \exp\left\{-\left(\frac{1}{2}s\right)^{2/\alpha}\right\}. \tag{31}$$

This implies that α-stable sub-Gaussian distributions are scale mixtures of multivariate normal distributions (Samorodnitsky and Taqqu, 1994). Note, for $\alpha = 2$, the characteristic function collapses to the Gaussian. For $G \sim \mathrm{N}(0, I_k)$, the characteristic function of Y in equation (29) simplifies to

$$\varphi_Y(u) = \mathsf{E}\left(iu^\top Y\right) = \exp\left(-\gamma^\alpha |u|^\alpha\right). \tag{32}$$

For the horizon of the investor, T, the estimated daily log-returns are summed to the chosen frequency:

$$\tilde{X} = TX \ \sim E_k(\alpha, 0, T\Gamma, T\delta, \psi) \tag{33}$$

For subsequent estimation, stability parameter $0 < \alpha \leq 2$, scale Γ and location δ need to be estimated, given that $-1 \leq \beta \leq 1$ can be assumed to be not significantly different from zero.

2.4 Parameter estimation

The utilization of α-stable laws implies that fractional moments of random variable X

$$\mathsf{E}|X|^p = \int_{-\infty}^{+\infty} |X|^p f(X)\mathrm{d}X \tag{34}$$

are finite for $0 < p < \alpha,\ p \in \mathbb{R}$ and infinite for $p \geq \alpha$. This implies that for the stable Paretian case, representing a slower decay than under the Gaussian, $0 < \alpha < 2$, the second moment $\mathsf{E}|X|^2 = \infty$ and higher moments such as skewness and kurtosis are infinite. For the empirical financial market returns $1 < \alpha < 2$ (see chapter 3), the first moment remains finite. For elliptically stable random variable $X \sim E_k(\alpha, 0, \Gamma, \delta, \psi))$ the expectation is

$$\mathsf{E}\,X = \delta < \infty. \tag{35}$$

In general, for univariate stable laws the mean is undefined for $\alpha \leq 1$ and $\mathsf{E}\,X = \delta - \beta\gamma\tan\left(\frac{\pi\alpha}{2}\right) < \infty$ for $\alpha > 1$. From the perspective of a data scientist, analyzing the sample, empirical moments are always finite. But under the assumptions of being α-stable distributed, fractional moments with $p \geq \alpha$ have no intrinsic meaning. As shown in section 2.1 higher moments behave erratic with increasing data points, contrary to moment convergence under Gaussianity.

For portfolio allocation the estimation of location and scale are crucial. Founding on the analysis of Chopra and Ziemba (1993), the mean represents the largest source of error for estimating the portfolio fraction. Their final implication is straightforward: "[...] the bulk of resources should be spent on obtaining the best estimates of expected returns of the asset classes under consideration".

Simulating from the class of elliptically stable distributions implies to estimate the stability parameter α, scaling matrix Γ and location δ, given that the skewness parameter β is zero. For the characteristic exponent α the method of Rachev and Mittnik (2000) is used:

i. Simulate $U_1, \dots, U_n$ uniformly i.i.d. random variables on the unit hypersphere S^{k-1}.
ii. Estimate the MLE for the index of stability $\hat{\alpha}_i$ (Nolan, 2001) for each i from 1 to n, $U_i^\top X_1, \dots, U_i^\top X_n$.
iii. Calculate the index of stability by $\hat{\alpha} = n^{-1} \sum_i^n \hat{\alpha}_i$.

By utilizing the MLE for the characteristic exponent α, severe estimation biases from e.g. the Hill estimator (Hill, 1975) are circumvented, see also McCulloch (1997) and Kearns and Pagan (1997). For the proposed nonparametric scaling approximation in section 2.5, the estimation of stability α will not be necessary.

Estimating the location vector $\delta \in \mathbb{R}^k$ of multidimensional variable $X \sim E_k(\alpha, 0, \Gamma, \delta, \psi)$ is of crucial importance for portfolio allocation, representing *the* driver for asset growth.
From the perspective of information theory, we aim to chose the parameter vector, which maximizes the probability of coming from the empirical dataset. From the perspective of decision theory, this method coincides with the minimization of expected loss under the 0-1 loss function:

$$L(\delta, \hat{\delta}) = 1(\delta \neq \hat{\delta}). \tag{36}$$

The according risk function is

$$\mathrm{R}(\delta, \hat{\delta}) = \mathsf{E}\left\{\mathrm{L}(\delta, \hat{\delta})\right\} = \mathsf{E}\left\{1(\delta \neq \hat{\delta})\right\} \tag{37}$$

Consequently the optimization

$$\delta^* = \underset{\delta^* \in \mathbb{R}^k}{\arg\min} \left[\mathsf{E}\left\{1(\delta \neq \hat{\delta})\right\}\right] \tag{38}$$

leads to the common Maximum Likelihood Estimate (MLE). If the loss function is not presumed to be $0-1$ loss, e.g. quadratic, the usual ML estimator may not be suitable. The inadmissability of the sample mean under the Gaussian for dimensions $k > 2$ has been first shown by Stein (1955), leading to the class of shrinkage estimators, starting with James and Stein (1961). An overview over the class of shrinkage estimators is given in Hansen (2015). To our knowledge, those results have not been extended to α-stable laws.

Following Nolan (2013), there are two methods to estimate the scale matrix Γ:

i. Given that X is elliptically stable,

$$\forall u,\ u^\top X \sim E_k\left(\alpha, 0, (u^\top \Gamma u)^{\frac{1}{2}}, u^\top \delta, \psi\right). \tag{39}$$

The $k(k+1)/2$ parameters of the scale matrix Γ are estimated by

$$\begin{aligned} \hat{\Gamma}_{j,j} &= \hat{\gamma}_j^2 \\ \hat{\Gamma}_{j,i} &= \tfrac{1}{2}\left\{\hat{\gamma}^2(1,1) - \hat{\gamma}_i^2 - \hat{\gamma}_j^2\right\}, \end{aligned} \tag{40}$$

where $\hat{\gamma}^2(1,1) = (1,1)^\top(X_j, X_i) = X_j + X_i$ and $\hat{\gamma}_j$ is the univariate scale ML estimate of asset j. Note that $\hat{\Gamma}_{j,i}$ depends solely on directions $(1,1)$,$(1,0)$ and $(0,1)$.

ii. As $\mathsf{E}\left\{\exp(iu^\top X)\right\} = \exp\left\{-\gamma(u)^\alpha\right\}$

$$\left\{-\log \mathsf{E}\exp(iu^\top X)\right\}^{\frac{2}{\alpha}} = u^\top \Gamma u = \sum_i u_i^2 \Gamma_{i,i} + 2\sum_{i<j} u_i u_j \Gamma_{i,j}, \tag{41}$$

so $\Gamma_{i,j}$ can be estimated as linear function via regression, taking more directions into account than the first method.

For the remainder of the paper, the first method is utilized due to its analytical tractability.

2.5 Nonparametric scaling approximation

Instead of simulating from the estimated elliptically contoured stable distribution, a fast scaling approximation is proposed. Three major drawbacks of elliptically stable distributions can be detected. The first one is the imposition of the elliptical shape of two random variables plotted on a two-dimensional projection as the scatter plot. Secondly, simulating from the elliptically stable distribution implies that

$$\begin{aligned} &\forall j = 1,\ldots,k \qquad \alpha_j = \alpha,\ 0 < \alpha < 2, \\ &\forall j = 1,\ldots,k \qquad \beta_j = 0. \end{aligned} \tag{42}$$

Thirdly, for empirical stock returns, quantiles $Q_\alpha,\ \alpha < 0.02$ are underestimated, implying that risk measures for large confidence levels are underestimated. Vice versa, quantiles $Q_\alpha,\ \alpha > 0.98$ are consequently overestimated, see figure (5). Reasons for the absence of extreme outliers as presumed stable distributions, may be exchange or central bank restrictions i.e. in times of crisis or market makers, smoothing the price changes.

The last two drawbacks are dealt with by using stable properties of the sample data-set. Assume that the daily data-set $X_t \sim E_k(\alpha, 0, \Gamma_t, \delta_t, \psi)$ is elliptically stable distributed. Then,

i. estimate location δ_t and scale $\Gamma_t = A_t A_t^\top$ of daily returns X_t as proposed in section 2.4.
ii. Normalize X_t to radially symmetric $Y \sim E_k(\alpha, 0, I_k, 0, \psi)$

$$Y = A_t^{-1} X_t - \delta_t. \tag{43}$$

iii. Rescale radially symmetric Y to distribution $X \sim E_k(\alpha, 0, \Gamma, \delta, \psi)$, $\Gamma = AA^\top$ with investment horizon T,

$$X = AY + \delta \tag{44}$$

with $\Gamma = T\Gamma_t$ and $\delta = T\delta_t$.

The resulting presumed distribution in horizon T, represented by convoluted daily distributions, is simply an affine transformation of its radially symmetric analogue, given its scaling nature. Given that $\beta = 0$, different stabilities α_j of the marginals have no effect on location δ and scale Γ.

3 Implementation

3.1 Data

The daily financial asset prices come from Bloomberg and cover the time span from 1997-12-12 to 2015-10-01, 4617 daily data points per asset. The assets with a linear payoff structure (stocks, bonds, commodities) are chosen to be a subset of the investment universe, aiming to represent its main drivers. The German and the American stock markets are represented by the stock market indices DAX 30 and Standard and Poor's 500. The bond market is reflected by the IBOXX EMU SOV 1-3 and JPMorgan EMBI Global Total Return Index for emerging markets. The Bloomberg Commodity Index it utilized as proxy for the commodity markets. For potential short-selling one asset with fixed interest - the interest rate the investor has to pay in order to lend - is calculated. The according descriptives including Maximum Likelihood Estimates (MLE) under α-stability (Nolan, 2001) are given in table 2.

The assets with a non-linear payoff structure are represented as long put options, written on the stock market indices DAX 30 and Standard and Poor's 500. As will be assumed for the representative investor in section 3.4, the maturity, and hence the investment horizon T, is chosen to be one year. No exchange-traded, liquid long put options are available for the chosen bond and commodity market proxies. The price of the long put options determines the price of the hedge and hence the reduction in wealth if the stocks close above the chosen strike levels. For the distribution of wealth in T, the put option price O_T at maturity is given by the inner value

$$O_T = \max\{0; K - S_T\}. \tag{45}$$

Solely for evaluating the price of the non-linear assets between $t = 1$ and horizon T, a pricing model is needed.

Descriptives	DAX30	S&P500	EMUSOV	EMBI	BCI
$\mu \times T$ (in %)	4.74	4.07	3.65	8.46	0
$\sigma \times T^{1/2}$ (in %)	24.88	20.22	1.48	9.25	16.6
Skewness	0.04	-0.12	0.17	-2.01	-0.21
Kurtosis	6.22	10.12	22.33	35.47	5.54
α	1.69 ± 0.06	1.59 ± 0.06	1.60 ± 0.06	1.44 ± 0.06	1.79 ± 0.05
β	-0.20 ± 0.17	-0.16 ± 0.16	0.00 ± 0.14	-0.08 ± 0.12	-0.14 ± 0.22
$\gamma \times T^{1/\alpha}$	0.24	0.22	0.02	0.11	0.14
$\delta \times T$	0.17	0.16	0.04	0.15	0.07

Table 2: Return descriptives with ML estimates under α-stability, 1997-2015 including confidence intervals (99%) for stability α and skewness β

3.2 Stable tests

In order to verify if the class of elliptically stable distributions is suitable for the financial assets, the following prerequisites have to be met:

- heavy tails beyond the Gaussian (Leptokurtic behavior)
- linear dependence structure between the margins
- comparable range of α_j (for simulation)
- skewness parameter β not coherently different from zero.

As examined descriptively in table 2, empirical financial market returns are significantly non-Gaussian. In figure 5 the densities of the normalized log-returns on log-scale are plotted for Gaussian ($\alpha = 2$), Stable ($\alpha = 1.7$), Cauchy ($\alpha = 1$) and the individual assets using Kernel Density Estimates. Within the stable framework all examined assets lie between Gaussian and Cauchy, $1 < \alpha < 2$. The stable fit for $\alpha = 1.7$ captures the tails adequately, although events are captured, which never took place in the data history. The range of characteristic exponents stands in line with results of Westerfield (1977), McCulloch (1997) or Nolan (2013).
The elliptical behavior is assessed by using two dimensional projections in the form of the scatter matrix of the empirical log-returns. The significance of the skewness parameters β_j is verified by the utilization of the Fisher information from the MLE. The respective confidence intervals for the individual parameters show that β_j are not consistently different from zero, given a confidence level of 99% (see table 2). For larger dimensions, Nolan (2013) reaches the same conclusion for the Dow Jones constituents.
Making use of the Nonparametric Scaling Approximation implies that there is no need to estimate one specific alpha for the elliptical stable distribution. As $\forall j \; 1 < \alpha_j < 2$ we can deny the null of Gaussianity coherently for the 99% confidence level (see table 2), speaking in favour of the α-stable hypothesis. This implies the assumption of infinite variance. As we are interested in the horizon distribution, constituted by the sum of daily random variables, the generalized CLT is utilized.

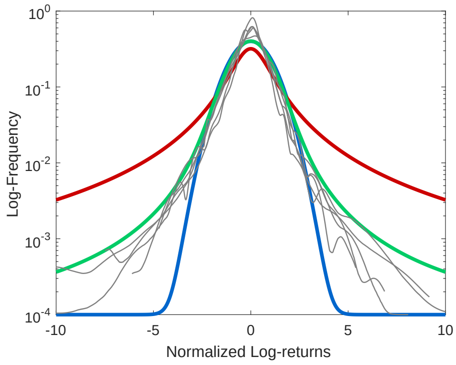

Fig. 5: Log-densities for Gaussian, $\alpha = 2$ (blue), Stable, $\alpha = 1.7$ (green), Cauchy, $\alpha = 1$ (red) and financial assets (gray)

Stable_Kelly_LogDensityFinancials

3.3 Stable estimation

Following section 2.4, the parameter estimates for the daily distribution $X_t \sim E_k(\alpha, \beta, \Gamma_t, \delta_t, \psi)$, $\beta = 0$ are scaled to the chosen horizon of one year. For the utilized scaling approximation, $X_T \sim E_k(\alpha, 0, \Gamma_T, \delta_T, \psi)$ for location $\delta_T = T\delta_t$ and scaling $\Gamma_T = T\Gamma_t$, given that $\beta = 0$. Location vector and scaling matrix, plotted as heat matrix, are given in table 3 and 4.

Asset	δ_T
DAX30	0.0474
S&P500	0.0407
EMUSOV	0.0365
EMBI	0.0846
BCI	0

Table 3: Estimate for location vector δ_T

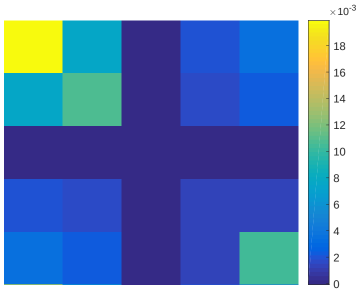

Table 4: Estimate for the scaling matrix Γ_T

3.4 Portfolio implementation

Exemplary, the representative investor has an investment horizon of one year. According to his regulators, no more than 15% ($b = 0.15$) of his wealth should be lost given probability 99.9%. In terms of the discrete wealth return distribution only $\alpha = 0.1\%$ of the wealth paths should end below -15%. The investor is able to lend capital for 3% per year, representing the sixth asset, which can only be short-selled. The two maximization problems, without ($k = 6$) and with options ($k = 134$), as special case of optimization in (12), are formulated within the framework of spectral measures:

$$\begin{aligned} f^* = \underset{f^* \in \mathbb{R}^k}{\arg\max}\ & G_{\phi_{\mathrm{Elog}}}\{W_T(f)\} \\ \text{s.t. } & S_{\phi_{\mathrm{Q}_\alpha}}\left\{1 - \frac{W_T(f)}{W_0}\right\} \leq 0.15, \\ & \sum_{j=1}^{k} f_j \leq 1. \end{aligned} \tag{46}$$

The resulting discrete wealth return distributions with according statistics is given in figure 6 and table 5. Including non-linear instruments into the restricted optimization proves to be beneficial for the Kelly Criterion, preserving the quantile restriction. The option investment is restricted to half of the initial capital. In terms of portfolio fractions (see table 6), the leverage is increased by investing in a combination of the according puts of DAX30 and S&P500 and the according underlyings, equaling the protective put strategy of section 1.2.

Portfolio (in %)	Without	With
Geometric mean	6.03	8.85
Arithmetic mean	6.33	10.90
Standard deviation	4.98	26.94
Minimum	−29.76	−28.19
$Q_{0.1\%}$	−15.00	−15.00
$Q_{1\%}$	−6.02	−8.10
$Q_{10\%}$	1.33	−0.95
$Q_{50\%}$	6.25	5.27
$Q_{90\%}$	11.30	23.35
$Q_{99\%}$	20.92	1.19
Maximum	67.26	871.57

Table 5: Discrete wealth return statistics (in %) without and with options

Fractions	Without	With
DAX30	0.01	0.74
S&P500	0	1.29
EMUSOV	2.36	1.88
EMBI	0.33	0.48
BCI	0	0
SHORT	−1.69	−3.89
Put DAX	/	0.14
Put S&P500	/	0.36

Table 6: Portfolio fractions without and with options

Extending the quantile constraint to the interval $0 \leq b \leq 1$, leads to a series of optimizations for all relevant quantile levels. $b = 0$ represents the risk free portfolio, whereas $b = 1$ implies that the investor can loose all of his fortune, given chosen confidence level.

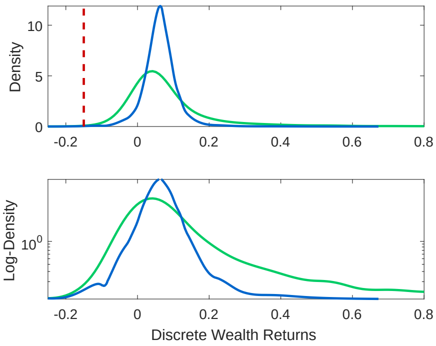

Fig. 6: Wealth densities without (blue) and with (green) put-options, VaR constraint (red)

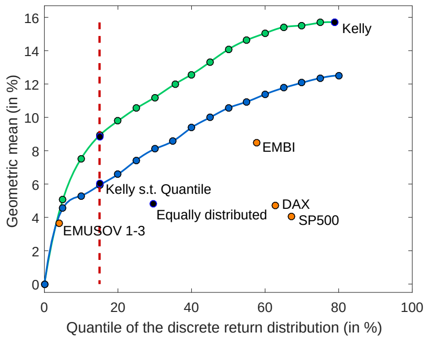

Fig. 7: Kelly-quantile frontier without (blue) and with (green) options, $\alpha = 0.001$

$$
\begin{aligned}
f^* = \underset{f^* \in \mathbb{R}^k}{\arg\max}\ & G_{\phi_{\mathrm{Elog}}} \{W_T(f)\} \\
\text{s.t. } & S_{\phi_{\mathrm{Q}_\alpha}} \left\{ 1 - \frac{W_T(f)}{W_0} \right\} \leq b,\ 0 \leq b \leq 1, \\
& \sum_{j=1}^{k} f_j \leq 1.
\end{aligned}
\tag{47}
$$

This series of restricted optimizations is mapped to the Kelly-quantile frontier (figure 7), in which each point represents a growth-optimal portfolio given quantile (VaR) constraint. The portfolio with (without) options, which can lose at most 15% with 99.9% probability is the point where the green (blue) frontier crosses the quantile constraint (red). Except for the risk-free portfolio, $0 < b \leq 1$, every restricted portfolio with options outperforms the portfolio without options in terms of the geometric mean. The unrestricted Kelly portfolio exhibits the highest geometric mean possible (15.69%), given a VaR of 78.82%. The investor, who allows for a larger security constraint b than the unrestricted Kelly solution, should still invest into the growth-optimal portfolio, as the geometric mean of the unrestricted Kelly portfolio cannot be surpassed.

4 Conclusion

Whereas the unrestricted Kelly portfolio ensures the asymptotic outperformance of the investor towards significantly different strategies, the presented model ensures growth-optimal investment subject to personal risk. The constrained optimization is formulated within the framework of spectral measures, inducing quantile (VaR) and Expected Shortfall as special cases. In order to allow for an asymmetric wealth distribution, long put options are included into the optimization.

Financial market returns are with large probability non-Gaussian. Founding on the work of Mandelbrot (1963), it can be observed that the stability parameter α is significantly smaller than two, speaking in favor of the class of α-stable distributions. Given a chosen investment horizon, the distribution of financial market returns is modelled as the sum of daily random variables. For stable laws with $\alpha < 2$ the variance of those random variables is infinite. Hence, the standard CLT does not apply and the generalized CLT of Gnedenko and Kolmogorov (1954) is applied. For the multidimensional estimation elliptical stable distributions, implying a linear dependence structure, are used. Instead of simulating from this class of distributions, a nonparametric scaling approximation is proposed. The resulting horizon distribution, represented by convoluted daily distributions, is simply an affine transformation of its normalized daily analogue, given its scaling nature.

Heavy tails beyond the Gaussian, linear dependence between the marginals and nonsignificant skewness are empirically supported. Correspondingly, the joint distribution of financial market returns for a specified horizon is estimated by elliptical stable distributions utilizing a nonparametric scaling approximation. The portfolio model is implemented for a representative investor with quantile (VaR) constraint. The resulting growth-optimum strategy maximizes the geometric mean, given his risk constraint. Including put options into the optimization levers the portfolio by a suitable protective put strategy, leading to an increased geometric mean for the same quantile. For the Kelly-quantile frontier, except for the risk-free portfolio, every restricted portfolio with options outperforms the portfolio without options in terms of the geometric mean.

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