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Coins with benefits: on existence, pricing kernel and risk premium of cryptocurrencies

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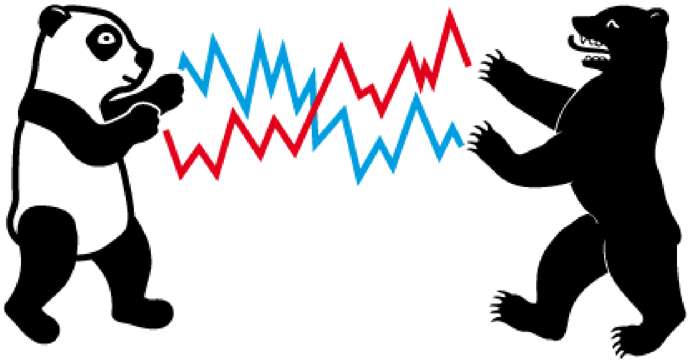

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Coins with benefits: on existence, pricing kernel and risk premium of cryptocurrencies
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Abstract

Cryptocurrencies come with benefits, such as anonymity of payments and positive network effects of user adoption, and transaction risks including unconfirmed transactions, hacks, and frauds. They compete with central-bank-regulated money but consumers may prefer one currency over the other. In our arbitrage-free world utility from consumption depends on benefits, which are governed by distinct stochastic processes, implying incomplete markets and distinct pricing kernels. We characterize the cryptocurrency kernels, evaluate the otherwise unobservable benefits, and show their contribution to pricing. The model explains both the co-existence of the two currencies and the high volatility of the cryptocurrency price.

Keywords: Bitcoin, cryptocurrency, pricing kernel, currency competition

JEL Classification: A1, D0, E21, G12

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1 Introduction

Cryptocurrencies are assets, ownership of which is recorded and verified by means of a cryptography-based digital technology (blockchain being an example), rather than by a financial institution like a bank or a central bank. The "currency" part of their name, and related notions of "coins", "wallets", and "tokens", all suggest they are meant to serve transactions, just like the central bank money. There are, however, important differences, such as lack of regulation, in particular for cross-border transfers, the anonymity of transactions in cryptocurrencies, etc. For some, these represent crucial benefits.[1] Transactions in cryptocurrencies keep growing: bitcoin alone is reported to serve about 300 000 transactions daily by the end of 2020.[2] At the same time, as an investment asset, cryptocurrencies make headlines by exhibiting poor predictability and high volatility of their exchange rate with traditional [central bank] currencies. In this paper, we investigate to what extent the transactional nature and the added benefits, including but not limited to anonymity, of cryptocurrencies can explain their co-existence with traditional monies as well as their return and volatility patterns.

Our key idea is that of segmented markets: the utility of consuming goods obtained in the market served by the central bank currency (e.g. dollars) differs from that of consuming the same goods purchased for a cryptocurrency (e.g. bitcoins). The difference in utilities is justified by added benefits. In real life, these may be small, like the satisfaction of one's curiosity or desire to feel part of the "crypto-community", which

[1]Especially so in illegal activities involving payments in cryptocurrencies. Although the Silk Road online black market for drugs was shut in 2013, cryptocurrencies continue to be used in illegal tenders, such as cyber-attacks on universities and hospitals, with the UCSF being in the center of the news in June 2020, reportedly paying 116.4 bitcoins (USD 1.14 million) to re-gain control over its systems after the hackers' attack "How hackers extorted $1.14m from University of California, San Francisco" by Joe Tidy, BBC, 29 June 2020 (https://www.bbc.com/news/technology-53214783). In August 2020, the University of Utah paid just under a half of this amount: "The University of Utah pays more than $450,000 in a ransomware attack on its computers", By Scott D. Pierce, The Salt Lake Tribune, August 21, 2020. Foley et al. (2019) give an annual estimate of about $76 billion of illegal activity in bitcoins, which is just under a half of all bitcoin transactions. A discussion of cryptocurrencies as assets and payment instruments is in the recent review by Giudici et al. (2020), who, in particular, assert the fundamental value of cryptocurrencies is effectively in their anonymity.

[2]"Number of daily Bitcoin transactions worldwide from 1st quarter 2016 to 2nd quarter 2020", Statista.com. https://www.statista.com/statistics/730806/daily-number-of-bitcoin-transactions

makes an ordinary T-shirt or a parking ticket paid with bitcoins more valuable than if they were paid in dollars. Benefits can be also large if one considers drugs, prostitution, ransom, or bribery, where the transacting parties want to preserve anonymity. We assume benefits of transacting in dollars and benefits of transacting in bitcoins are governed by different stochastic processes. Market segmentation then allows us to use the apparatus of international finance to characterize the arbitrage-free price of bitcoins in dollars. In particular, we show the two markets are governed by distinct pricing kernels. This is due to the incompleteness, which stems from an impediment to hedge risks of stochastic benefits: no hedging or insurance strategy in the dollar market can be constructed for a bitcoin transaction without breaking anonymity.[3] By establishing a relationship between the two pricing kernels, we characterize the expected return (change in the bitcoin-dollar exchange rate), its volatility, and the risk premium of bitcoins and end up with a building block – the co-existence of crypto- and central bank currencies, explained by the heterogeneity of benefits from transactions in them.

Benefits are the fundamentals in bitcoin pricing. As long as they are non-negligible, the cryptocurrency markets do not vanish.[4] The empirical challenge is however to identify them and to quantify the benefits-driven pricing kernel. Assuming the iceberg-style (Samuelson; 1954) effect of benefits on utility from consumption and the constant relative risk aversion (CRRA) functional form for utility, we are able to decompose the kernels into the *benefits-based* and *consumption-based* kernel components and distill the benefit-related component from the fundamental pricing equation. In particular, we show the benefits in the bitcoin market are empirically well captured by the number of coins sent through the blockchain network. Empirically, on data from 2012 to 2020, we find the bitcoin pricing kernel and its benefits-driven component exhibit an autoregressive behavior coupled with a moving average effect, and the difference between the benefits in the two

[3] Generally, market segmentation does not automatically imply incompleteness. It is the risks of the added benefits that create incompleteness: some states of nature in the bitcoin market are unobservable for other participants and players in the dollar market.

[4] Pagnotta (2021) highlights security of transactions as the fundamental, yet notes that a sufficient condition for bitcoins to be positively priced is that they are essential at least in some transactions. We focus exactly on the reasons why bitcoins are essential. Security and security threats, are special cases of what we call benefits.

markets becomes stable. Along with that, we observe a decline in the volatility of the bitcoin kernel, which is good news in terms of market stabilization. Noteworthy, while the Covid-19 shock produced a pronounced spike in the dollar pricing kernel, its impact on the bitcoin kernel was close to nil. This latter observation supports our theoretical approach of having two distinct pricing kernels for crypto- and centralized currencies.

Recently, Schilling and Uhlig (2019) proposed a model of competition between a crypto- and a traditional currency in a single market, with the key difference between the two being the stability of the purchasing power, ensured by the central bank for the traditional currency. The model yields a characterization of the pricing kernel, which is used to price the cryptomoney. Under an appropriate monetary policy regime, digital currency can vanish in a long-run equilibrium. We adopt a similar transaction perspective on the fundamental role of cryptocurrencies, yet we relax the perfect competition assumption: consumers in our model derive extra utility from using the cryptocurrency as a payment tool. We use anonymity as a leading example although we recognize there may be other benefits or drawbacks of using digital currencies, including overcoming local regulatory restrictions and risks of unconfirmed transactions, hacks and frauds. For example, in a concurrent work by Foley et al. (2021) drawbacks of using bitcoins for transactions arise due to congestion in data flows, resulting in longer transaction confirmation times, which have increased from a couple of minutes at the beginning of the bitcoin system to about an hour by now.

We obtain net expected benefits play a central role in determining the bitcoin return, yet in our model, it is the pricing of their risks that matters: the riskier (higher variation) the benefits of bitcoin transactions, the higher fluctuation of bitcoin returns. From a modeling perspective, benefits in our paper explicitly affect utility (consonant to the approach used in Bolt and Van Oordt; 2020), and subsequently the pricing kernel, rather than the budget constraint (the approach adopted, e.g., in Biais et al.; 2020). One of the advantages is that incorporating benefits in the pricing kernel allows us to use the fundamental pricing equation to empirically characterize the benefits kernel and its dynamics. Bolt and Van Oordt (2020) distinguish between adoption benefits (proportional to the number of users in the cryptocurrency network) and fixed benefits (technological preferences of individual consumers, reduced cross-border and another cost, privacy, and anonymity), yet the currency choice in consumers' decisions in their model is disjoint from consumption choices. In our model benefits form part of the utility, along with consumption, and the two types of benefits we consider capture the differences between the centralized and the decentralized markets.[5] A consumption-based approach is also followed by Biais et al. (2020). In their model, bitcoins compete with bonds (debt instruments) and conventional currency (fiat money) as means of savings, to transfer value between periods in an overlapping generation set up with a single consumption good; they are equally good with money as means of payment. Benefits of using bitcoins either expand or shrink the intertemporal budget constraint. This gives rise to an equilibrium pricing for bitcoins (in dollars) as an investment asset, through the discounted value of their expected net benefits over the infinite horizon. The solution is not unique: a random perturbation of the equilibrium path may be designed without violating the equilibrium condition. Empirically, the benefits, proxied by transaction costs and the number of shops accepting bitcoins as means of payment, explain only a small fraction of long-term bitcoin dynamics, hence from the perspective of this model it is the random perturbation of the equilibrium path (extrinsic noise) that drives the cryptocurrency dynamics.

Addressing the issue of incomplete markets, the closest to ours are the works by Bakshi et al. (2018) and Lustig and Verdelhan (2019). Following them, we focus on the wedge, i.e. the difference between the exchange rate growth and the pricing kernel differential (Backus et al.; 2001). The incompleteness in Lustig and Verdelhan (2019) stems from the trade restriction they impose in the model; the wedge effectively prices this trade barrier, and in econometric estimates can be proxied by taxes on cross-country capital movements. In Bakshi et al. (2018), the incompleteness manifests in differing stochastics of consumption and other macroeconomic variables across countries. In our

[5] This approach can be extended to incorporate different types of benefits in the crypto market, with no qualitative changes to theoretical results.

case, the incompleteness stems from the difference between stochastic benefits associated with dollars and bitcoins, which effectively exemplifies a trade barrier: risks (if any) of dollar benefits cannot be traded against risks of bitcoin benefits. The wedge then prices the heterogeneity of stochastic benefits. In an economic environment with two sources of uncertainty (consumption and benefits), to gauge an efficient decision in terms of the payment means, we elaborate the stochastic dominance relationship between heterogeneous stochastic benefits, and conclude that once agents perceive a higher probability of transaction benefits in the decentralized market relative to that in the counterpart market, the price of Bitcoin in terms of dollar and the risk premium of bitcoin move up. Empirically we find a persistent distributional dominance of benefits in decentralized over centralized markets from 2016 to the present.

The rest of the paper is organized as follows. Section 2 assumes cryptocurrencies and traditional currencies serve distinct parts of the economy, and derives some fundamental relationships, highlighting the essential role for market incompleteness. Section 3 explicitly introduces incompleteness by allowing for stochastic benefits in the two markets and through that models the price dynamics of bitcoin, its variance and the risk premium. Section 4 applies stochastic dominance analysis to benefits, in order to further characterize the difference between their kernels. The stochastic dominance analysis allows us to qualify the conditions under which bitcoins co-exist with dollars. Section 5 empirically evaluates the main parameters of the model from Bitcoin transaction data, estimates the [unobserved] benefits and the benefit-related kernel, and elaborates on the evolution of pricing kernels over time. The paper concludes with a summary and a discussion. All proofs and auxiliary details are in the Appendix.

2 Segmented markets: preliminaries

Consider a world, in which two currencies - call them dollars and bitcoins (BTC) - can be used as a means of payment in transactions in two separate markets, the *centralized* *market* using dollars and the *decentralized market* using BTC.[6] Agents can buy and sell goods in the two markets at no transaction costs. Time is discrete, time t consumption is $c_t^{\$}$ in the dollar-denominated market, and $c_t^{\B}$ in the BTC-denominated market. The physical good is assumed the same in the two markets, yet consumption brings different utilities, depending on the market where the good is obtained (e.g. consumption is not anonymous in the centralized market, but is anonymous in the decentralized market, therefore consumers who value anonymity derive higher utility in the latter), thus notations $c_t^{\$}$ and $c_t^{\B}$ effectively refer to the consumption-market tuple. Subjects derive utility $u(c_t^{\$}, c_t^{\B})$ from consumption at time t. Marginal utilities in the respective markets are $u'_{\$}(c_t^{\$}, c_t^{\B})$ and $u'_{\B}(c_t^{\$}, c_t^{\B})$, with the subscripts denoting consumption of the good served by the respective currency. If goods in the two markets are perfect substitutes, i.e. $u(c_t^{\$}, c_t^{\B}) = v(c_t^{\$} + c_t^{\B})$, and hence $u'_{\B}(c_t^{\$}, c_t^{\B}) = u'_{\$}(c_t^{\$}, c_t^{\B})$, we obtain the case of Schilling and Uhlig (2019) who consider one single market with one good and two currencies. Conversion of BTC to dollars is possible at the spot exchange rate S_t. For example, agents who desire anonymity may convert their dollar-denominated endowment into bitcoins for possessing and consuming the good in the decentralized market. The expected utility is then evaluated by the BTC-denominated marginal utility or pricing kernel.[7] Uncertainty about consumption is given by random shocks $\zeta_t = (\zeta_0, \cdots, \zeta_t)$ observable and measurable with respect to the filtration generated by the stochastic process. $P_t = P(\zeta_t)$ is the price of the consumption good in dollars; $B_t = B(\zeta_t)$ the price of the consumption good in bitcoins. In Section 3 we introduce further sources of uncertainty.

Introduce the following notation for the intertemporal marginal rates of substitution (IMRS) in the dollar- and the bitcoin-served markets respectively:

$$M_{t+1}^{\$} = \frac{u'_{\$}(c_{t+1}^{\$}, c_{t+1}^{\B})}{u'_{\$}(c_t^{\$}, c_t^{\B})}, \quad M_{t+1}^{\B} = \frac{u'_{\B}(c_{t+1}^{\$}, c_{t+1}^{\B})}{u'_{\B}(c_t^{\$}, c_t^{\B})}. \tag{2.1}$$

[6]The centralized market encompasses the domestic and the foreign markets in the conventional international trading mechanism, while the decentralized market launches new payment techniques to purchase goods and services without third-party authorities' supervision. The key innovation of decentralized markets is the implementation of cryptographic identification techniques into a distributed ledger, known as blockchain.

[7]To ensure trade between agents and currency exchange, as in Schilling and Uhlig (2019), there are two types of agents, whose generations alternate in consuming and producing the consumption goods.

Currencies serve two purposes: storage of value (savings) between t and $t+1$ and transfer of value between owners of assets and goods within each t. As a storage of value, they serve the intertemporal substitution of consumption between t and $t+1$, which is key for the following fundamental pricing result, highlighting the potential existence of two pricing kernels. The latter is due to imperfect competition between payment technologies: consumption preferences depend on the currency in which payments are made.

Proposition 1 (Fundamental pricing equation). *In the absence of arbitrage, there exist admissible pricing kernels for the consumption goods denominated in dollars and in BTC, respectively, in the two disparate markets, $M^{\$}$ and $M^{\text{Ƀ}}$, such that at any time t holds*

$$\mathsf{E}_t\left[M^{\$}_{t+1}\frac{P_t}{P_{t+1}}\right] = \mathsf{E}_t\left[M^{\text{Ƀ}}_{t+1}\frac{B_t}{B_{t+1}}\right] = 1. \tag{2.2}$$

As a means of cross-sectional transfer, the two currencies can be priced against each other. In general, if pricing kernels differ, the following holds for the nominal exchange rate S_t in dollars per bitcoin:

$$\frac{S_{t+1}}{S_t} = \frac{P_{t+1}}{P_t}\frac{B_t}{B_{t+1}}\frac{M^{\text{Ƀ}}_{t+1}}{M^{\$}_{t+1}}, \tag{2.3}$$

or, denoting $Q_t = S_t\frac{B_t}{P_t}$ the real exchange rate, and Δq_{t+1} the log change in the real exchange rate, $\frac{Q_{t+1}}{Q_t}$, we obtain

$$\Delta q_{t+1} = m^{\text{Ƀ}}_{t+1} - m^{\$}_{t+1}, \tag{2.4}$$

where lowercase letters are natural logs of respective capitalized variables. This result is standard in international finance for two countries with complete markets.

However, in a world with a single source of uncertainty the two kernels coincide. Under discrete states of nature, SDFs can be interpreted as state prices divided by probabilities of the states; an asset is then priced by the probability-weighted average of its state-contingent payoffs multiplied with these probability-adjusted state prices (see, e.g., Campbell; 2000). If the dollar-served markets are complete, Arrow securities can be constructed for all states of nature covered by the process ζ. The same stochastic process governs the bitcoin-served segment, and hence the same states of nature occur with the same probabilities, and the same system of Arrow securities suffices to price assets. If markets are complete, state prices are unique, and so should be the pricing kernels: this obtains if for any state of nature, given by the realization of ζ_t, holds $M^{\$}_{t+1} \frac{P_t(\zeta_t)}{P_{t+1}(\zeta_{t+1})} = M^{B}_{t+1} \frac{B_t(\zeta_t)}{B_{t+1}(\zeta_{t+1})}$.

In our segmented markets, bitcoins co-exist with dollars as long as consumers prefer to transact in bitcoins at least for some strictly positive quantity of consumption. A condition for the bitcoin disappearance results in Schilling and Uhlig (2019) is that at least at some date there is no goods transaction against bitcoins. This does not hold in our setting if the utility function rules corner solutions out. A particular case where a corner solution may arise is the perfect substitution between the two currencies, the case considered in Schilling and Uhlig (2019). Similarly, in Benigno et al. (2019) the cryptocurrency and the centralized money are equally liquid but the cryptocurrency offers a better return, which generates a crowding-out issue. In our model, two features prevent this: the preference structure, as explained above, and the incompleteness of markets (introduced in the next section) which requires both currencies are in use to span all states of nature.

Although preferences for bitcoins justify the co-existence of centralized and decentralized markets, the segmented markets approach degenerates to the case of a single market if markets are complete. Of course, the dollar-denominated markets may also be incomplete *per se*, in which case the pricing of the dollar-denominated assets would not be unique, yet as long as all states of nature occurring in the bitcoin market are replicable by dollar assets, bitcoin price is driven solely by the developments in the dollar market. Our interest is, however, in Bitcoin-specific pricing fundamentals, appearing in (2.2) in the form of the pricing kernel for the bitcoin-consumption market. The two pricing kernels may only be distinct if incompleteness is essential to the overall bitcoin-dollar system, which in our case means shocks to prices in the bitcoin market cannot be hedged by assets in the centralized market. With anonymity and illegal activities in cryptocurrencies being our leading example, bitcoin consumption shocks are indeed not hedgeable: there is no way to insure against a failure to complete a ransom deal, nor there is any way to hedge against a failed anonymous transaction without disclosing anonymity. These shocks are themselves random. This leads us to focus on the case of the incomplete market with two distinct pricing kernels, which obtains if benefits from transacting in the bitcoin market follow a stochastic process distinct from that in the dollar market.

3 Pricing kernel under stochastic benefits

In the above, benefits from transacting in bitcoins were deterministic, given by consumer preferences. In a realistic case, benefits are not deterministic but rather stochastic, as in Biais et al. (2020). Benefits reflect developments in the real economy that are relevant to the usefulness of cryptocurrency, e.g. darknet activities, e-commerce, ransomware or privacy-preserving transactions. Cong et al. (2021) consider the network effect as an exogenous variable in pricing cryptocurrencies or more broadly tokens; our approach explains the network effect through benefits arising from it: the more consumers use cryptocurrency, the easier it is for any consumer to find a transaction counterpart, the higher the utility she derives and the more valuable the coins/tokens are. However, the adoption rate itself is stochastic. To keep tractability, we now restrict our attention to the case where consumers can consume by transacting in one of the two markets but not in both simultaneously.[8] Consumers derive utility $u_j(c_t^j)$ if transaction takes place in market $j \in \{\$, \text{\B}\}$. To introduce market-specific benefits, we make the following assumption on

[8]Our previous formulation $u(c_t^{\$}, c_t^{\text{\B}})$ was more general, allowing for consumption from both markets simultaneously. By re-defining, in a slight abuse of notation, $u(c_t^{\$}) := u(c_t^{\$}, 0)$ and $u(c_t^{\text{\B}}) := u(0, c_t^{\text{\B}})$, we arrive at the case of consumption confined to one market solely, which is typically the case when a consumer decides to make an indivisible purchase. It looks like our results would also hold if consumers can consume in both markets and derive utility $v(c_t^{\$}, c_t^{\text{\B}}) = u(c_t^{\$}) + u(c_t^{\text{\B}})$. The function is separable, marginal utility with respect to $c_t^{\$}$ or $c_t^{\text{\B}}$ is the derivative of the first or the second term. This looks less restrictive, and justifies why consumers may consume different levels of consumption in one and the other market. The consumption bundle is identical, it is the level of consumption that differs.

the utility function:

Assumption 1 (Benefits-adjusted utility)**.**

$$u_j(c_t^j) = \frac{(c_t^j)^{1-\gamma} - 1}{1-\gamma} \tag{3.1}$$

where $c_t^j \equiv \theta_{j,t} C_{j,t}$*,* $\theta_j \in (0, \infty)$ *stands for transaction benefits,and* γ *is the coefficient of the relative risk aversion.*

First, in Assumption 1, utility depends on the product of two random variables $c_t^j \equiv \theta_{j,t} C_{j,t}$, representing the benefit-adjusted (or benefit-distorted) consumption, whereby the consumption level $C_{j,t}$ is distorted by the stochastic benefits $\theta_{j,t}$. The inclusion of the $\theta_{j,t}$-term is in the spirit of the "iceberg" model of international trade (Samuelson; 1954; Dornbusch et al.; 1977): an amount C of the good purchased in and transported from a foreign market arrives in the amount $g(C) < C$ in the domestic market where it is consumed[9]; the underway "meltdown" reflects transportation cost. In our case, $\theta_{j,t}$ represents a distortion to the utility derived from the amount physically consumed, $C_{j,t}$. As consumers can choose the market, as well as the consumption level, they maximize utility by choosing the resulting effective consumption c_t^j. We reserve $\theta_{j,t}$ to represent non-hedgeable shocks due to which the system of markets is incomplete. In contrast, all shocks that can be hedged by dollar-market instruments are in $C_{j,t}$.

Second, by Assumption 1 utility is CRRA, common in the consumption and decision theory. Key results in the previous section hold generally, with minimum assumptions on preferences. As will become clear in a few steps, the CRRA utility significantly simplifies the exposition and yields useful explicit representations in our model.

Consumption good in our framework corresponds to an average consumption bundle, which is identical across the two markets[10]. A consumer decides on how much they

[9] $g(\cdot)$ is a real-valued function in the support [0,1].

[10] An average consumption bundle (in the sense of averaging consumption bundles across all consumers) would include ordinary goods (current and durable consumption) and also drugs, sex services, "services" of removing malicious software and unblocking hacked databases (ransom), etc. - a "representative" or

should consume in the bitcoin market or in the dollar market. The probability of reverting a transaction when purchasing ordinary goods, or the probability of being caught for illegal use of drugs or a weapon trade may differ between the dollar and the bitcoin markets. The ex-post utility from consumption depends on the realization of random factors describing whether this average consumer can successfully receive online orders, revert transactions, retain anonymity in illegal or unethical activities, etc, which we call *benefits*. A state of nature determines for each element in the consumption bundle the realization of these random factors.

Agents may enjoy transaction benefits in each market, albeit with different uncertainties on benefits. We assume the amount consumed and the benefit are statistically independent:

Assumption 2. *$\theta_{j,t}$ and $C_{j,t}$ are statistically independent for any $j \in \{\$, B\}$.*

To justify assumption 2 consider, for example, anonymity: if anonymity is not preserved, consumption takes place anyway, yet potentially at some dis-utility ($\theta_{j,t} \leq 1$). [11] Similarly, a failed transaction may be represented with $\theta_{j,t} = 0$ so that $\theta_{j,t} \cdot C_{j,t} = 0$ but the decision to consume (and money spent) $C_{j,t}$ does not depend on the realization of $\theta_{j,t}$. With this in mind, $C_{j,t}$ can be seen as consumption that would take place in a world with no extra benefits ($\theta_{j,t} = 1$ deterministically). Recall that we reserved $\theta_{j,t}$ to absorb all shocks that are non-hedgeable due to incompleteness, hence deterministic $\theta_{j,t}$ describe a world with complete markets.

Lemma 1. *With utility $u_j(c_t^j) = \frac{(\theta_{j,t} \times C_{j,t})^{1-\gamma} - 1}{1-\gamma}$, the SDFs in the two markets are* [12]*:*

$$M_{t+1}^{B} = \Big(\frac{\theta_{B,t+1} C_{B,t+1}}{\theta_{B,t} C_{B,t}}\Big)^{-\gamma} = M_{t+1}^{C_B} M_{t+1}^{\theta_B} \tag{3.2}$$

a "median" consumer may find those latter ethically unacceptable and illegal, but they exist and enter the average consumption basket. Foley et al. (2019) provide an account of their spread in the bitcoin segment.

[11] It's also possible to have a sharper dis-utility by letting $\theta_{j,t} \leq 0$.

[12] Marginal utility is computed with respect to c_t^j, the product of two random variables.

and

$$M^{\$}_{t+1} = \Big(\frac{\theta_{\$,t+1}C_{\$,t+1}}{\theta_{\$,t}C_{\$,t}}\Big)^{-\gamma} = M^{C_{\$}}_{t+1}M^{\theta_{\$}}_{t+1}, \tag{3.3}$$

where $M^{C_j}_{t+1} = \Big(\frac{C_{j,t+1}}{C_{j,t}}\Big)^{-\gamma}$, $M^{\theta_j}_{t+1} = \Big(\frac{\theta_{j,t+1}}{\theta_{j,t}}\Big)^{-\gamma}$ *with* $j \in \{\$, \text{Ƀ}\}$

Lemma 1 formalizes the multiplicative decomposition of kernels characterizing the two distinct markets. It is instrumental in that it decomposes the kernels into two components, the complete-market consumption kernel $M^{C_j}_{t+1}$ and the incompleteness increment $M^{\theta_j}_{t+1}$. Given the multiplicative nature of most relationships, we will use small letters to denote logs of capitalized variables, e.g. $m^{\text{Ƀ}}_{t+1} = \log M^{\text{Ƀ}}_{t+1}$. The additional assumption below is needed for any operation requiring moment-generating functions.

Assumption 3. *The log SDFs in the two markets,* $m^{\text{Ƀ}}$, $m^{\$}$, $m^{\theta_{\text{Ƀ}}}$, $m^{\theta_{\$}}$, $m^{C_{\text{Ƀ}}}$ *and* $m^{C_{\$}}$ *are normally distributed.*

3.1 Incompleteness

Let $\Omega = \{\omega_s\}$ be the set of states of nature. Extending the previous notation, associate benefits with states of nature: $\theta_{j,s}$ is the benefit realized when the state of nature s occurs; if the state of nature $s = s(t)$ occurs at time t, we would write $\theta_{j,t} = \theta_{j,s(t)}$. Denote $E^{\$}_k = \{\omega_s|\theta_{\$,s} = \theta_k\}$ the event (set of states of nature) that the transaction in the dollar market is characterized by benefit θ_k. Similarly, denote $E^{\text{Ƀ}}_k = \{\omega_s|\theta_{\text{Ƀ},s} = \theta_k\}$ the event that the transaction in the bitcoin market is characterized by benefit θ_k. Since θ_k is a non-negative real number, $E^{\$}_k \cap E^{\$}_n = \varnothing$ for any k and n; i.e. event sets $E^{\$}_k$ form a partition on Ω (same holds for the bitcoin sets). We can now formalize the nature of the incompleteness in the model.

Proposition 2. *If there exists at least one tuple* (k, k_1, k_2) *such that* $E^{\$}_k \supseteq E^{\text{Ƀ}}_{k_1} \cup E^{\text{Ƀ}}_{k_2}$, *then a complete set of state contingent claims is unfeasible in the dollar market.*

Condition $E^{\$}_k \supseteq E^{\text{Ƀ}}_{k_1} \cup E^{\text{Ƀ}}_{k_2}$ means that events $E^{\text{Ƀ}}_{k_1}$ and $E^{\text{Ƀ}}_{k_2}$ are indistinguishable for observers in the dollar market: a player in the dollar market who observes $E^{\$}_k$ cannot judge whether θ_{k_1} or θ_{k_2} occurs in the bitcoin market.[13] Contingent claims to hedge these events therefore cannot be constructed in the dollar market even if the dollar market is complete in the sense that all events in the dollar market can be covered. The lack of contingent claims or incapability of synthesizing contingent claims to reflect or hedge transaction benefits gives rise to the consequence that payoffs cannot span all the possible states of nature and some SDFs may lie outside the space of traded assets. Generally, in incomplete markets there may be infinitely many pricing kernels. Proposition 1 in Backus et al. (2001) demonstrates one can select from the set of all admissible kernels those that satisfy conditions one would normally expect in the set of complete markets. The key to their result is the possibility to use instruments that exist in one market to hedge, or replicate, returns on investments in the second market. This implicitly assumes states of nature that realize in the first market cover all states realizable in the second market, *and* the realization of the states of nature in the second market is observable. Anonymity of transactions in the bitcoin market is a feature that violates observability, unless consumers wish to break it. If they break anonymity, the benefits of using bitcoins, relative to dollars, disappear. In our formulation, this would mean transaction benefits are governed by the same stochastic process, covering the same events, and the issue of incompleteness does not arise.

By our assumption, incompleteness stems from the stochastic benefits, and hence stochastic benefits determine the wedge in the sense of Lustig and Verdelhan (2019). Once the wedge can be quantified and filtered out from the pricing kernel, the kernels from the two markets are no longer contaminated by the benefit-related shocks, but only reflect the consumption shocks. For this reason, the "benefit-free" kernel is unique. This gives rise to the following relationship between the bitcoin and dollar-denominated pricing kernels:

[13] One can replace Ƀ and \$ indexes with i and $-i$ respectively, to denote some "designated" player i and the rest of the market. In this case player i is able to distinguish between events E_{k_1} and E_{k_2} but cannot hedge or insure against them separately because for the rest of the market they look as one event E_k.

Proposition 3. *There exists a unique discount factor given by benefit-free kernels* $\frac{M^{\text{Ƀ}}_{t+1}}{M^{\theta_{\text{Ƀ}}}_{t+1}}$ *and* $\frac{M^{\$}_{t+1}}{M^{\theta_{\$}}_{t+1}}$*:*

$$\frac{M^{\text{Ƀ}}_{t+1}}{M^{\theta_{\text{Ƀ}}}_{t+1}} = \frac{M^{\$}_{t+1}}{M^{\theta_{\$}}_{t+1}} \frac{Q_{t+1}}{Q_t}, \tag{3.4}$$

where Q_t is the real exchange rate. Equation 3.4 is the counterpart of (2.4) recognizing the differences between the two market kernels. In the log form, (3.4) becomes

$$\Delta q_{t+1} = m^{\text{Ƀ}}_{t+1} - m^{\$}_{t+1} - m^{\theta_{\text{Ƀ}}}_{t+1} + m^{\theta_{\$}}_{t+1}. \tag{3.5}$$

To highlight the logic of Proposition 3, recall that $\theta_{\text{Ƀ},t+1}$ and $\theta_{\$,t+1}$ capture incompleteness. If both benefit kernels equal unity then by Lemma 1 holds $m^{\text{Ƀ}}_{t+1} = m^{C_{\text{Ƀ}}}_{t+1}$, $m^{\$}_{t+1} = m^{C_{\$}}_{t+1}$. At the same time, for complete markets (2.4) implies $\Delta q_{t+1} = m^{\text{Ƀ}}_{t+1} - m^{\$}_{t+1} = m^{C_{\text{Ƀ}}}_{t+1} - m^{C_{\$}}_{t+1}$. This latter equality also holds in our incomplete markets case exactly because kernels $m^{C_{\text{Ƀ}}}_{t+1}$ and $m^{C_{\$}}_{t+1}$ are constructed to be free of incompleteness. For arbitrary $m^{\theta_{\text{Ƀ}}}_{t+1}$ and $m^{\theta_{\$}}_{t+1}$, by (3.2) and (3.3) in Lemma 1, holds $m^{C_{\text{Ƀ}}}_{t+1} = m^{\text{Ƀ}}_{t+1} - m^{\theta_{\text{Ƀ}}}_{t+1}$ and $m^{C_{\$}}_{t+1} = m^{\$}_{t+1} - m^{\theta_{\$}}_{t+1}$, which leads directly to (3.5), and hence to (3.4) in Proposition 3.

The Euler equation in Proposition 1 can now be recast to account for the benefit contamination and heterogeneity:

$$\mathsf{E}_t\Big[\frac{M^{\$}_{t+1}}{M^{\theta_{\$}}_{t+1}} \frac{P_t}{P_{t+1}}\Big] = \mathsf{E}_t\Big[\frac{M^{\text{Ƀ}}_{t+1}}{M^{\theta_{\text{Ƀ}}}_{t+1}} \frac{B_t}{B_{t+1}}\Big] = \mathsf{E}_t\Big[\frac{M^{\$}_{t+1}}{M^{\theta_{\$}}_{t+1}} \frac{Q_{t+1}}{Q_t} \frac{B_t}{B_{t+1}}\Big] = 1 \tag{3.6}$$

Note that Proposition 3 holds at each moment of time (in each state of nature), not only in expectations.

3.2 Return and risk of Bitcoin under incompleteness

We are now ready to re-formulate the fundamental equations from Section 2 with an explicit account of benefits obtainable in both markets. Utilizing Proposition 3 and concerning incompleteness, the Euler equation for BTC holders can now be written as follows:

$$\mathsf{E}_t\Big[M^{\$}_{t+1}\frac{Q_{t+1}}{Q_t}\Big] = \mathsf{E}_t\Big[M^{\text{Ƀ}}_{t+1}\frac{M^{\theta_{\$}}_{t+1}}{M^{\theta_{\text{Ƀ}}}_{t+1}}\Big] = \frac{1}{R^{\text{Ƀ}}_{t+1}} \tag{3.7}$$

The Euler equation for dollar holders reads as:

$$\mathsf{E}_t\Big[M^{\text{Ƀ}}_{t+1}\frac{Q_t}{Q_{t+1}}\Big] = \mathsf{E}_t\Big[M^{\$}_{t+1}\frac{M^{\theta_{\text{Ƀ}}}_{t+1}}{M^{\theta_{\$}}_{t+1}}\Big] = \frac{1}{R^{\$}_{t+1}} \tag{3.8}$$

$R^{\text{Ƀ}}_{t+1}$ and $R^{\$}_{t+1}$ are the risk-free rate (in an exponential form) in the bitcoin and dollar market, respectively. See more discussion in the Appendix (Proposition A). We follow Lustig and Verdelhan (2019) to pin down the "perturbed SDF", i.e. the kernel multiplied by the wedge, $\frac{M^{\theta_{\$}}_{t+1}}{M^{\theta_{\text{Ƀ}}}_{t+1}}$, or by its reciprocal. The expected value of the perturbed SDF, like the conventional SDF, is the risk-free rate.

Proposition 4. *Under the assumptions in this section, the arbitrage-free expected return on bitcoins is:*

$$\mathsf{E}_t(\Delta q_{t+1}) = r^{\$}_t - r^{\text{Ƀ}}_t + \frac{1}{2}\Big[\,Var_t(m^{\$}_{t+1}) - \,Var_t(m^{\text{Ƀ}}_{t+1})\Big] + \mathsf{E}_t m^{\theta_{\$}}_{t+1} - \mathsf{E}_t m^{\theta_{\text{Ƀ}}}_{t+1}. \tag{3.9}$$

Both the first component of (3.9) in Proposition 4, i.e. the interest rate differential, and the second component, i.e. the difference in the variance risk of two pricing kernels, are rather common in the conventional asset pricing theory. The novelty is the last term, the benefit-driven deviation of the market risk of two pricing kernels. This term arises due to incomplete markets, as the incompleteness is benefits-driven. If benefits are not present (complete markets) or are driven by the same stochastic process (in which case the incompleteness is irrelevant as events that cannot be distinguished from each other will never realize), the benefit-kernel differential turns zero.

Equation (3.9) implies that a relative rise $\mathsf{E}_t m^{\theta_{\$}}_{t+1}$ over $\mathsf{E}_t m^{\theta_{\text{Ƀ}}}_{t+1}$ leads to an expected appreciation of bitcoins. $\mathsf{E}_t m^{\theta_{\$}}_{t+1}$ and $\mathsf{E}_t m^{\theta_{\text{Ƀ}}}_{t+1}$ can be seen as the market price of benefit-related kernels in the two respective markets. It implies a potential depreciation once the market risk of benefit-related kernel in the decentralized market exceeds that of the centralized market. In Section 4, we will show the difference $\mathsf{E}_t m_{t+1}^{\theta_\$} - \mathsf{E}_t m_{t+1}^{\theta_{\text{B}}}$ gets an interpretation of the stochastic benefit differential with the help of a dominance relationship between distributions of θ_{B} and that of $\theta_\$$.

Proposition 5. *Under the assumptions in this section, the arbitrage-free variance of bitcoin return is*

$$Var_t(\Delta q_{t+1}) = Var_t(m_{t+1}^{\text{B}}) + Var_t(m_{t+1}^{\$}) + Var_t(m_{t+1}^{\theta_{\text{B}}}) - Var_t(m_{t+1}^{\theta_\$}) - 2Cov_t(m_{t+1}^{\text{B}}, m_{t+1}^{\$}) \tag{3.10}$$

Proposition 5 indicates that a high fluctuation of Bitcoin price (in dollars) is attributed to the variance risk of the benefit-related kernel in the decentralized market, relative to the variance of the benefit-related kernel in the centralized market. Uncertainty associated with benefits from bitcoin transactions adds to the volatility of the bitcoin exchange rate. Whenever consumers perceive benefit-driven uncertainty in the decentralized market to be high, the exchange rate becomes more volatile in response. As in the conventional international finance literature, $\text{Cov}_t(m_{t+1}^{\text{B}}, m_{t+1}^{\$})$ is inversely related to the fluctuation of the bitcoin/dollar exchange rate: the stronger the negative correlation between the kernels, the higher the volatility of the exchange rate. In the following section, we elaborate, by matching the realized exchange rate volatility with the variance of the implied kernels, the association between $\text{Cov}_t(m_{t+1}^{\text{B}}, m_{t+1}^{\$})$ and the second moment deviation of the benefit-driven kernels, $\text{Var}_t(m_{t+1}^{\theta_{\text{B}}}) - \text{Var}_t(m_{t+1}^{\theta_\$})$.

3.3 Risk premium of bitcoins

The notion of risk premium or excess return for the Bitcoin market is rather heuristic as the cryptocurrency world lacks a well-accepted interest rate or a regulated money market.[14] We make use of the synthetic interest rate, the reciprocal of the expected

[14]There has been a striking growth of the decentralized finance (DeFi), known as crypto lending market. Such lending and borrowing activities are operated in Permissionless Lending Protocols, systems that allow users to lend and borrow various digital assets typically through so-called smart contracts

pricing kernel, explained in Proposition A in the Appendix, to derive the moment-based determinants of the risk premium. For a modeling convenience, we assume the interest rates in both markets exist and can be used in terms of lending and borrowing. Consider the following strategy: borrow funds in the centralized market at $r_t^{\$}$, convert them to Bitcoins, lend Bitcoins in the decentralized market, earn $r_t^{\text{Ƀ}}$, then convert back to the centralized market at $t+1$. This strategy defines the Bitcoin excess return, r^e:

$$\mathsf{E}_t(r_{t+1}^e) = r_t^{\text{Ƀ}} - r_t^{\$} + \mathsf{E}_t(\Delta q_{t+1}) \tag{3.11}$$

The Bitcoin risk premium is exactly this excess return: it shows the gains of lending in bitcoins relative to those of lending in dollars. If markets were complete and there was no difference in risk, the interest rate parity would imply $\mathsf{E}_t(\Delta q_{t+1}) = r_t^{\$} - r_t^{\text{Ƀ}}$ and hence a zero excess return. The following corollaries characterize the excess return on bitcoins, and in particular articulate the role of benefits, which are the source of incompleteness in our model:

Corollary 1. *Under the assumptions in this section, the expected arbitrage-free excess return on bitcoins is:*

$$\mathsf{E}_t(r_{t+1}^e) = \frac{1}{2}\Big[Var_t(m_{t+1}^{\$}) - Var_t(m_{t+1}^{\text{Ƀ}})\Big] + \mathsf{E}_t m_{t+1}^{\theta\$} - \mathsf{E}_t m_{t+1}^{\theta\text{Ƀ}} \tag{3.12}$$

The first term, the difference of the second moments of kernels, coincides with the conventional asset pricing theory, while the second term underscores the impact of incompleteness on the risk premium. Like (3.9), a relative rise $\mathsf{E}_t m_{t+1}^{\theta\$}$ over $\mathsf{E}_t m_{t+1}^{\theta\text{Ƀ}}$ leads

on the Ethereum blockchain. Smart contracts are not contracts in the legal sense; rather, they are computer codes that effectively lives and executes on the Ethereum blockchain. Users that lend assets are distributed the interest earned from their loans. Borrowers are required to post collateral in the form of designated cryptos, typically featured by over-collateralization (value of collateral greater than the value of the loan) to reduce credit risk, and pay interest rate by smart contract agreement to lender. The risk in Permissionless Lending Protocols can be assessed by DeFi Score, a composite score comprises of smart contract risk referring a risk that a smart contract is hacked, financial risk referring to the volatility or downside risk of collateral; and intermediary risk about protocol Administration risk. Three risk account for 45%, 35% and 25% roughly. Among the smart lending contracts, Dai savings rate is considered as the safest (risk-free) one and is used as the base rate in the permissionless or trustless lending markets, given the fact that by construction its coin value is pegged to U.S. dollar known as stablecoin.

to an increase of Bitcoin risk premium. In the case of a high market price of benefit-related kernels in the decentralized market, $\mathsf{E}_t m_{t+1}^{\theta_B}$, or high variance risk of $\text{Var}_t(m_{t+1}^{B})$, the Bitcoin users gain less excess returns.

4 Pricing the benefits: the stochastic dominance view

Decision makers in our model deal with two sources of uncertainties or two state variables, consumption shocks and transaction benefits, constituting *multi-dimensional* uncertainty. In our model, having different sources of uncertainty is a key feature of incomplete markets, manifesting in the benefit-related kernels $m_{t+1}^{\theta_B}$ and $m_{t+1}^{\theta_\$}$ and playing a building block in the bitcoin pricing. In this section, to get a deeper insight, we offer a stochastic dominance view on the benefit-related kernels.

4.1 Two-dimensional uncertainty

We re-frame the multi-period model of Levy (1976) to accommodate two state variables (consumption and benefits, both represent uncertainty) in our context and derive the sufficient and necessary condition for the efficient consumption strategy. The notion of "efficient" in Levy (1976) corresponds to strict preferences towards one or another alternative, and is formulated in terms of the stochastic dominance analysis. This notion of "efficiency" in our context defines consumers (and transactions) for whom bitcoins are strictly preferred to dollars. Existence of these preferences (and contexts in which they arise) explains the existence of the bitcoin market.

The utility function is of the specific form of $u(x) = \log(x)$, a special case of the CRRA utility used in Section 3 with the risk aversion coefficient equal to unity. It is worth noting that without loss of generality, the multi-uncertainty decision rule can be applied to any kind of utility functions. However, the conditions to be not only sufficient but also necessary hinge on the specification of the utility function, see more discussion in

Levy and Paroush (1974). With the logarithmic utility, we obtain the following additively separable representation:[15]

$$u(c \cdot \theta) = \log(c \cdot \theta) = u_1(c) + u_2(\theta) \tag{4.1}$$

where $u_1(c) = \log(c)$ and $u_2(\theta) = \log(\theta)$. u_1 and u_2 are non-negative under a certain range and non-decreasing function whose expected values exist with respect to F_1 and G_1 for u_1 and F_2 and G_2 for u_2. F_1 and F_2 are marginals of the joint distribution $F(c, \theta)$ in the decentralized market, while G_1 and G_2 are marginals of the joint distribution $G(c, \theta)$ in the centralized market. The logarithmic form of utility has nice properties such as non-decreasing $(u'(\cdot) > 0)$ and non-increasing marginal utility $(u''(\cdot) < 0)$, a concave shape of function alike. Note that the specification of utility in (4.1) implies that consumption and transaction benefit are neither substitutes nor complements.

Given the utility functions, joint and marginal distributions of bivariate random variables, the consumption strategy in the decentralized market with a joint probability $F(c, \theta)$, is preferred to the strategy in the centralized market with $G(c, \theta)$, if

$$\mathsf{E}_F u(c, \theta) - \mathsf{E}_G u(c, \theta) = \int (G_1(t) - F_1(t)) du_1(t) + \int (G_2(s) - F_2(s)) du_2(s) \tag{4.2}$$

$$= \int (G_1(t) - F_1(t)) \frac{1}{t} dt + \int (G_2(s) - F_2(s)) \frac{1}{s} ds \geq 0 \tag{4.3}$$

Proposition 6 (First-order dominance). *With the logarithmic utility functions which are additively separable and joint probability $F(c, \theta)$, $G(c, \theta)$, marginal distribution $F_1(c)$, $G_1(c)$, $F_2(\theta)$, $G_2(\theta)$, a necessary and sufficient condition for the dominance of decentralized-type transaction over the centralized alternative in the first-order fashion is $G_1(c) \geq F_1(c)$ and $G_2(\theta) \geq F_2(\theta)$, with strong inequality for at least some values in c or in θ.*

For the second-order dominance (which is implied in Proposition 6), we separately

[15] Additively separable logarithmic utility functions was introduced by Samuelson (1954) and is picked up again in the recent work by Miyake (2016).

integrate the first and the second terms of (4.3) by parts to yield:

$$
\begin{aligned}
u_1\prime(\infty)\int_1^{\infty}(G_1(t)-F_1(t))dt-\int_1^{\infty}u_1^{''}(t)\int_1^{t}(G_1(s)-F_1(s))dsdt &= \\
\int_1^{\infty}\frac{1}{t^2}\int_1^{t}(G_1(s)-F_1(s))dsdt &\geq 0 \\
u_2\prime(\infty)\int_1^{\infty}(G_2(t)-F_2(t))dt-\int_1^{\infty}u_2^{''}(t)\int_1^{t}(G_2(s)-F_2(s))dsdt &= \\
\int_1^{\infty}\frac{1}{t^2}\int_1^{t}(G_2(s)-F_2(s))dsdt &\geq 0
\end{aligned}
$$

In the above two equations we make use of $u_1\prime(\infty) = u_2\prime(\infty) = 0$ and of the twice differentiable logarithmic utility. This yields the following second-order dominance result.

Proposition 7 **(Second-order dominance).** *Under assumptions of Proposition 6 and $u_1^{''} < 0, u_2^{''} < 0$, a necessary and sufficient condition for the second-order dominance of the decentralized-type transaction over the centralized alternative is $\int_t^{\infty}(F_1(s)-G_1(s))ds \geq 0$ and $\int_t^{\infty}(F_2(s)-G_2(s))ds \geq 0$ for all $s > t$ with at least one strict inequality.*

Under additive utility, the first- and the second-order dominance conditions for efficient currency choice underscore the importance of marginal distributions .Combining the above general formulation with an assumption of comparable consumption uncertainties in the two markets (as we assumed in the main model), $F_1 \simeq G_1$, we obtain the uncertainty about transaction benefits plays a pivotal role. Agents' choice to make Bitcoin payments reflects the projected expected utility with respect to the distribution of bitcoin transaction benefits, F_2.

4.2 Implications for Bitcoin pricing

To gain the insight into the extent to which the dollar value of Bitcoin reflects the dominance relationship between the distribution of $\theta_{\text{Ƀ}}(F_2)$ and that of $\theta_{\$}(G_2)$, we re-visit Eq. (3.9), with a focus on $\mathsf{E}_t m_{t+1}^{\theta_{\$}}$ and $\mathsf{E}_t m_{t+1}^{\theta_{\text{Ƀ}}}$. If a dominance of F_2 over G_2 is anticipated, one should be able to infer $\mathsf{E}_t m_{t+1}^{\theta_{\$}} \geq \mathsf{E}_t m_{t+1}^{\theta_{\text{Ƀ}}}$ and subsequently a rise of $\mathsf{E}_t(\Delta q_{t+1})$. The proof is sketched below:

$$\begin{aligned}
\mathsf{E}_t m_{t+1}^{\theta_\$} - \mathsf{E}_t m_{t+1}^{\theta_{\text{Ƀ}}} &= \theta_{\$,t}\mathsf{E}_t\Big(\frac{1}{\theta_{\$,t+1}}\Big) - \theta_{\text{Ƀ},t}\mathsf{E}_t\Big(\frac{1}{\theta_{\text{Ƀ},t+1}}\Big) \\
&= \int \frac{1}{t} dG_2(t) - \int \frac{1}{t} dF_2(t) \\
&= \int \frac{1}{t} d(G_2 - F_2) \\
&= \int (F_2 - G_2) d\frac{1}{t} \\
&= -\int (F_2 - G_2)\frac{1}{t^2} dt
\end{aligned}$$

We assume $\theta_{\text{Ƀ},t} = \theta_{\$,t} = 1$ or any scalar since they can be observed at t. The fourth equality is derived after applying the Lemma of Levy and Paroush (1974). As long as F_2 dominates G_2 in the first-order sense indicated in Proposition 6, having $G_2(\theta) \geq F_2(\theta)$ ensures $\mathsf{E}_t m_{t+1}^{\theta_\$} - \mathsf{E}_t m_{t+1}^{\theta_{\text{Ƀ}}} > 0$. In sum, $\mathsf{E}_t m_{t+1}^{\theta_\$} \geq \mathsf{E}_t m_{t+1}^{\theta_{\text{Ƀ}}}$ implies a dominance of F_2 over G_2 and vice versa. Once agents perceive a higher probability of transaction benefits in the decentralized market relative to that in the counterpart market, the price of Bitcoin in terms of dollar in (3.9) moves up. This also holds for the expected log-excess return of Bitcoins in (3.12).

5 Empirical application and discussion

To make sense of the model and understand the empirical counterparts of the key parameters, we first calibrate key parameters using interest rate data (T-bills for dollars and the long-short spread in bitcoin futures) and use the implied wedge to get insights on incompleteness. We then proceed with an empirical estimate of the benefit-related kernels, the fundamental factors in our pricing approach, by using data on the number of coins sent, assuming a positive association between the number of coins sent through the Blockchain networks with benefits from trading in bitcoins. This approach further helps understand the dynamics of the bitcoin price through the dynamics of underlying kernels. Going ahead, our data stretches up to the Covid-19 pandemic, showing a pronounced impact of the latter on the dollar-consumption kernel accompanied by a much milder effect on the bitcoin pricing kernel and on the benefits kernels.

5.1 Calibration and interpretation

This section aims to calibrate the implied moments of the benefits kernels. To do this, we need the following: currency risk premium, maximum Sharpe ratio of risk premium acting as the bound for volatility of the pricing kernel and the exchange rate between BTC and dollar. Implied values $\mathsf{E}_t m_{t+1}^{\theta_\$}$, $\mathsf{E}_t m_{t+1}^{\theta_{\text{Ƀ}}}$, $\mathrm{Var}_t(m_{t+1}^{\theta_\$})$, and $\mathrm{Var}_t(m_{t+1}^{\theta_{\text{Ƀ}}})$ are obtained from the theoretical results formulated above, as we demonstrate in this section. We will use data for the period between February 2018 and September 2020, which allows us to make use of Bitcoin futures data. The annualized volatility of BTCUSD in this period has been around 80%, which does not come as a big surprise as coins like Bitcoin are known to be notoriously volatile. Yet the period we focus on avoids extreme turbulence periods (which some would refer to as the bubble period) as well as the extremely thin trading period in the early-adoption phase. Note that this exchange rate between the decentralized markets and the centralized markets exhibits more than seven times the volatility of the exchange rates within the centralized markets themselves.

Bitcoin risk-free rate. For many purposes, we need a characterization of the risk-free Bitcoin interest rate. Dollar investors in the centralized markets can buy government-backed bonds to earn risk-free yields, in contrast to the decentralized markets, which lack purely risk-free assets such as those guaranteed by governments. One way to synthesize the Bitcoin risk-free rate is to resort to the regulated derivative markets such as the Chicago Mercantile Exchange (CME). CME launched Bitcoin futures contracts in the end of 2017 to offer a marketplace for hedgers and speculators to manage their Bitcoin exposure. A synthetic Bitcoin risk-free rate obtains as a spread of taking a long position in spot bitcoins and simultaneously taking a short position in Bitcoin futures: the yield from this strategy is essentially risk-free. Instead of going long in the

BTC spot market, we take a long position in the front-month contract and short the back-month contract. For this exercise, we collect the near-term CME futures prices and the futures contracts in longer terms, such as 3 months, expiring on a daily basis and calculate the yield. The resulting average annualized interest rate is $r^{\text{Ƀ}} = 2.45\%$.

Currency risk premium. Having the synthetic risk-free rate derived from the BTC futures, one can quantify Bitcoin risk premium, r^e denoted in Corollary 1, as the average carry trade excess return obtained by borrowing in low interest rate currency and investing in high interest rate currency. The average risk premium corresponding to the average carry trade excess return is 48.118% with the standard deviation 80.2%. All numbers are annualized. The Bitcoin is thus characterized by a strictly positive risk premium of $r^e = 48.118\%$. For comparison, the equity market risk premium is in the range 5-6%.[16] Given the risk-free rates are low, the major part of the excess return comes from the exchange rate fluctuations.

Sharpe ratio as the lower bound of the variability of a pricing kernel. Recall that the maximum Sharpe ratio obtainable from the carry trade excess return determines a lower bound on the variability of a pricing kernel, see Hansen and Jagannathan (1991) using the covariance decomposition and Cauchy-Schwarz inequality. The Sharpe ratio of bitcoin excess return imposes the restrictions on the volatility of pricing kernel, more explicitly, $\frac{\sigma(m^{\text{Ƀ}})}{\mathsf{E}m^{\text{Ƀ}}} \geq \frac{\mathsf{E}r^e}{\sigma(r^e)} = \frac{0.481}{0.802} = 0.6$.

Second moments of composite kernels. The maximum Sharpe ratio paves the way for the theoretical pricing model so that it can be vital under the pricing errors. The imposed lower bound to the volatility of $m^{\text{Ƀ}}$, from the derived maximum annual Sharpe ratio, is 0.6, which is a relatively conservative estimate compared to the squared weekly Sharpe-ratio 0.057 in Liu et al. (2019). The Sharpe ratio in the Bitcoin asset is economically sizable. Following Lustig and Verdelhan (2019), we set the volatility of $m^{\$}$ equal to $\sigma(m^{\$}) = 0.5$, which is a bit lower than volatility of $m^{\text{Ƀ}}$. We agree with

[16]See, e.g., the KPMG Equity Market Risk Premium Research Summary, https://assets.kpmg/content/dam/kpmg/nl/pdf/2019/advisory/equity-market-risk-premium-research-summary-31032019.pdf

Lustig and Verdelhan (2019) in terms of preserving the volatility bound of kernels in a conservative manner, in a consideration of not raising the bar for incomplete model too much. In Lustig and Verdelhan (2019), the variance of wedge is proportionally increasing with the variance of kernels, that is, a higher variance bound of kernels, a higher variance of wedge.

Implied second moments of benefit kernels. The implied difference between $\mathrm{Var}_t(m_{t+1}^{\theta_{\text{Ƀ}}})$ and $\mathrm{Var}_t(m_{t+1}^{\theta_{\text{Ƀ}}})$ can be inferred by re-arranging (3.10) in Proposition 5:

$$\begin{aligned}
\mathrm{Var}_t(m_{t+1}^{\theta_{\text{Ƀ}}}) - \mathrm{Var}_t(m_{t+1}^{\theta_{\$}}) &= \mathrm{Var}_t(\Delta q_{t+1}) - \mathrm{Var}_t(m_{t+1}^{\text{Ƀ}}) - \mathrm{Var}_t(m_{t+1}^{\$}) + 2\mathrm{Cov}_t(m_{t+1}^{\text{Ƀ}}, m_{t+1}^{\$}) \\
&= 0.80^2 - 0.6^2 - 0.5^2 + 2\mathrm{Cov}_t(m_{t+1}^{\text{Ƀ}}, m_{t+1}^{\$}) \\
&= 0.03 + 2\mathrm{Cov}_t(m_{t+1}^{\text{Ƀ}}, m_{t+1}^{\$})
\end{aligned}$$

By matching the realized volatility of the exchange rate with the second moment of implied kernels from a maximum Sharpe ratio, we obtain the benefits kernels in the two markets are characterised by different volatilities if the covariance (correlation) of kernels deviates from -0.015 (-0.06). In other words, if $\mathrm{Cor}_t(m_{t+1}^{\text{Ƀ}}, m_{t+1}^{\$})$ is around -0.06, then the two benefit kernel processes behave similarly in terms of the second moment of their distributions. More explicitly, one could pay less attention to the wedge if the correlation is around -0.06 but better be aware of the emerging wedge caused by a high positive or negative correlation. To elaborate how the correlation of kernels implies the difference between the volatility of the benefits kernels and the subsequent implied wedge in the second moment notion, we depict in Figure 5.1 the difference between the variance of the implied benefits kernels in the two markets against the implied correlation between $m_{t+1}^{\text{Ƀ}}$ and $m_{t+1}^{\$}$. It would be counterintuitive to assume the benefits kernel in the dollar market is more volatile than that in the bitcoin market. Therefore, one should not expect the correlation between the two kernels to be far below -0.06. In short, the derived insight enables us to attribute a relatively higher fluctuation in the bitcoin benefits kernel, if it is observed in the data, to an interdependence of the two kernels.

First moments of benefits kernels. We switch our focus to the calibration of the implied first moment of benefit-related kernels. Recall Corollary 1 for a relationship between the Bitcoin risk premium and the expected benefit-kernel differential $\mathsf{E}_t m_{t+1}^{\theta_\$} - \mathsf{E}_t m_{t+1}^{\theta_B}$:

$$\mathsf{E}_t(r_{t+1}^e) = \frac{1}{2}\Big[\mathrm{Var}_t(m_{t+1}) - \mathrm{Var}_t(m_{t+1}^B)\Big] + \mathsf{E}_t m_{t+1}^{\theta_\$} - \mathsf{E}_t m_{t+1}^{\theta_B}.$$

The benefit-kernel differential characterizes the degree of market incompleteness (the wedge): if $\mathsf{E}_t m_{t+1}^{\theta_\$} = \mathsf{E}_t m_{t+1}^{\theta_B}$, we are back in the complete markets case. The expected bitcoin excess return $\mathsf{E}_t(r_{t+1}^e)$ has been calibrated above at 48.118%. Using the same parameters for the second moments as above (see Figure 5.1), we obtain

$$\begin{aligned} 0.481 = \mathsf{E}_t(r_{t+1}^e) &= \frac{1}{2}\Big[\mathrm{Var}_t(m_{t+1}) - \mathrm{Var}_t(m_{t+1}^B)\Big] + \mathsf{E}_t m_{t+1}^{\theta_\$} - \mathsf{E}_t m_{t+1}^{\theta_B} \\ &= \frac{1}{2}(0.5^2 - 0.6^2) + \mathsf{E}_t m_{t+1}^{\theta_\$} - \mathsf{E}_t m_{t+1}^{\theta_B} \\ &= -0.055 + (\mathsf{E}_t m_{t+1}^{\theta_\$} - \mathsf{E}_t m_{t+1}^{\theta_B}) \end{aligned}$$

The calibrated value of $\mathsf{E}_t(r_{t+1}^e)$ fits its theoretical decomposition only if the wedge is strictly positive, $\mathsf{E}_t m_{t+1}^{\theta_\$} - \mathsf{E}_t m_{t+1}^{\theta_B} = 0.536$, as illustrated in Figure 5.2. The implied wedge is qualitatively comparable with the empirical estimation in Figure 5.6 which will be discussed later.

The implied wedge of 0.536 highlights incompleteness. From the stochastic dominance analysis between the benefit distribution in the decentralized markets (F_2) and in the centralized market (G_2), we infer that F_2 dominates G_2, given a strictly positive wedge. In the economic context, agents perceive the probability of large transaction benefits to be higher in the decentralized market than in the centralized market, which may answer where the excess return comes from.

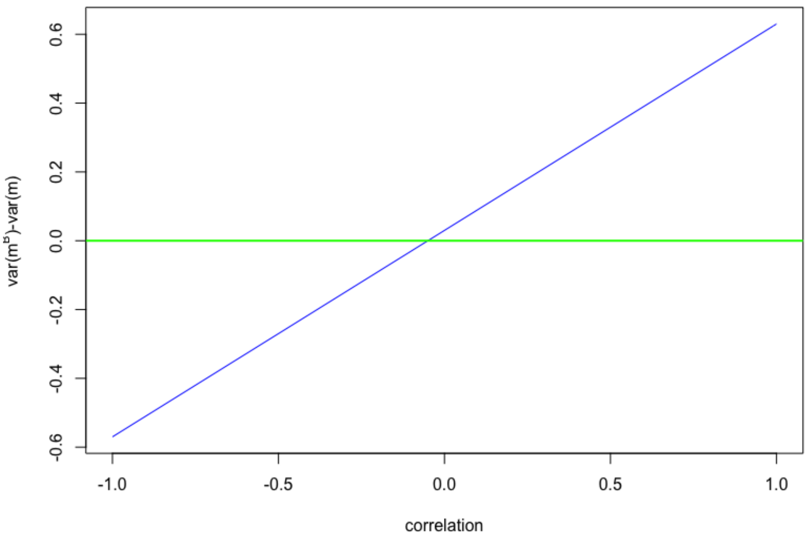


Figure 5.1: The implied second moment of the benefits kernels and the correlation of the kernels

The vertical axis is $\mathrm{Var}_t(m^{\theta_{B}}_{t+1}) - \mathrm{Var}_t(m^{\theta_{\$}}_{t+1})$ along with $\mathrm{Cor}(m^{B}_{t+1}, m^{\$}_{t+1}) \in [-1, 1]$ in the horizontal axis. By (3.10), this figure is depicted assuming a maximum Sharpe ratio 0.6 in the Bitcoin market and 0.5 in the dollar market which follows Lustig and Verdelhan (2019) to obtain $\mathrm{Var}_t(\Delta q_{t+1}) = 0.64$, $\mathrm{Var}_t(m^{B}_{t+1}) = 0.36$ and $\mathrm{Var}_t(m^{\$}_{t+1}) = 0.25$.

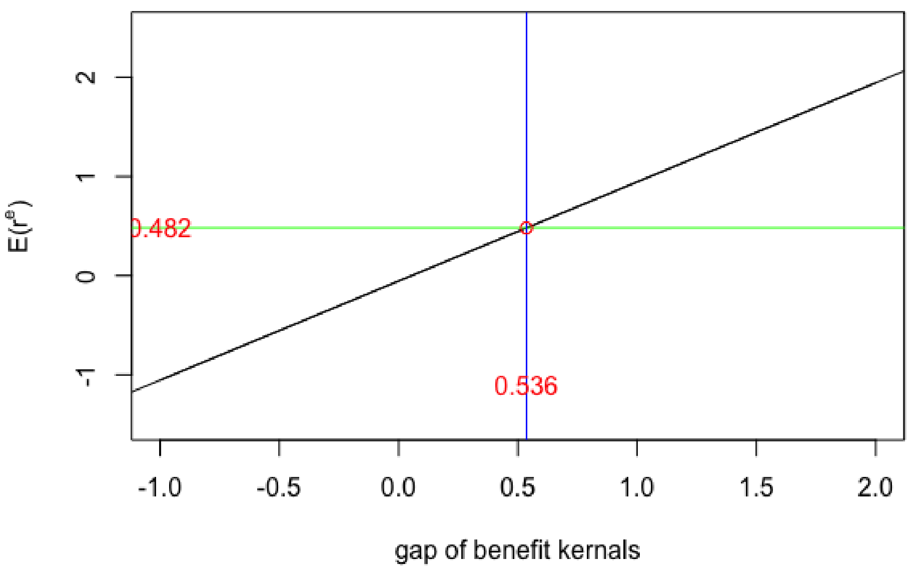


Figure 5.2: Theoretical relation between $\mathsf{E}_t(r^e_{t+1})$ and $\mathsf{E}_t m^{\theta_{\$}}_{t+1} - \mathsf{E}_t m^{\theta_{B}}_{t+1}$

$\mathsf{E}_t(r^e_{t+1}) = -0.055 + (\mathsf{E}_t m^{\theta_{\$}}_{t+1} - \mathsf{E}_t m^{\theta_{B}}_{t+1})$. Green line is the mean log excess return in practice while the blue line shows the implied wedge used to yield a coherence between theorems and practice.

5.2 Explaining bitcoin price dynamics with benefits

In our framework, consumption and benefits are both state variables but only consumption can be observed. To infer the benefit-related kernels, one can instead exploit the data that encode real transaction activities in the Blockchain network to parametrize a relation between the hidden benefits and the observed data. A model-free estimation method like GMM can cope with estimation challenges. Recall the Euler equation in Proposition 1 as the underpinning of the model:

$$\mathsf{E}_t\left[M^{B}_{t+1}\frac{B_t}{B_{t+1}}\right] = E_t\left[M^{\theta_B}_{t+1}M^{C}_{t+1}\frac{B_t}{B_{t+1}}\right] = 1$$

Using Lemma 1, we obtain

$$E_t\left[\Big(\frac{\theta_{B,t+1}}{\theta_{B,t}}\Big)^{-\gamma}\Big(\frac{C_{t+1}}{C_t}\Big)^{-\gamma}\frac{B_t}{B_{t+1}}\right] = 1 \tag{5.1}$$

Our strategy is to parametrize the pricing kernels by the GMM method, using the observed transaction data reflecting real economic activities in Blockchain network.[17] In (5.1) we have dropped the market-specific index for the consumption kernel. The actual consumption data for the bitcoin market is not readily available, see Foley et al. (2019) for details on the elaboration of the consumption patterns. In our model, the link between the consumption kernels in the dollar and the bitcoin markets is given by (3.5), however, this link involves the USD/BTC exchange rate. We could therefore use the real exchange rate data to construct the time series of the bitcoin-consumption kernel using (3.5), in order to test the internal consistency of the model. Instead, we seek to evaluate the potential role of benefits in explaining the bitcoin price dynamics. Recall, benefits were introduced to capture shocks that cannot be hedged by dollar instruments. We now go one step further and focus on those consumption shocks that are identical in both markets, namely the dollar-consumption kernel for its measurable feature. The benefits term in (5.1) would thus absorb all other shocks. Although we may expect this estimate

[17]Note that we opt for this strategy because there is no options data for cryptocurrencies with the quality that we can use for a non-parametric characterization (Aıt-Sahalia and Lo; 2000).

overstates the volatility of the benefits kernel, the insights from this exploitation remain robust. For C_t we will use the consumption spending data from the St Louis Fed.

Bitcoin inflation

The ratio $\frac{B_t}{B_{t+1}}$ in (5.1) reflects the purchasing power of Bitcoins. Colloquially, it is often measured by the Bitcoin Big Mac index, which shows how many Big Mac hamburgers one can buy with one bitcoin. This data is readily available from from `BitcoinPPI`[18]. However, the BigMac index is computed from a local BigMac price in a local currency (US dollars in our case) by converting it in bitcoins; as such, it is intrinsically based on the BTC exchange rate Q_t. If the BigMac price is relatively stable, the index effectively measures the growth rate of the BTC price in USD, which is what we aim to explain. On the one hand, the purpose of our exercise here is to estimate the benefit-related pricing kernel, hence using the BigMac index may be instructive in testing the consistency of our construct.[19] On the other hand, the Internet marketplaces that accept bitcoins as payment for goods (often luxury) and services (e.g. airport parking), obtainable in the dollar market, set bitcoin prices as conversions from current dollar prices, using current BTC-USD rate, exactly as done in the BigMac index.[20] If these goods were the only transactions in BTC, the BigMac index would correctly describe the bitcoin price level for our purposes.

Instead, evidence suggests, and in our model we assume, there exist anonymous undisclosed deals, for which we do not observe prices, but they enter the "average consumption bundle" in the bitcoin market. We therefore seek an alternative measure for the

[18] `http://bitcoinppi.com/`

[19] The BigMac index was introduced to measure the purchasing power parity, which compares the price of an identical good in two different markets, where prices are set by separate equilibrium processes; the prediction is then that the exchange rate should reflect the difference in prices. This is not applicable in our case as we have two distinct markets, but the good (BigMac) is priced only in one market, and then the price is converted from dollars to BTC. The standard purchasing parity is thus met by construction. Using the BigMac index in the kernel estimation is equivalent to assuming a market friction: in the absence of a distinct market equilibrium mechanism in the BTC segment, sellers use the PPP to compute the BTC price of goods; this only applies to goods and services available in *both* segments.

[20] Often they offer a discount for purchases in BTC, but still the price is a calculation from the dollar price. The constant discount is irrelevant for the growth rate calculation.

bitcoin price level. Formally, the price level is an average price of all transactions in the market, or a hypothetical price of an average consumption bundle. With this in mind, we resort to the quantity theory of money (QTM)[21]: equality $MV = PT$ holds in any period, here M is the total amount of money, V is the velocity of money circulation, P is the price level and T is the number of transactions. Applying this to the bitcoin market solely, we can define price level B as MV/T, where the total stock of bitcoins M and the number of transactions in each month T are available. Velocity V can be estimated as the number of coins sent N, divided by the total stock of bitcoins M. It follows that the bitcoin price level in each month is

$$B_t = \frac{N_t}{T_t}, \tag{5.2}$$

where N_t is the number of coins sent in period t, and T_t is the total number of transactions in period t. This gives the "average" price level in the bitcoin market, including anonymous undisclosed transactions. If there was a big ransom deal, for example, the average price B_t in that period would be high, this would add volatility to the price level. From this we can compute the BTC inflation B_{t+1}/B_t.

We use both the QTM and the BigMac methods, as they represent two extreme estimates of B_t: the latter assumes cross-sectionally and intertemporally homogeneous consumption bundle, yet no distinct equilibrium mechanism in the bitcoin market, while the former allows for a separate equilibrium in the bitcoin market, yet lacks the homogeneity of consumption. For a well diversified consumption bundle, if all sellers and buyers use the same bitcoin rate to convert prices from dollars to bitcoins, as is currently done by online shops, the two measures would coincide. If the QTM yields a different

[21] An interesting deviation from QTM is discussed in Pagnotta (2021), where an increase in the amount of bitcoins does not necessarily devalue them because any such increase is intrinsically related to the activity of miners, rewarding their contribution to the security of transactions, which in turn is valued by bitcoin buyers. The value of bitcoins in this observation corresponds to the BTC/USD exchange rate in our model. Our interest in this section is in the value of bitcoins in terms of goods and services circulating in the bitcoin-served market. The two values coincide only if the purchasing power parity holds. As we have shown, this is not the case in our model. The value of security is captured by θ, as part of the benefits, and as such does affect the exchange rate. Similarly, it affects the number of transactions, and thus the potential dis-proportionality between the bitcoin growth rate and its value is also implicitly captured in the elaborations in this section.

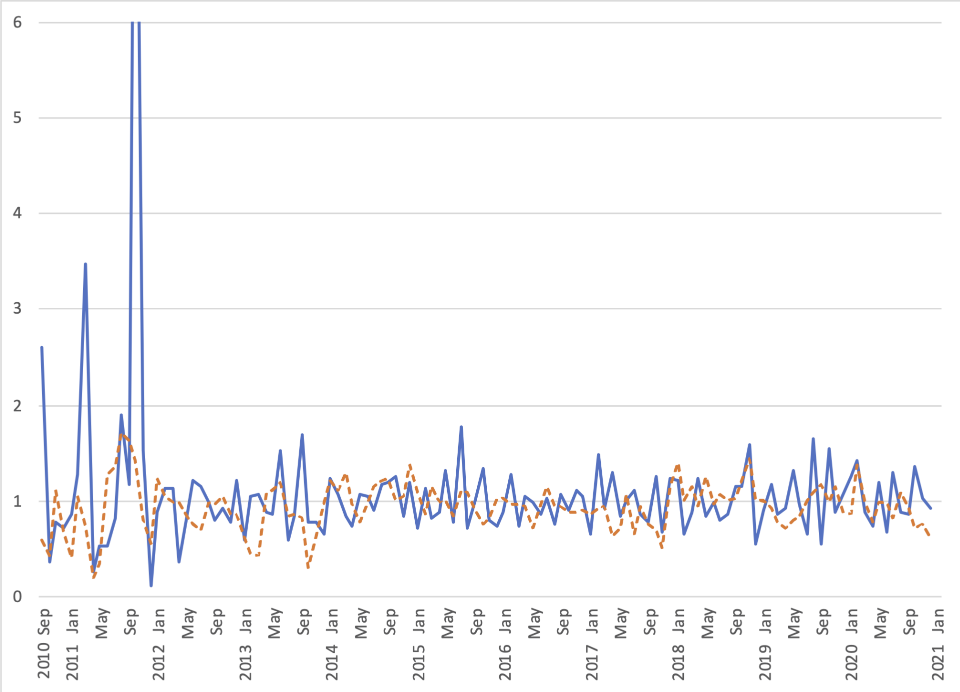


Figure 5.3: Bitcoin-market inflation B_{t+1}/B_t: September 2010 to December 2020. Solid line is the monthly QTM bitcoin inflation with B_t defined in (5.2). Dashed line is the monthly USD per BTC exchange rate growth Q_{t+1}/Q_t. Data source: blockchain.com.

price level, this shows there are transactions that are concluded at a price that differs from the one dictated by the PPP.

In Figure 5.3 we use data from blockchain.com to compute the BTC price level as "total BTC value of confirmed transactions" / "number of transactions" - this gives the average value of a BTC market transaction in bitcoins, which corresponds to the definition of a price level in (5.2). The data is collapsed to have monthly totals from Sep 2010 onwards. From these we computed an average monthly inflation, i.e. B_{t+1}/B_t and the exchange rate growth Q_{t+1}/Q_t where Q_t is the BTC rate in USD. After the November 2011 spike both exhibit rather bounded volatility around a horizontal trend with the average BTC inflation from 2012 onwards being around 99.5% (signifying a monthly deflation of 0.5% on average), and the average exchange rate growth (which would be what the Big Mac index would give, adjusted by the Big Mac price growth) about 1.13 (i.e. on average a 13% monthly growth, quite different from the QTM estimate).[22]

[22]Reports on historical Big Mac prices vary, yet even the most pessimistic of them give at most a 50% price growth over the last 10 years, translating in about 0.3-0.4% average monthly Big Mac price growth. For this reason, the 13% exchange rate growth figure is a good representation for the BigMac inflation. Discarding the spikes in 2013 and 2104 makes no substantial changes to the estimates. Using

Estimating benefit-related kernels

Although benefits are intangible and cannot be straightforwardly measured, they themselves are key drivers of real activities and incentivize transactions completed in decentralized markets. As an indicator of activity on the bitcoin network, we choose the number of coins N_t sent through the blockchain network at time t, collected from BitInfoCharts. This number reflects the real activity that involves spending bitcoins in all on-chain transactions (such as trades) between two addresses in the blockchain network. This approach thus captures our view of Bitcoins as a medium of exchange and the benefits associated with this role.

To parametrize, we assume the following relationship between benefits and bitcoin transactions:

$$\frac{\theta_{B,t+1}}{\theta_{B,t}} = \kappa_0 \Big(\frac{\mathrm{N}_{t+1}}{\mathrm{N}_t}\Big)^{\kappa_1}. \tag{5.3}$$

This functional form is not causal, i.e. we do not assume benefits depend on the number of users or transactions in the network; we only posit a relationship between benefits and the total number of coins sent. Log-linearity is numerically convenient given our utility specification, and yet preserves the flexibility needed for the parametrization exercise. The κ_1 coefficient governs a relationship between the hidden and the observed variables. With the specification in (5.3), the Euler equation turns into

$$E_t \left[\Big(\kappa_0 \Big(\frac{\mathrm{N}_{t+1}}{\mathrm{N}_t}\Big)^{\kappa_1}\Big)^{-\gamma} \Big(\frac{C_{t+1}}{C_t}\Big)^{-\gamma} \frac{B_t}{B_{t+1}} \right] = 1. \tag{5.4}$$

Now the residual term in the GMM is defined as:

$$e_{t+1} = \left(\kappa_0 \left(\frac{\mathrm{N}_{t+1}}{\mathrm{N}_t}\right)^{\kappa_1}\right)^{-\gamma} \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \frac{B_t}{B_{t+1}} - 1, \tag{5.5}$$

where $\kappa_0, \kappa_1, \gamma$ are the parameters to be estimated. The instrumental variables that are

the BigMac index thus appears to overestimate the bitcoin price inflation, as compared to that given by the QTM.

Parameter	coefficient	(standard error)	t value
Panel A: estimates with BigMac-based BTC inflation			
κ_0	1.225	(0.079)	17.383
κ_1	0.828	(0.158)	5.243
γ	2.056	(0.812)	2.532
J-test(p-value)	1079(0.0001)		
Panel B: estimates with QTM-based BTC inflation			
κ_0	2.362	(0.182)	12.996
κ_1	0.750	(0.085)	8.815
γ	2.089	(0.702)	2.975
J-test(p-value)	1082(0.0001)		

Table 1: Bitcoin kernel parameters: GMM estimates for (5.4).

used in the estimation are all adapted to the information set at t and $t-1$. Bitcoin price level is proxied by the BigMac index and by the QTM estimate, as explained above. All variables are monthly, the sample period is from July 2011 to April 2020, totaling 107 monthly observations.

Estimation results are in Table 1 and are qualitatively the same for both measures of bitcoin inflation: the number of coins sent is positively related to benefits, as expected.[23] The relationship is non-linear, $\kappa_1 < 1$, signifying, on the one hand, a diminishing marginal effect of an acceleration in network activity on benefits associated with bitcoin payment;, on the other hand, an increasing marginal effect of shocks to benefits on transactions: bitcoin transactions are sensitive to large changes in benefits. The coefficient of the relative risk aversion, $\gamma \approx 2$ is comfortably within the most commonly reported range of values (typically between 1 and 3, see, e.g., Gandelman et al.; 2015), adding external validity to our approach.

Using the estimated coefficients, Figures 5.4(a) and 5.4(b) depict the relationships separately for the two measures of inflation. Again, results are independent of the choice of the inflation measure. The upper panels of Figures 5.4(a) and (b) depict the association

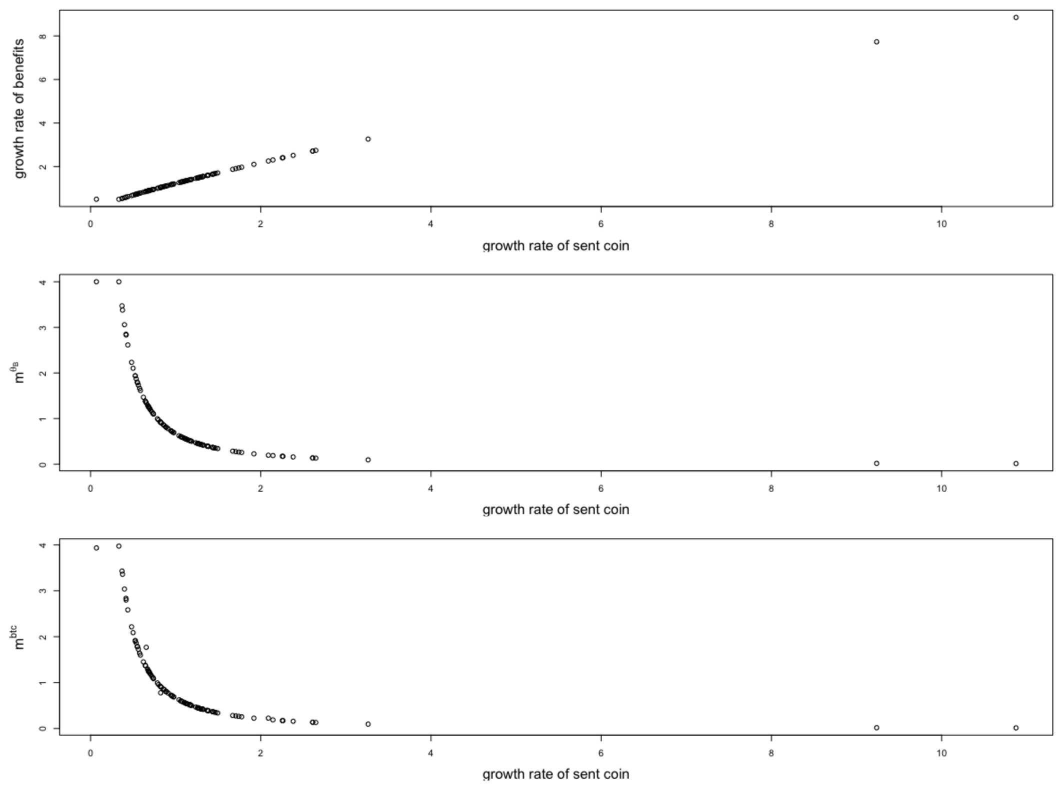
[23] In the GMM we apply the quadratic spectral kernel to compute the covariance matrix of the vector of the sample.

between the growth rate of benefits and the resulting growth rate of sent coins by plotting the computed benefits growth against actual transactions data. Similarly, the middle and the bottom panels display, respectively, the association between the growth rate of sent coins and the implied benefit-related kernels, $m_t^{\theta_{B}}$, and the consolidated bitcoin kernel, m_t^{B}. Both kernels inversely co-move with real activities, measured by numbers of sent coins, in the on-chain transactions. Economically speaking, in the bad times, with a low growth rate of sent coins, one observes increases in the pricing kernel.

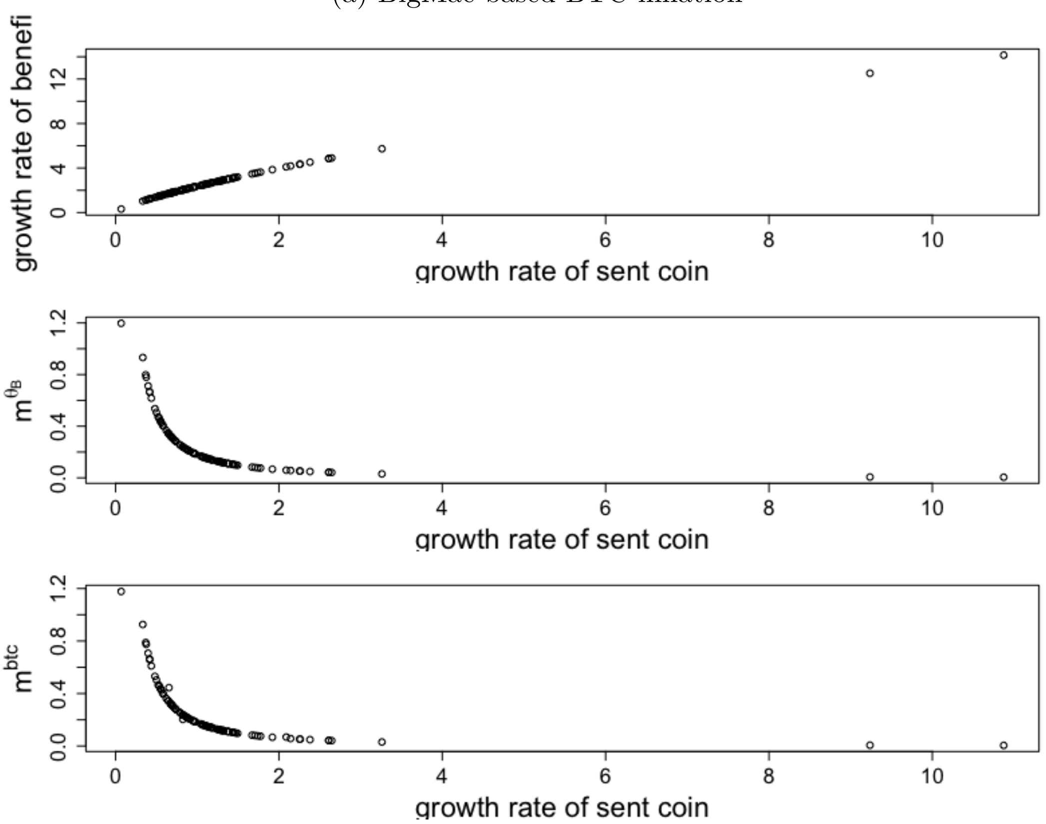
Dynamics of kernels, benefits and prices

With the above parametrization, the dynamics of the benefits-related kernel are displayed in Figure 5.5 over time, along with the time variation of growth rate of sent coins and the Big Mac index of Bitcoin. The dynamics of the QTM-inflation and the exchange rate are already depicted in Figure 5.3. Given the observed dynamics of consumption and the benefits indicator, using (5.4) we are able to characterize the development of the kernels over time. As kernel estimation is effectively independent of the choice of the inflation measure, we only present one figure. In line with Figure 5.4, an inverse relation between sent coins and benefit kernels is evident. In an early-adoption phase, roughly from 2012 to 2013, we observe several spikes in $m_t^{\theta_{B}}$, signaling a bad time in Bitcoin markets because the Darknet market that brought the popularity of cryptocurrency in the drug, weapon, cyber-crime and other illegal transactions had been shut down by the U.S. law enforcement. One of cases is the shutdown of the Silk Road in 2013, an online black market and the first modern darknet market, best known as a platform for selling illegal drugs.

In 2014, there was a sequence of cyber attacks on cryptocurrency exchanges. The most impressive one was the collapse of MtGox, the former biggest cryptocurrency exchange, collapsed and shut down in February 2014. As a result, the transaction benefits shrunk and the kernels increased. Parts of the benefits can be attributed to the illicit activities that require anonymity, resulting in the exuberance of on-chain transactions



(a) BigMac-based BTC inflation



(b) QTM-based BTC inflation

Figure 5.4: Benefit-related kernels versus the growth rate of coins used in real transaction

The top panel displays the growth rate of benefits implied by the growth rate of the numbers of coins being sent in the Bitcoin network; the middle panel depicts the association between the benefits kernel in the bitcoin market with the growth rate of the sent coins; the bottom one is to associate this growth rate with $m^{\text{Ƀ}}_{t+1}$.

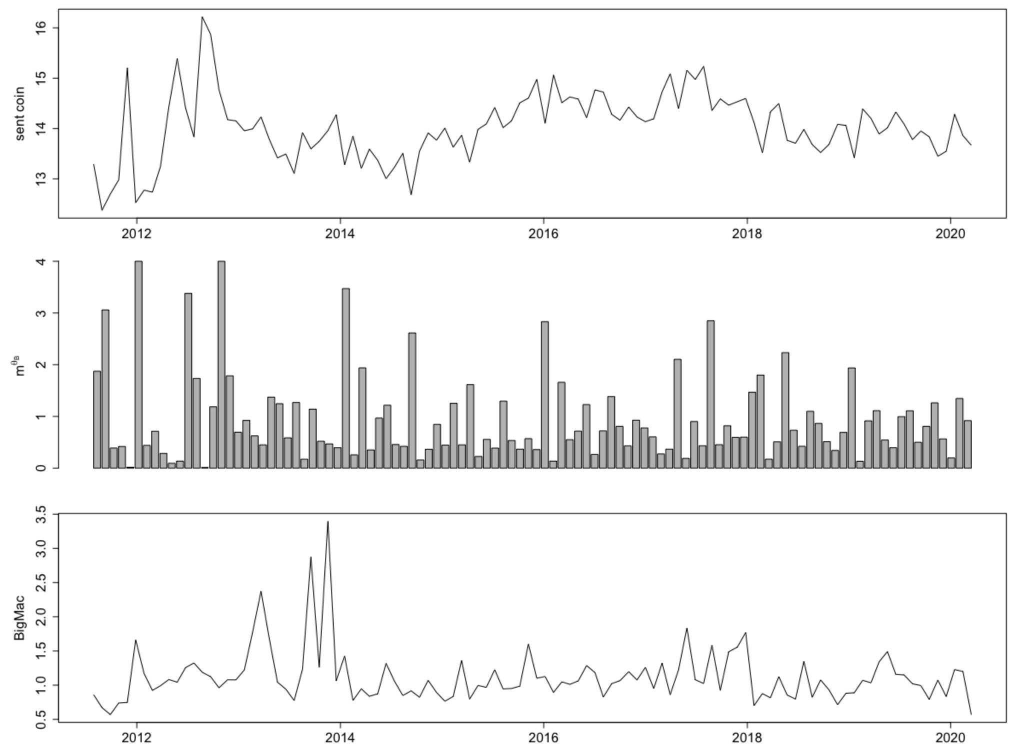


Figure 5.5: Evolution of the implied benefits-related kernel.
The top panel displays the dynamics of the sent coins; the middle panel is the dynamics of the implied benefits kernel in the bitcoin market using the sent coins as a measure of benefits; the bottom one is the dynamics of the BigMac index.

which is evident by the numbers of sent coins. We see a similar pattern between sent coins time series and the time series of the estimated illegal Bitcoin users in Figure 4 of Foley et al. (2019). This similarity supports the use of the number of sent coins as a measure of real activities. The parametrization of the kernel in the dollar market is undertaken in the same fashion but is much simpler since the inflation rate in the dollar market is directly measurable. We use the consumer price index from St Louis Fed. Empirically, concerning that the benefits kernel in the centralized market, due to its long-established transaction mechanism, may often fall into the steady status and behave less stochastically, we impose a steady benefits kernel in the centralized market for numerical tractability. Hence, the difference between the two benefits kernels is mainly driven by the decentralized one.[24]

Characterizing the excess return is impeded by the lack of the risk-free rate in the decentralized market. However, with Corollary 1, one can alternatively tackle it by

[24]The cost we pay for this presumption is a likelihood of overestimating wedge.

characterizing the conditional moments of pricing kernels in the two markets, $\mathrm{Var}_t(m^{\$}_{t+1})$, $\mathrm{Var}_t(m^{\text{\textit{Ƀ}}}_{t+1})$, $\mathsf{E}_t m^{\theta_{Ƀ}}_{t+1}$ and $\mathsf{E}_t m^{\theta_{\$}}_{t+1}$. For this purpose, we fit the characterized kernels series to the time series model, an ARIMA specification for the first moment and the generalized autoregressive conditional heteroskedasticity (GARCH) for the second moment. The goodness-of-fit is carried out by the BIC criteria. In the presence of fat tails in the distribution of innovations of $m^{\theta_{Ƀ}}$ as well as $m^{Ƀ}$, t-distribution, compared to Gaussian, is more capable of capturing tail risk.[25] Table 2 summarizes the estimated parameters across the three kernels, again separately for the two measures of inflation. With the BigMac-based bitcoin inflation measure (Table 2, Panel A), in the mean equation specification, both $m^{\theta_{Ƀ}}$ and $m^{Ƀ}$ exhibit a positive AR(1) and negative AR(2) process, but a negative MA(1) and positive MA(2) process. For the conditional volatility specification, we observe a remarkable persistence of volatility with $\beta = 0.991$ in both $m^{\theta_{Ƀ}}$ and $m^{Ƀ}$. With the QTM-based inflation measure (Table 2, Panel B), we observe more consistency in the specification for dollar, bitcoin and benefit kernels - all are MA(1). This shows it is mainly shocks of the previous period that affect the kernel.

The past innovations, however, have no influence on the variance generating functions, resulting in a smooth declining volatility of $m^{Ƀ}$ in Figure 5.6, where again we show two cases, for two inflation measures. A diminishing conditional volatility in both cases is consistent with a lower variation of the characterized kernels in the later-adoption than in the early-adoption phase shown in Figure 5.5. It also indicates a shrink of the variance risk of pricing kernels in the decentralized market. A low variation of cryptocurrency kernels in the recent years reflects the low variation in the dynamics of state variables. The benefits inferred by the number of sent coins, become rather steady, suggesting substantial economic activities using cryptocurrencies and considerable benefits. According to the Internet organized crime threat assessment 2020, with the effort from the law enforcement, regulators, and traditional financial institutions, as well as with the development of trade in standard goods, criminal activities shrunk to 1.1% of total cryptocurrency transactions, whereas using digital currencies for investment, trading and payment has

[25] $m^{\$}$ is free of this concern and fits Gaussian well.

Parameter	$m_t^{\theta Ƀ}$		$m_t^{Ƀ}$		$m_t^{\$}$	
Panel A: estimates with BigMac-based BTC inflation						
μ	0.719	(0.056)	0.710	(0.059)	0.992	(0.006)
$ar(1)$	1.119	(0.084)	1.121	(0.092)	-0.234	(0.268)
$ar(2)$	-0.905	(0.076)	-0.903	(0.080)		
$ma(1)$	-1.242	(0.044)	-1.242	(0.049)	0.947	(0.010)
$ma(2)$	0.954	(0.046)	0.952	(0.048)		
α	0.000	(0.030)	0.000	(0.143)	0.623	(0.205)
β	0.991	(0.005)	0.991	(0.005)	0.375	(0.277)
ν	3.044	(0.568)	2.990	(0.557)		
logLikelihood	110.55		110.81		291.526	
Panel B: estimates with QTM-based BTC inflation						
μ	0.205	(0.012)	0.203	(0.012)	0.992	(0.006)
$ar(1)$	0.217	(0.209)	0.212	(0.221)	-0.234	(0.268)
$ma(1)$	-0.438	(0.224)	-0.425	(0.221)	0.947	(0.010)
α	0.000	(0.030)	0.000	(0.143)	0.623	(0.205)
β	0.983	(0.069)	0.987	(0.028)	0.375	(0.277)
ν	3.608	(1.312)	3.593	(0.891)		
logLikelihood	40.722		40.378		91.526	

Table 2: Time series specification for kernels

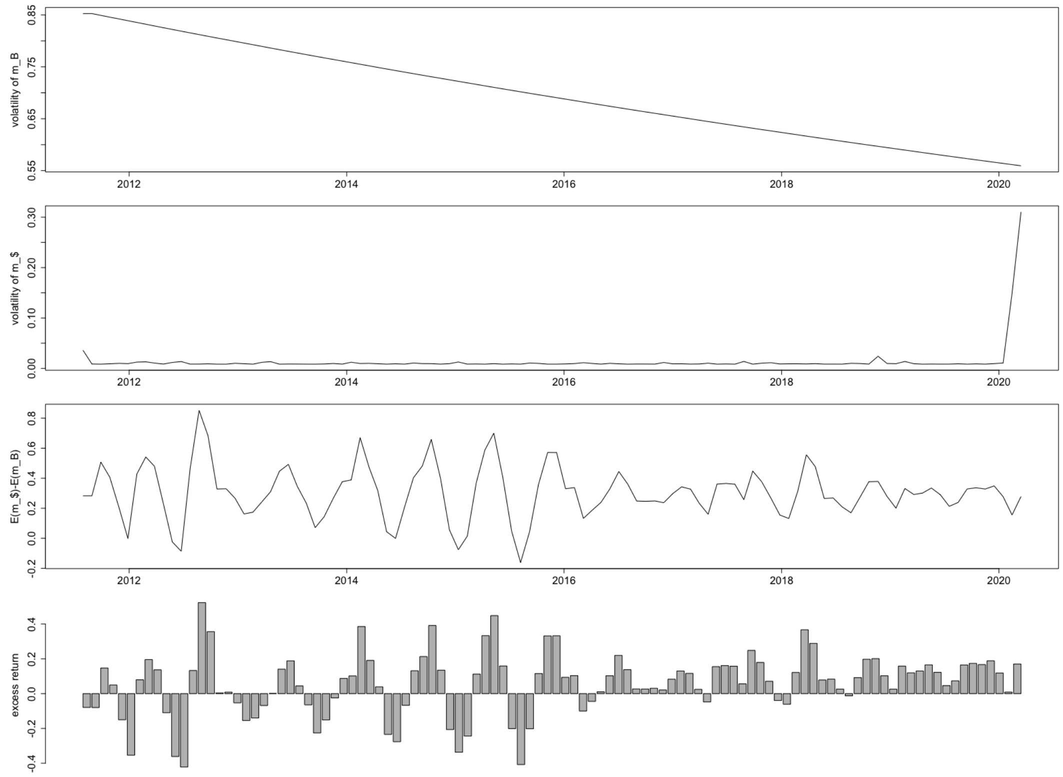
ν is the shape parameter in t distribution. μ, $ar(1)$, $ar(2)$, $ma(1)$, $ma(2)$ are the ARIMA parameters, while α and β are the parameters of variance equation specified by GARCH model. The standard errors are in parentheses.

become mainstream adoption.[26] The fact that 98.9% of transactions appear legitimate further encourages mainstream adoption as the medium of exchange. By contrast, in 2020 a spike of volatility of $m^{\$}$, mainly pertaining to the consumption-related kernel, is caused by a consumption shock due to Covid-19 pandemic. However, the impact of such shock on the benefits kernel in the decentralized markets is relatively modest, which further supports Assumption 2.

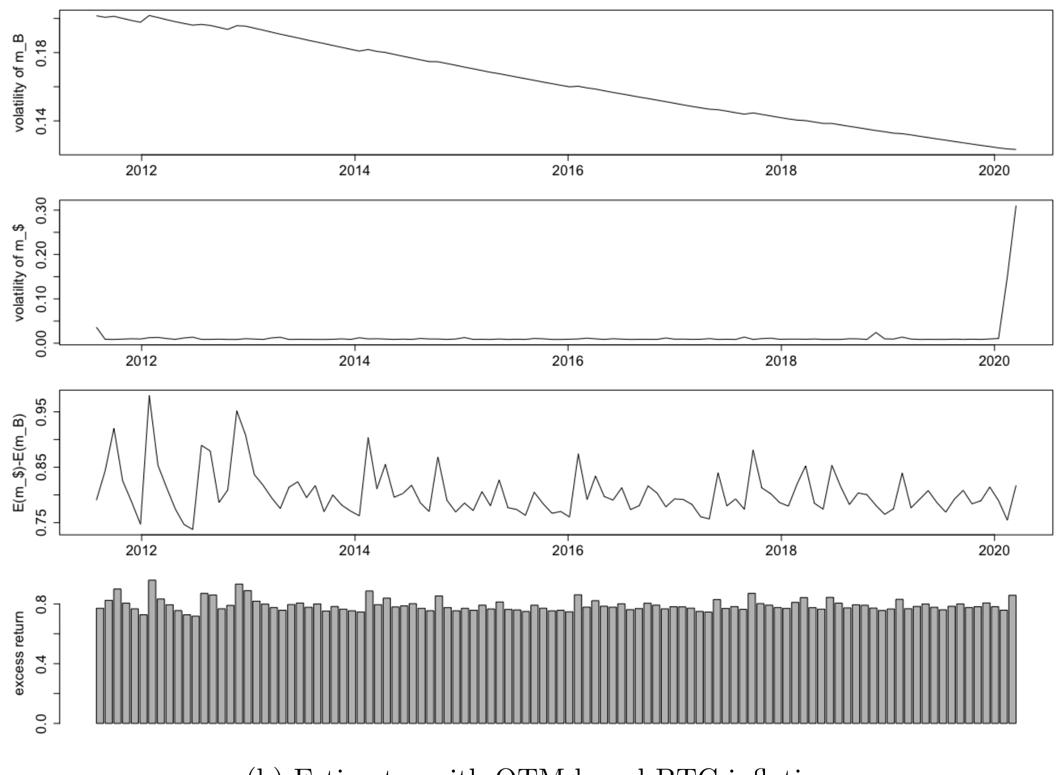
Characterizing the excess return on cryptocurrency market via (3.12) also requires information on the conditional mean of kernels and their difference. One observes that the deviation of the benefit-driven kernels, signaling a distributional dominance of benefits in the decentralized over centralized markets, moved erratically before 2016 but turns more stable in the recent years. In the case of the BigMac inflation measure, we obtain a mean value of the benefit difference around 0.473, while for QTM-inflation this value is around 0.79. Our calibrated implied wedge, 0.536 in Fig. 5.2 is of comparable size and falls within the range given by the two estimations. In this respect, the empirical estimation is consistent with the model calibration, and the reality is somewhere between the two extreme cases given by the two approaches to estimate the bitcoin inflation.

Owing to a diminishing variance risk of $m^{Ƀ}$ and the persistent positive wedge, we note that starting from 2016 even these conservative estimates of excess returns (based on the BigMac inflation) are overwhelmingly positive. Also worth noting is that the sharply rising variance risk of $m^{\$}$ in 2020, caused by the consumption shock in the centralized market, pumps up the excess returns. As a result, the cryptocurrency holders may count on a strictly positive excess return, even by conservative estimates.

[26] The assessment can be downloaded from `https://www.europol.europa.eu/activities-services/main-reports/internet-organised-crime-threat-assessment-iocta-2020`



(a) Estimates with BigMac-based BTC inflation



(b) Estimates with QTM-based BTC inflation

Figure 5.6: Kernel volatility, kernel difference, and the excess return on bitcoins.
The top two panels display the conditional volatility of $m^{\text{Ƀ}}_{t+1}$ and $m^{\$}_{t+1}$, respectively, followed by the difference of the conditional means of the kernels in the two segmented markets; the bottom one is the dynamics of the implied excess return.

6 Conclusion

Unlike conventional financial assets, cryptocurrencies do not represent claims on physical assets or future profits, thus the underlying fundamental is unclear. In this respect they are similar to monies issued by central banks to serve transactions. However, unlike conventional money, cryptocurrencies exhibit high return and volatility. We have developed a model in which cryptocurrencies compete with money as a means of payment, yet this competition is imperfect in the sense that the two means of payment are not identical: consumers' utility is affected by the type of currency used in transactions. This study explains existence of cryptocurrencies by expectations of benefits they may bring about. An example of benefits is anonymity of transactions. Furthermore, when benefits of using cryptocurrencies are driven by a stochastic process defined on events that cannot be observed by other market participants, the system of markets is essentially incomplete. The non-observability of events may also be linked to anonymity: in an anonymous transaction the parties by definition wish to keep the transaction unseen and undisclosed, thus it is impossible for other market participants (including hedgers and insurers) to observe the realized outcome and to insure against it. With incomplete markets, cryptocurrency pricing is not anymore fully explainable by the factors inherent to the centralized market, such as consumption shocks. The pricing of unobserved benefits is the essential fundamental in the cryptocurrency pricing equation.

Although benefits are not observed, transactions are - at least in the form of the number of coins sent over the system. We distill unobserved benefits from the fundamental pricing equation and show their pricing kernel is well characterized by the dynamics of the number of sent coins. This suggests a useful proxy for the unobserved benefits in the empirical pricing of cryptocurrencies. Our estimates show the benefit-related kernel is rather independent of the consumption kernel: while the latter was substantially affected by the recent Covid-19 shock, the former did not strongly respond to it. Moreover, the volatility of the benefits-related kernel has been diminishing over time between 2012 and 2020, which may be linked to the reduction of illegal activities in the market. The cryptocurrency holders are rewarded by the excess returns attributable to the diminishing kernel variance risk and to the substantial probability of high benefits in the decentralized marketplace.

As a final note, we have used to rather extreme proxies for bitcoin inflation. The good news is that for a large part of the analysis the choice between the two does not really make a difference. However the kernel dynamics crucially depends on the dynamics of inflation: with BigMac-based inflation we obtain both the autoregressive and the moving average characterization, as the kernel is intrinsically related to the exchange rate embedded in the BigMac index. It is not the case for QTM-based estimates, where the autoregressive behavior is not present. The other difference appears in estimated excess returns: QTM-inflation leads to persistently positive excess returns, which is not the case with BigMac-inflation. Which of the two measures should we rely on? On the balance of evidence, we argue in favor of the quantity theory one: its construction is independent of the exchange rate, it rules out the autoregressive feature and ensures consistency in the dynamics across all kernels. Persistently positive (and growing) bitcoin excess return obtains for the other inflation measure, too, from 2018 onwards, and as such does not appear to be driven by the choice of the inflation measure. Further research is required and more information on cryptocurrency transactions are needed to better identify price dynamics in this market segment, in order to ultimately better understand the fundamentals of this asset class.

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7 Appendix

7.1 Auxiliary results

The following proposition is a corollary to Proposition 3.8 allowing us to introduce risk-free rates in the model.

Proposition A (Risk-free rate). *For any* $t \geq 0$ *there exist real numbers* $R^{\$}_{t+1} > 0$ *and* $R^{\text{Ƀ}}_{t+1} > 0$ *such that*

$$\mathsf{E}_t \left[M^{\$}_{t+1} \frac{P_t}{P_{t+1}} \right] = R^{\$}_{t+1} \cdot \mathsf{E}_t \left[M^{\$}_{t+1} \right] \tag{7.1}$$

and

$$\mathsf{E}_t \left[M^{\text{Ƀ}}_{t+1} \frac{B_t}{B_{t+1}} \right] = R^{\text{Ƀ}}_{t+1} \cdot \mathsf{E}_t \left[M^{\text{Ƀ}}_{t+1} \right] \tag{7.2}$$

An obvious interpretation of $R^{\$}_{t+1}$ is that of a risk-free interest rate. This interpretation arises by applying (2.2) to (7.1 - 7.2):

$$\mathsf{E}_t \left[M^{\$}_{t+1} \right] = \frac{1}{R^{\$}_{t+1}} \text{ and } \mathsf{E}_t \left[M^{\text{Ƀ}}_{t+1} \right] = \frac{1}{R^{\text{Ƀ}}_{t+1}}, \tag{7.3}$$

where the expected value of the pricing kernel is, as usual, the risk-free interest rate in the respective market (note, by our definition of SDF, we use the reciprocal of the interest rate). We assumed both currencies can be used for savings, however savings in either currency does not have to be free of risk. Numbers $R^{\$}_{t+1}$ and $R^{\text{Ƀ}}_{t+1}$ we introduce now effectively describe interest rate on the risk-free savings. We do not assume risk-free instruments are available and traded, however it may be useful to *interpret* $R^{\$}_{t+1}$ and $R^{\text{Ƀ}}_{t+1}$ as *synthetic* risk-free rates. Note that if $M^{\$}_{t+1}$ and $\frac{P_t}{P_{t+1}}$ were statistically independent, $R^{\$}_{t+1}$ would reflect the expected [gross] inflation rate $\mathsf{E}_t \left[\frac{P_t}{P_{t+1}} \right]$. The same interpretation holds for the bitcoin market and bitcoin-inflation. The real-life counterparts of $R^{\$}_{t+1}$ and $R^{\text{Ƀ}}_{t+1}$ are discussed in Section 5.

The next proposition is needed for further proofs.

Proposition B. *Given (3.4) and under Assumption 2 and 3, we arrive at the following propositions:*

$$Cov_t(m^{\text{Ƀ}}_{t+1} - m^{\theta_{\text{Ƀ}}}_{t+1}, m^{\theta_{\$}}_{t+1}) = \mathsf{E}_t m^{\theta_{\text{Ƀ}}}_{t+1} - \mathsf{E}_t m^{\theta_{\$}}_{t+1} - \frac{1}{2}\Big[Var_t(m^{\theta_{\text{Ƀ}}}_{t+1}) + Var_t(m^{\theta_{\$}}_{t+1})\Big] + Cov_t(m^{\text{Ƀ}}_{t+1}, m^{\theta_{\text{Ƀ}}}_{t+1}) \tag{7.4}$$

$$Cov_t(m^{\$}_{t+1} + m^{\theta_{\text{Ƀ}}}_{t+1}, m^{\theta_{\$}}_{t+1}) = \mathsf{E}_t m^{\theta_{\text{Ƀ}}}_{t+1} - \mathsf{E}_t m^{\theta_{\$}}_{t+1} + \frac{1}{2}\Big[Var_t(m^{\theta_{\text{Ƀ}}}_{t+1}) + Var_t(m^{\theta_{\$}}_{t+1})\Big] + Cov_t(m^{\$}_{t+1}, m^{\theta_{\text{Ƀ}}}_{t+1}) \tag{7.5}$$

7.2 Proofs

Propositions 2, 3, 6 and 7 are explained in the text of the paper. Remaining proofs are below.

Proof of Proposition 1. Due to imperfect substitution, utility from consumption differs both intertemporally and cross-sectionally (across markets), for which reason one has to distinguish between four consumption components: in dollars and bitcoins, both today and tomorrow, captured by vector $\left(c^{\$}_t, c^{\$}_{t+1}, c^{\text{Ƀ}}_t, c^{\text{Ƀ}}_{t+1}\right)$, bound by budget constraints

$$c^{\$}_t \cdot P_t + c^{\text{Ƀ}}_t \cdot (B_t \cdot b_t) = w_t - s^{\$}_t, \tag{7.6}$$

and

$$c^{\$}_{t+1} \cdot P_{t+1} + c^{\text{Ƀ}}_{t+1} \cdot (B_{t+1} \cdot b_{t+1}) = w_{t+1} + s^{\$}_t, \tag{7.7}$$

where w_t and w_{t+1} are dollar endowments in each time period respectively (may be seen as income in dollars), $s^{\$}_t$ is the amount of savings in dollars at time t, and the rate of return on savings is assumed zero as we consider cash holdings. As both currencies store value intertemporally, consumers can keep part $s^{\$}_t$ of savings in dollar cash and part $s^{\text{Ƀ}}_t$ in bitcoins. For simplicity, we only consider the case of all savings in either currency; indifference between these two cases implies a mixed strategy offers the same payoff.

The portfolio choice problem is thus separated from the consumption choice.[27]. While consumers exhibit preferences towards the payment technology, which is reflected in the utility function and in the adjusted PPP relationship, they are indifferent between the storage technologies. Formally, the latter is reflected in savings entering the decision problem only in the budget constraint, linearly, and hence consumers would seek to maximise return on savings and choose the storage technology that offers the higher return.

For consumption, the first-order conditions require marginal utilities of $c_t^{\$}, c_{t+1}^{\$}, c_t^{Ƀ}, c_{t+1}^{Ƀ}$ equal prices $P_t, B_t \cdot b_t, P_{t+1}$ and $B_{t+1} \cdot b_{t+1}$ respectively, implying price ratios equal respective marginal rates of substitution. These intertemporal marginal rates of substitution (IMRS) determine the pricing kernels in Proposition 1, and therefore holds

$$\mathsf{E}_t \left[m_{t+1}^{\$} \frac{P_t}{P_{t+1}} \right] = 1 = \mathsf{E}_t \left[m_{t+1}^{Ƀ} \frac{B_t \cdot b_t}{B_{t+1} \cdot b_{t+1}} \right] \tag{7.8}$$

Now let consumers transfer value from t to $t+1$ using bitcoins, which turns budget constraints into

$$c_t^{\$} \cdot \frac{P_t}{b_t} + c_t^{Ƀ} \cdot B_t = \frac{w_t}{b_t} - s_t^{Ƀ}, \tag{7.9}$$

and

$$c_{t+1}^{\$} \cdot \frac{P_{t+1}}{b_{t+1}} + c_{t+1}^{Ƀ} \cdot B_{t+1} = \frac{w_{t+1}}{b_{t+1}} + s_t^{Ƀ}, \tag{7.10}$$

The same IRMS $m_{t+1}^{\$}$ and $m_{t+1}^{Ƀ}$ now equal new price ratios, yielding

$$\mathsf{E}_t \left[m_{t+1}^{\$} \frac{P_t / b_t}{P_{t+1} / b_{t+1}} \right] = 1 = \mathsf{E}_t \left[m_{t+1}^{Ƀ} \frac{B_t}{B_{t+1}} \right]. \tag{7.11}$$

□

[27]Fisher's separation theorem separates investment decisions of the firm from investors' preferences. The former corresponds to the optimal portfolio choice in our case, and the latter - to consumption decisions.

Proof of of Proposition 4. We utilize the proof for Proposition B, recall (3.4) and $\Delta q_{t+1} = m^{\text{Ƀ}}_{t+1} - m^{\$}_{t+1} - m^{\theta_{\text{Ƀ}}}_{t+1} + m^{\theta_{\$}}_{t+1}$.

$$
\begin{aligned}
\mathsf{E}_t\Delta q_{t+1} &= \mathsf{E}_t m^{\text{Ƀ}}_{t+1} - \mathsf{E}_t m^{\theta_{\text{Ƀ}}}_{t+1} - \mathsf{E}_t m^{\$}_{t+1} + \mathsf{E}_t m^{\theta_{\$}}_{t+1} \\
&= r^{\$}_t - r^{\text{Ƀ}}_t + \mathsf{E}_t m^{\theta_{\text{Ƀ}}}_{t+1} - \mathsf{E}_t m^{\theta_{\$}}_{t+1} + \frac{1}{2}\Big[\mathrm{Var}_t(m^{\$}_{t+1}) - \mathrm{Var}_t(m^{\text{Ƀ}}_{t+1})\Big] + \mathrm{Cov}_t(m^{\$}_{t+1}, m^{\theta_{\text{Ƀ}}}_{t+1}) \\
&\quad + \mathrm{Cov}_t(m^{\text{Ƀ}}_{t+1}, m^{\theta_{\text{Ƀ}}}_{t+1}) - \mathrm{Cov}_t(m^{\text{Ƀ}}_{t+1} - m^{\theta_{\text{Ƀ}}}_{t+1}, m^{\theta_{\$}}_{t+1}) - \mathrm{Cov}_t(m^{\$}_{t+1} + m^{\theta_{\text{Ƀ}}}_{t+1}, m^{\theta_{\$}}_{t+1}) \\
&= r^{\$}_t - r^{\text{Ƀ}}_t + \frac{1}{2}\Big[\mathrm{Var}_t(m^{\$}_{t+1}) - \mathrm{Var}_t(m^{\text{Ƀ}}_{t+1})\Big] + \mathsf{E}_t m^{\theta_{\$}}_{t+1} - \mathsf{E}_t m^{\theta_{\text{Ƀ}}}_{t+1}
\end{aligned}
$$

□

Proof of Proposition 5. It makes use of Proposition B , recalls (3.4) and under Assumption 2 and $\Delta q_{t+1} = m^{\text{Ƀ}}_{t+1} - m^{\$}_{t+1} - m^{\theta_{\text{Ƀ}}}_{t+1} + m^{\theta_{\$}}_{t+1}$.

$$
\begin{aligned}
\mathrm{Var}_t(\Delta q_{t+1}) &= \mathrm{Var}_t(m^{\text{Ƀ}}_{t+1}) + \mathrm{Var}_t(m^{\$}_{t+1}) - \mathrm{Var}_t(m^{\theta_{\text{Ƀ}}}_{t+1}) - \mathrm{Var}_t(m^{\theta_{\$}}_{t+1}) + \\
&\quad 2\Big[\mathrm{Cov}_t(m^{\text{Ƀ}}_{t+1}, m^{\theta_{\text{Ƀ}}}_{t+1}) + \mathrm{Cov}_t(m^{\theta_{\text{Ƀ}}}_{t+1}, m^{\theta_{\$}}_{t+1}) - \mathrm{Cov}_t(m^{\theta_{\text{Ƀ}}}_{t+1}, m^{\$}_{t+1}) - \mathrm{Cov}_t(m^{\text{Ƀ}}_{t+1}, m^{\$}_{t+1})\Big] \\
&= \mathrm{Var}_t(m^{\text{Ƀ}}_{t+1}) + \mathrm{Var}_t(m^{\$}_{t+1}) - \mathrm{Var}_t(m^{\theta_{\text{Ƀ}}}_{t+1}) - \mathrm{Var}_t(m^{\theta_{\$}}_{t+1}) + \\
&\quad 2\Big[\mathrm{Var}_t(m^{\theta_{\text{Ƀ}}}_{t+1}) + \mathrm{Cov}_t(m^{c^{\text{Ƀ}}}_{t+1}, m^{\theta_{\text{Ƀ}}}_{t+1}) - \mathrm{Cov}_t(m^{c^{\$}}_{t+1}, m^{\theta_{\text{Ƀ}}}_{t+1}) - \mathrm{Cov}_t(m^{\text{Ƀ}}_{t+1}, m^{\$}_{t+1})\Big] \\
&= \mathrm{Var}_t(m^{\text{Ƀ}}_{t+1}) + \mathrm{Var}_t(m^{\$}_{t+1}) + \mathrm{Var}_t(m^{\theta_{\text{Ƀ}}}_{t+1}) - \mathrm{Var}_t(m^{\theta_{\$}}_{t+1}) - 2\mathrm{Cov}_t(m^{\text{Ƀ}}_{t+1}, m^{\$}_{t+1})
\end{aligned}
$$

□

Proof of Proposition B. Proof of Proposition B makes use of (3.7):

$$
\mathsf{E}_t\Big[M^{\$}_{t+1}\frac{Q_{t+1}}{Q_t}\Big] = \mathsf{E}_t\Big[\frac{M^{\text{Ƀ}}_{t+1}}{M^{\theta_{\text{Ƀ}}}_{t+1}} M^{\theta_{\$}}_{t+1}\Big] = \frac{1}{R^{\text{Ƀ}}_t}
$$

$$
\begin{aligned}
-r^{\text{Ƀ}}_t &= \mathsf{E}_t m^{\text{Ƀ}}_{t+1} - \mathsf{E}_t m^{\theta_{\text{Ƀ}}}_{t+1} + \mathsf{E}_t m^{\theta_{\$}}_{t+1} + \frac{1}{2}\Big[\mathrm{Var}_t(m^{\text{Ƀ}}_{t+1}) + \mathrm{Var}_t(m^{\theta_{\text{Ƀ}}}_{t+1}) - 2\mathrm{Cov}_t(m^{\text{Ƀ}}_{t+1}, m^{\theta_{\text{Ƀ}}}_{t+1}) + \mathrm{Var}_t(m^{\theta_{\$}}_{t+1})\Big] \\
&\quad + \mathrm{Cov}_t(m^{\text{Ƀ}}_{t+1} - m^{\theta_{\text{Ƀ}}}_{t+1}, m^{\theta_{\$}}_{t+1})
\end{aligned}
$$

Given $-r_t^{\text{Ƀ}} = \mathsf{E}_t m_{t+1}^{\text{Ƀ}} + \frac{1}{2}\mathrm{Var}_t(m_{t+1}^{\text{Ƀ}})$, the equality ensures

$$\mathrm{Cov}_t(m_{t+1}^{\text{Ƀ}} - m_{t+1}^{\theta_{\text{Ƀ}}}, m_{t+1}^{\theta_{\$}}) = \mathsf{E}_t m_{t+1}^{\theta_{\text{Ƀ}}} - \mathsf{E}_t m_{t+1}^{\theta_{\$}} - \frac{1}{2}\Big[\mathrm{Var}_t(m_{t+1}^{\theta_{\text{Ƀ}}}) + \mathrm{Var}_t(m_{t+1}^{\theta_{\$}})\Big] + \mathrm{Cov}_t(m_{t+1}^{\text{Ƀ}}, m_{t+1}^{\theta_{\text{Ƀ}}}).$$

Likewise, we can prove

$$\mathrm{Cov}_t(m_{t+1}^{\$} + m_{t+1}^{\theta_{\text{Ƀ}}}, m_{t+1}^{\theta_{\$}}) = \mathsf{E}_t m_{t+1}^{\theta_{\text{Ƀ}}} - \mathsf{E}_t m_{t+1}^{\theta_{\$}} + \frac{1}{2}\Big[\mathrm{Var}_t(m_{t+1}^{\theta_{\text{Ƀ}}}) + \mathrm{Var}_t(m_{t+1}^{\theta_{\$}})\Big] + \mathrm{Cov}_t(m_{t+1}^{\$}, m_{t+1}^{\theta_{\text{Ƀ}}}).$$

□

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