Analysis of Production Decisions under Budget Limitations

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Abstract. The issue of when to intervene in the evolution of a production system is the focus of this study. The interventions take the form of changes to production depending on the current value of the products. Each change incurs a charge representing costs such as physical expansion, overtime or new hiring when production increases and costs such as severance or shut down when production decreases. The goal is to maximize the expected return subject to these intervention costs over at most a finite number of intervention cycles. This paper determines for a large class of problems an explicit formula for the value function and a set of optimal times at which to increase and decrease production. The optimization is over a very general class of stopping times and proves that an optimal set of times in this general class is given as the hitting times of various levels, depending on the number of remaining interventions. These optimal hitting levels are characterized as a maximizing point for a high-dimensional nonlinear function and can be efficiently and iteratively determined as the solutions of successive one-dimensional nonlinear maximization problems. The solution method is illustrated on some examples, including mean-reverting processes.

Keywords. reversible investment, reversible disinvestment, entry-and-exit, infinite-dimensional linear programming, nonlinear optimization.

1 Introduction

Consider the situation of a company which seeks to determine optimal times at which to increase a line of production when the return rate is good and to reduce production when times are bad. Associated with each decision is a cost to increase production so the manager is willing to initially delay an increase in "good times" in light of the concern that "bad times" are just around the corner. Similarly, there is a cost to reduce production so management will continue production in "bad times" with the hope that they are not too "bad" and "better times" are coming shortly. The company directors have set a limit on the funds designated for these switching costs and hence the increase and decrease of production are limited to at most a finite number of occurrences. Let N denote the number of production increase-decrease cycles; N remains fixed throughout this paper.

For ease of terminology, we refer to the increase of production as "start-up" and the decrease as "mothballing." These terms indicate one type of increase and decrease. The results of this paper, however, apply to more general increases and decreases in production.

To establish the main goal of the problem, let X denote the process for the value of production and let Y be an indicator process denoting that production is running (1) or

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mothballed (0). In our framework, we allow the evolution of the process X to depend on the process Y so that there are different dynamics when the line is mothballed from when it is in production. Let $r(x, y)$ denote the net rate of income when (X, Y) takes value (x, y) and let $\alpha > 0$ denote the discount rate. Throughout this paper for simplicity, we assume the company has not yet invested in production so the first decision is when to do so. For $k = 1, \dots, N$, let $c_k^{(in)} \geq 0$ denote the fixed cost to open production for the kth time and similarly let $c_k^{(out)} \geq 0$ be the cost to mothball; we assume $c_k^{(in)} + c_k^{(out)} > 0$ and $c_k^{(out)} + c_{k+1}^{(in)} > 0$. The aim of the production manager is to select decision times $\{\tau_k^{(in)} : k = 1, \dots, N\}$ and $\{\tau_k^{(out)} : k = 1, \dots, N\}$ at which to open up and to mothball production so as to maximize

$$\mathbb{E}\left[\int_0^\infty e^{-\alpha s} r(X(s), Y(s))\, ds - \sum_{k=1}^N e^{-\alpha \tau_k^{(in)}} c_k^{(in)} - \sum_{k=1}^N e^{-\alpha \tau_k^{(out)}} c_k^{(out)}\right]. \tag{1.1}$$

This problem is a modification of well-studied problems in which one seeks to maximize the expected discounted reward earned over either a single decision or over an infinite number of cycles when there are entry and exit costs. In the forestry literature, the problem of interest relates to the timing of harvests and rewards rather than costs accrued at harvest times; the single decision problem is known as the Wicksell problem, whereas the infinite-cycle problem is the Faustmann problem (see e.g., [2, 26, 29]). These problems also appear, for example, in the economics literature [7, 8, 15, 27, 28] and management and operations research literature (see e.g., [20, 25]). Mathematically, these models are versions of optimal stopping, impulse control or more general stochastic control problems. The problems are usually analyzed by solving the Hamilton-Jacobi-Bellman (HJB) equation associated with the given problem. Since there are different solution concepts for HJB-equations [11], these models have also attracted the interest of mathematicians and control theorists; see [3, 6, 9, 21, 30] and references therein, for a short selection of recent publications. The paper [23] utilizes a viscosity solutions approach to solve the infinite switching problem for a two-regime geometric Brownian motion model, while [17] address a single-regime diffusion process having non-smooth data with a more general discount mechanism by constructing an explicit solution to the HJB equation.

This paper analyzes a modification in which only a finite number of cycles is allowed, a problem also considered by Øksendal and Sulem (see Chapter 7 of [22]) and Bayraktar and Egami [4]. The view taken in these analyses is that the finite-cycle problem is an approximation to the infinite-cycle problem. These approaches rely on dynamic programming and determine an iterative scheme for computing the N-cycle value function. (This latter result is similar to our iterative approach in Section 4 though our iteration scheme arises directly from optimizing a nonlinear function.) Furthermore, the paper [4] constructs the value function for an infinite number of cycles for a class of diffusion processes on $(0, \infty)$ using a generalized convexity approach to characterize the excessive functions.

This paper approaches the problem quite differently. Under the assumptions detailed in the Section 1.1, the explicit form of the value function is given as a (to-be-determined) constant factor of the increasing solution ψ_0 of Condition 1.2. Moreover, two methods are derived to identify this constant. The first is by maximizing a nonlinear function of $2N$ variables, while the second iteratively solves $2N$ one-dimensional maximization problems. A maximizer's coordinates are then used to specify a set of optimal switching times as the hitting times of these levels. Of particular significance is the fact that, though the original stochastic problem allows very general switching times, an optimal set of times occur as the hitting times of the process at various levels (Theorem 3.2). This optimality result arises from the imbedding of the stochastic problem in an infinite-dimensional linear program on a collection of measures for which an optimizing measure is determined from the maximizer of a finite-dimensional nonlinear function (Theorem A.1).

When one considers either *single-cycle* or *infinite-cycle* versions of this problem, the solutions have the same structure, namely there exist two values a^* and b^* with $a^* < b^*$ such that the optimal time(s) to initiate production is(are) when the process X hits or exceeds level b^* with production mothballed and the optimal time(s) to mothball occurs(occur) when X hits or lies below a^* while production is in process [13, 14, 27]. The optimal levels differ between the single-cycle and infinite-cycle problems with a corresponding increase in the optimal value for the infinite-cycle problem indicating the benefit of additional production/mothballing opportunities.

As indicated above, the structure of the solution to the finite multiple-intervention problem is more complex. Again, there are levels so that the production starts when X reaches or exceeds these specified upper values while production is mothballed and mothballing occurs when X is at or below lower values during production. However, these optimal production-initiating levels differ and depend on the number of remaining cycles and similarly for the mothballing levels. The paper [13] uses the same type of linear programming imbedding, but both single-cycle and infinite-cycle problems are simpler in that they reduce to the optimization of a nonlinear function of two variables. The current paper reduces the problem to one of maximizing a nonlinear function of $2N$ variables and establishes a tractable iterative scheme to obtain the optimal solution; iteration is not needed for the single-cycle problem and is not available for the infinite-cycle problem, except through a limiting argument.

The paper is organized as follows. Section 1.1 gives a careful formulation of the problem. We then derive in Section 2 an infinite-dimensional linear programming problem in which the original stochastic problem is imbedded. Utilizing the structure of the dynamics, in Section 3 we relax the constraints to form an auxiliary linear program (also infinite-dimensional) for which the optimal value can be characterized in terms of the maximization of a $2N$-dimensional nonlinear optimization problem. An optimal production/mothballing policy is determined from the solution to the nonlinear problem. We then investigate an iterative formulation of the problem which leads to a more efficient optimization procedure.

1.1 General Formulation

It is helpful to set a general framework for the processes and rewards. As indicated above, let Y be the indicator process that takes value 0 when production is mothballed and value 1 when running. Let $(x_l, x_r) \subseteq \mathbb{R}$. For $y \in \{0, 1\}$ and an initial probability distribution π on (x_l, x_r), the process X_y is a diffusion process satisfying the stochastic differential equation

$$dX_y(t) = \mu(X_y(t), y)\, dt + \sigma(X_y(t), y)\, dW(t), \qquad X_y(0) \sim \pi, \tag{1.2}$$

in which W is a standard Brownian motion that provides the fluctuations to the process. For each $y \in \{0, 1\}$, the drift coefficient $\mu(\cdot, y)$ and diffusion coefficient $\sigma(\cdot, y)$ are assumed to be continuous and such that X_y takes values in the interval (x_l, x_r); the process X_y has generator A_y given by $A_y f(x) = (1/2)\sigma^2(x,y)f''(x) + \mu(x,y)f'(x))$ operating on $f \in C^2((x_l, x_r))$ (see [5, **9**, p. 17] for sufficient conditions). Further assume X_y is a weak solution of (1.2) while $X_y(t) \in (x_l, x_r)$ (see Ethier and Kurtz [10, Section 5.3, p. 291] or Rogers and Williams [24, V.16.1, p. 149] for details) and that the solution to (1.2) is unique in distribution. This uniqueness implies that the martingale problem for A_y, $y \in \{0, 1\}$, is well-posed and hence that X_y is a strong Markov process (see [10, Theorem 4.4.2, p. 184] or [24, V.21.2, p. 162]). We denote the filtration for the weak solutions by $\{\mathcal{F}_t\}$.

The paired process (X, Y) is constructed for each collection of $\{\mathcal{F}_t\}$-stopping times using the strong Markov property of each X_y to switch between these processes and change the value of Y at the switching times. For example, the process Y starts in state 0 and evolves according to X_0; then at time $\tau_1^{(in)}$, the X process has value of $X_0(\tau_1^{(in)})$ so the process switches to the X_1 process with initial distribution being that of $X_0(\tau_1^{(in)})$. Throughout the paper we denote the initial position of the X process (of the pair) by x_0 and assume $x_0 \neq x_l, x_r$. With regard to the initial value of Y, it seems natural for the first decision to be a time to begin production, possibly with a high initial cost, so we have assumed $Y(0) = 0$. The solution when $Y(0) = 1$ can be easily determined using the same methodology; we leave the details to the reader.

Before proceeding further, we briefly digress to consider the boundary points. We restrict the models to those for which x_l is either a natural boundary point or an entrance but not an exit boundary point (see [5, II.10, p. 14-19] or [16, p. 128-131]). When x_l is either an entrance-not-exit or natural boundary, X will almost surely never reach x_l in finite time so the process is defined for all time. The distinction between entrance and natural boundaries is that the process will immediately enter the interval (x_l, x_r) when $x_0 = x_l$ is an entrance point (we assume $x_0 \neq x_l$) after which it will never return to the boundary and thus x_l is in the state space of the process. This behavior does not happen with a natural boundary point so such an x_l will not be in the state space of X. We place the same restrictions on the model for x_r.

The production/mothballing decisions are made based on some reward earned while producing and costs charged to start-up production or to mothball it. The switching times are $\{\mathcal{F}_t\}$-stopping times $\{\tau_k^{(in)} : k = 1, \ldots, N\}$ and $\{\tau_k^{(out)} : k = 1, \ldots, N\}$ with $\tau_1^{(in)} \leq \tau_1^{(out)} \leq \tau_2^{(in)} \leq \tau_2^{(out)} \leq \ldots \leq \tau_N^{(in)} \leq \tau_N^{(out)}$; each of these inequalities is strict on the set where the left-hand random variable is finite. The requirement of strict inequality when the switching times are finite places no restriction on the problem. Since the number of switches is limited to at most $2N$ but is not required to be exactly $2N$, any switching times for which two consecutive times are finite and coincide induce an instantaneous strictly positive charge and leaves the process in exactly the same state as before these switches. A better value is obtained by omitting these switches. Let $\mathcal{A}$ denote this collection of stopping times and denote by $\tilde{\tau}$ an element of $\mathcal{A}$. The type of reward function can be quite general; let $r : (x_l, x_r) \times \{0, 1\} \to \mathbb{R}$ denote the net income rate. For the problem to be meaningful, we impose the following integrability condition throughout the paper.

Condition 1.1 *The coefficients μ and σ of the diffusion X and the income rate function r*

are such that

$$\mathbb{E}\left[\int_0^\infty e^{-\alpha s}[|r(X(s),0)| + |r(X(s),1)|]\,ds\right] < \infty.$$

Condition 1.1 implies the total discounted reward available to the investor is finite.

A key additional assumption on the coefficients is required, which we separate out for later reference.

Condition 1.2 *For $y = 0, 1$, the eigenvalue problem $Af(\cdot, y) = \alpha f(\cdot, y)$ has both a positive, strictly decreasing solution ϕ_y and a non-negative, strictly increasing solution ψ_y.*

The conditions assumed in this paper are sufficient to imply Condition 1.2 (see Itô and McKean [16, pp. 128-131] or Borodin and Salminen [5, II.10, p. 18,19]). The functions ϕ_y and ψ_y, $y = 0, 1$, depend on the discount factor α; since we assume the discount factor is fixed, we omit this dependence from the notation. The properties of significance for this paper are that $\psi_y(x_l) \geq 0$ and $\phi_y(x_l+) = \infty$ (see [5, pp. 14-19]). Symmetric properties hold for x_r with the roles of ϕ_y and ψ_y reversed.

We also highlight an important identity involving ϕ_y and ψ_y. Suppose $X_y(0) = x \in (x_l, x_r)$. Let $c \in (x_l, x_r)$ and define $\tau_c = \inf\{t \geq 0 : X_y(t) = c\}$. Then (see [5, p. 18])

$$E[e^{-\alpha\tau_c}] = \begin{cases} \phi_y(x)/\phi_y(c) & \text{for } c \leq x, \\ \psi_y(x)/\psi_y(c) & \text{for } c \geq x. \end{cases} \tag{1.3}$$

The diffusion processes under consideration in this paper exclude reflection at a boundary and killing in the interior of (x_l, x_r). The inclusion of reflected processes would require either restricting the domain of the test functions to capture the reflection or adding a second operator B that adjusts the evolution of the process X when reflection occurs. The latter approach adapts well to the linear programming methodology and is an example of one type of singular behavior. A future paper will examine entry-and-exit problems for singular stochastic processes. The exclusion of killing is merely for convenience so as to clearly observe the effect discounting has on the problem. The generator A for a killed diffusion is $Af(x, y) = (1/2)\sigma^2(x, y)\frac{\partial^2 f}{\partial x^2}(x, y) + \mu(x, y)\frac{\partial f}{\partial x}(x, y) - c(x, y)f(x, y)$, where $c \geq 0$ gives the state-dependent killing rate; when discounting is also included, the operator of interest is $Af(x, y) - \alpha f(x, y) = (1/2)\sigma^2(x, y)\frac{\partial^2 f}{\partial x^2}(x, y) + \mu(x, y)\frac{\partial f}{\partial x}(x, y) - (c(x, y) + \alpha)f(x)$. The key requirement for this paper is that Condition 1.2 be satisfied. Thus the results of this paper easily extend to diffusions having state-dependent killing rates.

We wish to reformulate the criterion (1.1) as a multiple optimal stopping problem without the running reward term by adjusting the "costs" accrued at times $\tau_k^{(in)}$ and $\tau_k^{(out)}$, $k = 1, \ldots, N$. To simplify the notation for the argument of the following proposition, let x and y denote the initial values of X and Y, respectively. Also let $\tilde{\tau} := (\tau_1, \tau_2, \tau_3, \ldots, \tau_{2N}) := (\tau_1^{(in)}, \tau_1^{(out)}, \tau_2^{(in)}, \ldots, \tau_N^{(out)})$ be an admissible set of switching times, denoted by $\tilde{\tau} \in \mathcal{A}$.

Proposition 1.3 *Assume the reward function r satisfies Condition 1.1. Then there exists a function f_r with the property that for all $\tilde{\tau} \in \mathcal{A}$, initial values $x \in (x_\ell, x_r)$, $y = 0$ and the* *corresponding paired process* $(X(s), Y(s))$,

$$\mathbb{E}_{xy}\left[\int_0^\infty e^{-\alpha s} r(X(s), Y(s))\, ds\right] \tag{1.4}$$
$$= -\sum_{j=1}^{2N} \mathbb{E}_{xy}\left[I_{\{\tau_j<\infty\}} e^{-\alpha\tau_j}\Big(f_r(X(\tau_j), 1-Y(\tau_j-)) - f_r(X(\tau_j), Y(\tau_j-))\Big)\right] - f_r(x,y).$$

Proof. We begin by concentrating on the process X_y where the value of y is fixed. Recall the generator of X_y is $A_y f(x) = \frac{\sigma^2(x,y)}{2} f''(x) + \mu(x,y) f'(x)$. Noticing in particular the negative sign in the expression on the right-hand side below, define

$$f_y(x) := -\mathbb{E}_x\left[\int_0^\infty e^{-\alpha s} r(X(s), y)\, ds\right].$$

Condition 1.1 implies that $\int_0^\infty e^{-\alpha s} r(X_y(s), y)\, ds$ is integrable and hence that f_y is well-defined. Now for the next argument only, let τ be any $\{\mathcal{F}_t\}$-stopping time (rather than a sequence). By the strong Markov property we get

$$\begin{aligned}
\mathbb{E}_x\left[I_{\{\tau<\infty\}}\int_\tau^\infty e^{-\alpha s} r(X(s), y)\, ds\right] &= \mathbb{E}_x\left[I_{\{\tau<\infty\}}\, \mathbb{E}_x\left[\int_\tau^\infty e^{-\alpha s} r(X(s), y)\, ds \,\middle|\, \mathcal{F}_\tau\right]\right] \\
&= \mathbb{E}_x\left[I_{\{\tau<\infty\}} e^{-\alpha\tau}\, \mathbb{E}_x\left[\int_0^\infty e^{-\alpha u} r(X(\tau+u), y)\, du \,\middle|\, X(\tau)\right]\right] \\
&= \mathbb{E}_x\left[I_{\{\tau<\infty\}} e^{-\alpha\tau} \mathbb{E}_{X(\tau)}\left[\int_0^\infty e^{-\alpha u} r(X(u), y)\, du\right]\right] \\
&= -\mathbb{E}_x\left[I_{\{\tau<\infty\}} e^{-\alpha\tau} f_y(X(\tau))\right]. \qquad (1.5)
\end{aligned}$$

Suppose τ and $\widehat{\tau}$ are two stopping times with $\tau < \widehat{\tau}$. It then follows immediately that

$$\mathbb{E}_x\left[I_{\{\tau<\infty\}}\int_\tau^{\widehat{\tau}} e^{-\alpha s} r(X(s), y)\, ds\right] = \mathbb{E}_x\left[I_{\{\widehat{\tau}<\infty\}} e^{-\alpha\widehat{\tau}} f_y(X(\widehat{\tau}))\right] - \mathbb{E}_x\left[I_{\{\tau<\infty\}} e^{-\alpha\tau} f_y(X(\tau))\right].$$

Now recall $\tilde{\tau} = (\tau_1, \tau_2, \tau_3, \ldots, \tau_{2N}) \in \mathcal{A}$ and set $\tau_0 = 0$ and $\tau_{2N+1} = \infty$. Let $x \in (x_\ell, x_r)$ be chosen arbitrarily and consider the paired process (X, Y). We denote the dependence of the expectations on the initial positions x and $y = 0$ of the paired process (X, Y) by $\mathbb{E}_{x,0}[\cdot]$ and on the initial position of the X_y process by $\mathbb{E}_x[\cdot]$; recall X is defined in terms of the processes X_0 and X_1. Define the function $f_r(x, u) = f_0(x)(1-u) + f_1(x)u$ with $u = 0, 1$. Observe

$$\begin{aligned}
&\mathbb{E}_{x,0}\left[\int_0^\infty e^{-\alpha s} r(X(s), Y(s))\, ds\right] \\
&= \sum_{j=0}^{N} \mathbb{E}_{x,0}\left[I_{\{\tau_{2j}<\infty\}}\int_{\tau_{2j}}^{\tau_{2j+1}} e^{-\alpha s} r(X(s), Y(s))\, ds\right] \\
&\qquad + \sum_{j=0}^{N-1} \mathbb{E}_{x,0}\left[I_{\{\tau_{2j+1}<\infty\}}\int_{\tau_{2j+1}}^{\tau_{2j+2}} e^{-\alpha s} r(X(s), Y(s))\, ds\right] \\
&= \sum_{j=0}^{N} \mathbb{E}_x\left[I_{\{\tau_{2j}<\infty\}}\int_{\tau_{2j}}^{\tau_{2j+1}} e^{-\alpha s} r(X_0(s), 0)\, ds\right] + \sum_{j=0}^{N-1} \mathbb{E}_x\left[I_{\{\tau_{2j+1}<\infty\}}\int_{\tau_{2j+1}}^{\tau_{2j+2}} e^{-\alpha s} r(X_1(s), 1)\, ds\right]
\end{aligned}$$

$$= \sum_{j=0}^{N} \left(\mathbb{E}_x \left[I_{\{\tau_{2j+1}<\infty\}} e^{-\alpha\tau_{2j+1}} f_0(X_0(\tau_{2j+1})) \right] - \mathbb{E}_x \left[I_{\{\tau_{2j}<\infty\}} e^{-\alpha\tau_{2j}} f_0(X_0(\tau_{2j})) \right] \right)$$

$$+ \sum_{j=0}^{N-1} \left(\mathbb{E}_x \left[I_{\{\tau_{2j+2}<\infty\}} e^{-\alpha\tau_{2j+2}} f_1(X_1(\tau_{2j+2})) \right] - \mathbb{E}_x \left[I_{\{\tau_{2j+1}<\infty\}} e^{-\alpha\tau_{2j+1}} f_1(X_1(\tau_{2j+1})) \right] \right)$$

$$= \sum_{j=0}^{N-1} \mathbb{E}_x \left[I_{\{\tau_{2j+1}<\infty\}} e^{-\alpha\tau_{2j+1}} \left(f_0(X_0(\tau_{2j+1})) - f_1(X_1(\tau_{2j+1})) \right) \right]$$

$$+ \sum_{j=0}^{N-1} \mathbb{E}_x \left[I_{\{\tau_{2j+2}<\infty\}} e^{-\alpha\tau_{2j+2}} \left(f_1(X(\tau_{2j+2})) - f_0(X_0(\tau_{2j+2})) \right) \right] - f_r(x,0)$$

$$= \sum_{j=0}^{N-1} \mathbb{E}_x \left[I_{\{\tau_{2j+1}<\infty\}} e^{-\alpha\tau_{2j+1}} \Big(f_r(X(\tau_{2j+1}),0) - f_r(X(\tau_{2j+1}),1) \Big) \right]$$

$$+ \sum_{j=0}^{N-1} \mathbb{E}_x \left[I_{\{\tau_{2j+2}<\infty\}} e^{-\alpha\tau_{2j+2}} \Big(f_r(X(\tau_{2j+2}),1) - f_r(X(\tau_{2j+2}),0) \Big) \right] - f_r(x,0)$$

$$= \sum_{j=1}^{2N} \mathbb{E}_{x,0} \left[I_{\{\tau_j<\infty\}} e^{-\alpha\tau_j} \Big(f_r(X(\tau_j), 1-Y(\tau_j)) - f_r(X(\tau_j), Y(\tau_j)) \Big) \right] - f_r(x,0)$$

$$= - \sum_{j=1}^{2N} \mathbb{E}_{x,0} \left[I_{\{\tau_j<\infty\}} e^{-\alpha\tau_j} \Big(f_r(X(\tau_j), 1-Y(\tau_j-)) - f_r(X(\tau_j), Y(\tau_j-)) \Big) \right] - f_r(x,0),$$

where the last equality (notice the negative sign before the summation sign) follows from the fact that $Y(\tau_j) = 1 - Y(\tau_j-)$. □

Remark 1.4 *The proof of Proposition 1.3 only requires the reward rate r to satisfy Condition 1.1, not that it be continuous. However, when the reward rate function $r(\cdot, y)$ is bounded and continuous, f_y has an integral representation in terms of $r(\cdot, y)$, the Green kernel and the speed measure of the process (see [5, II.1.4,II.1.9,II.1.11]). Using the continuity of $r(\cdot, y)$, this representation shows that $f_r(\cdot, y)$ is twice-differentiable and is a particular solution of $A_y f - \alpha f = r(\cdot, y)$.*

It immediately follows from Proposition 1.3 that the maximization of (1.3) can be reformulated as maximizing

$$\begin{aligned} J(\tilde{\tau}; x_0, y_0) = &- \sum_{k=1}^{N} \mathbb{E} \left[I_{\{\tau_k^{(in)}<\infty\}} e^{-\alpha\tau_k^{(in)}} [f_r(X(\tau_k^{(in)}),1) - f_r(X(\tau_k^{(in)}),0) + c_k^{(in)}] \right] \\ &- \sum_{k=1}^{N} \mathbb{E} \left[I_{\{\tau_k^{(out)}<\infty\}} e^{-\alpha\tau_k^{(out)}} [f_r(X(\tau_k^{(out)}),0) - f_r(X(\tau_k^{(out)}),1) + c_k^{(out)}] \right] \\ &- f_r(x_0, y_0). \end{aligned}$$

The constant $-f_r(x_0, y_0)$ does not affect the solution to the optimization but is required to determine the correct value. *We therefore eliminate this term in our further analysis but* *include it when stating the value of the original production/mothballing problem.* Now for $k = 1, \ldots, N$, define $g_k^{(in)}(x, 0) = -(f_r(x, 1) - f_r(x, 0) + c_k^{(in)}) = f_r(x, 0) - f_r(x, 1) - c_k^{(in)}$ and $g_k^{(out)}(x, 1) = -(f_r(x, 0) - f_r(x, 1) + c_k^{(out)}) = f_r(x, 1) - f_r(x, 0) - c_k^{(out)}$. Thus we analyze the optimization problem of maximizing

$$J(\tilde{\tau}; x_0, 0) := \sum_{k=1}^{N} \mathbb{E}[I_{\{\tau_k^{(in)} < \infty\}} e^{-\alpha \tau_k^{(in)}} g_k^{(in)}(X(\tau_k^{(in)}), Y(\tau_k^{(in)}-))] \\ + \sum_{k=1}^{N} \mathbb{E}[I_{\{\tau_k^{(out)} < \infty\}} e^{-\alpha \tau_k^{(out)}} g_k^{(out)}(X(\tau_k^{(out)}), Y(\tau_k^{(out)}-))] \tag{1.6}$$

over admissible decision times $\tilde{\tau} \in \mathcal{A}$. As is traditional, we denote the value by $V(x_0, 0) = \sup_{\tilde{\tau} \in \mathcal{A}} J(\tilde{\tau}; x_0, 0)$.

A final observation is important. For each $x \in (x_l, x_r)$, the sum

$$\begin{aligned} g_k^{(in)}(x, 0) + g_k^{(out)}(x, 1) &= f_r(x, 0) - f_r(x, 1) - c_k^{(in)} + f_r(x, 1) - f_r(x, 0) - c_k^{(out)} \\ &= -(c_k^{(in)} + c_k^{(out)}) < 0. \end{aligned}$$

This indicates that a decision to instantaneously switch on production and to mothball it incurs a non-zero charge.

We impose a slightly stronger condition on the admissible switching times. We assume that there is a value $\overline{w} \in (x_l, x_r)$ such that $X(\tau_k^{(in)}) \geq \overline{w}$ on the set $\{\tau_k^{(in)} < \infty\}$ and $X(\tau_k^{(out)}) \leq \overline{w}$ on the set $\{\tau_k^{(out)} < \infty\}$, for each k. The collection of admissible switching times are therefore such that the decisions to start-up production are always made when the inventory is small and mothballing production occurs when the inventory is relatively large. Denote this restricted class of stopping times by $\mathcal{A}_0$. We therefore analyze the problem of maximizing (1.6) over $\tilde{\tau} \in \mathcal{A}_0$. We also restrict attention to initial values $X(0) = x_0 \leq \overline{w}$.

The following set of conditions on the payoff functions $g_k^{(in)}$ and $g_k^{(out)}$ will imply the existence of a finite optimal value and the corresponding existence of optimal stopping times $\{\tau_k^{(in)}\}$ and $\{\tau_k^{(out)}\}$. Recall from Condition 1.2 that ψ_0 and ϕ_1 satisfy $Af(x, y) - \alpha f(x, y) = 0$ with $y = 0$ and $y = 1$, respectively.

Condition 1.5 **(a)** *For $k = 1, \ldots, N$, there exists some values $x_k^{(in)} > \overline{w}$ and $x_k^{(out)} < \overline{w}$ in (x_l, x_r) such that $g_k^{(in)}(x_k^{(in)}, 0) > 0$ and $g_k^{(out)}(x_k^{(out)}, 1) > 0$.*

(b) *For $k = 1, \ldots, N$,* $\displaystyle\lim_{x \nearrow x_r} \frac{g_k^{(in)}(x, 0)}{\psi_0(x)} = 0.$

(c) *For $k = 1, \ldots, N$,* $\displaystyle\lim_{x \searrow x_l} \frac{g_k^{(out)}(x, 1)}{\phi_1(x)} = 0.$

In light of Condition 1.5(a), the objective is maximized when all switches are utilized, at least on a set of paths having positive probability though possibly not on almost all paths. Notice also that Condition 1.5(b,c) imposes a restriction on which solution f_r of the differential equation in Remark 1.4 is allowed. A solution f_r is composed of a particular solution and a linear combination $a_0\phi_0(x)I_{\{0\}}(y) + b_0\psi_0(x)I_{\{0\}}(y) + a_1\phi_1(x)I_{\{1\}}(y) + b_1\psi_1(x)I_{\{1\}}(y)$ of the solutions from Condition 1.2. In particular, a_1 needs to be 0 in order to satisfy Condition 1.5(c) and $b_0 = 0$ is required by Condition 1.5(b). Solving the differential equation provides an alternate way to determine the function f_r than using the definition in Proposition 1.3, as will be seen in the examples.

2 Linear Programming Imbedding

In this section, we reformulate the problem by replacing the stochastic processes with measures induced on the state space as the fundamental quantities. These measures become the variables in an infinite-dimensional linear program and are seen to be feasible points. The key to the linear program is the constraint (2.3) which is imposed for a large collection of test functions. Though the measures are defined by expectations related to the pair of processes (X, Y) and $\tilde{\tau} \in \mathcal{A}_0$, this collection is sufficiently large so that (2.3) captures the stochasticity of the processes [18, 19]. However, we only need the result that the processes (X, Y) along with an admissible strategy $\tilde{\tau}$ induce measures satisfying (2.3) and thus the optimal value is bounded above by the resulting linear program so we do not claim equivalence between the stochastic problem and the linear program. This paper does, however, demonstrate that the values of the two problems are equal and moreover identifies an optimal solution for each problem.

Choose $\tilde{\tau} \in \mathcal{A}_0$ arbitrarily. To simplify the following expressions, we use a different notation for the stopping times. For $k = 1, \ldots, N$, define $\tau_{2k} = \tau^{(in)}_{N-k+1}$ and $\tau_{2k-1} = \tau^{(out)}_{N-k+1}$; also let $\tau_{2N+1} = 0$. In this way, the stopping times are indexed by the set $\{1, \ldots, 2N, 2N+1\}$ with $0 = \tau_{2N+1} \leq \tau_{2N} \leq \tau_{2N-1} \leq \cdots \leq \tau_1 \leq \infty$ having strict inequality whenever the smaller stopping time is finite; the stopping time τ_k represents the time at which the k^{th}-from-the-end change in production level is made. Our analysis will show that these optimal switching times will be characterized as hitting times of the process in the set $\Xi_{\overline{w}}$ defined by $\Xi_{\overline{w}} = \{(x_1, y_1, x_2, y_2 \ldots, x_{2N}, y_{2N}) : x_{2j-1} \leq \overline{w}, y_{2j-1} = 1, x_{2j} \geq \overline{w}, y_{2j} = 0, j = 1, 2, \ldots, N\}$. Moreover, the process Y takes value 0 from an odd-indexed stopping time to an even-indexed stopping time and has value 1 from the even-indexed to the odd-indexed stopping times. This choice of indices will prove beneficial since an optimal set of decision times $\tilde{\tau}^* \in \mathcal{A}_0$ will be seen to depend on the number of remaining decisions.

Now define the process $\lambda_{\tilde{\tau}}$, which counts the number of changes in production, by

$$\lambda_{\tilde{\tau}}(t) = \sum_{k=1}^{2N} I_{[0,t]}(\tau_k) = \sum_{k=1}^{2N} I_{[\tau_k,\infty)}(t). \tag{2.1}$$

Define a new process Z to have $Z(0) = 2N =: z_0$ and to decrease by 1 at each finite intervention time τ_k, $k = 1, \ldots, 2N$. The purpose of the process Z is to record the number of remaining interventions. Observe that when $Z(t)$ is even, $Y(t) = 0$ and when $Z(t)$ is odd, $Y(t) = 1$. Let $f \in C^2_c((x_l, x_r) \times \{0, 1\} \times \{0, \ldots, 2N\})$; that is, f is a twice-continuously differentiable function having compact support. Let $\tilde{A}$ denote the operator $\tilde{A}f(x, y, z) = (\sigma^2(x, y)/2)\frac{\partial^2 f}{\partial x^2}(x, y, z) + \mu(x, y)\frac{\partial f}{\partial x}(x, y, z)$. Recall, $\tau_{2N+1} = 0$. Then applying Itô's formula yields

$$\begin{aligned}
&e^{-\alpha t}f(X(t),Y(t),Z(t))\\
&= f(x_0,y_0,z_0)+\int_0^t e^{-\alpha s}[\tilde{A}f(X(s),Y(s),Z(s))-\alpha f(X(s),Y(s),Z(s))]\,ds\\
&\quad+\int_0^t e^{-\alpha s}\sigma(X(s),Y(s))\tfrac{\partial f}{\partial x}(X(s),Y(s),Z(s))\,dW(s)\\
&\quad+\int_{[0,t]} e^{-\alpha s}[f(X(s),1-Y(s-),Z(s-)-1)-f(X(s),Y(s-),Z(s-))]\,d\lambda_{\tilde{\tau}}(s).
\end{aligned}$$

Taking expectations and then letting $t\to\infty$ results in

$$\begin{aligned}
0 &= f(x_0,y_0,z_0)+\mathbb{E}\left[\int_0^\infty e^{-\alpha s}[\tilde{A}f(X(s),Y(s),Z(s))-\alpha f(X(s),Y(s),Z(s))]\,ds\right]\\
&\quad+\mathbb{E}\left[\int_{[0,\infty]} e^{-\alpha s}[f(X(s),1-Y(s-),Z(s-)-1)-f(X(s),Y(s-),Z(s-))]\,d\lambda_{\tilde{\tau}}(s)\right]\\
&= f(x_0,y_0,z_0)+\mathbb{E}\left[\int_0^\infty e^{-\alpha s}[\tilde{A}f(X(s),Y(s),Z(s))-\alpha f(X(s),Y(s),Z(s))]\,ds\right] \qquad (2.2)\\
&\quad+\sum_{k=1}^{2N}\mathbb{E}\left[\int_{(\tau_{k+1},\tau_k]} e^{-\alpha s}[f(X(s),1-Y(s-),Z(s-)-1)-f(X(s),Y(s-),Z(s-))]\,d\lambda_{\tilde{\tau}}(s)\right],
\end{aligned}$$

where by a slight abuse of notation, the interval $(\tau_{2N+1},\tau_{2N}]$ is defined to be $[0,\tau_{2N}]$ and on the set $\{\tau_k=\infty\}$, $(\tau_{k+1},\tau_k]=(\tau_{k+1},\infty)$.

Define the finite measures $\mu^{\tilde{\tau}}$ and $\nu_k^{\tilde{\tau}}$, $k=1,\ldots,2N$, on $(x_l,x_r)\times\{0,1\}\times\{1,\ldots,2N\}$ such that for $G\in\mathcal{B}((x_l,x_r)\times\{0,1\}\times\{1,\ldots,2N\})$,

$$\begin{aligned}
\mu^{\tilde{\tau}}(G) &= \mathbb{E}\left[\int_0^\infty e^{-\alpha s}I_G(X(s),Y(s),Z(s)))\,ds\right], \text{ and}\\
\nu_k^{\tilde{\tau}}(G) &= \mathbb{E}\left[\int_{[0,\infty]} e^{-\alpha s}I_G(X(s)),Y(s-),Z(s-))I_{\{k\}}(Z(s-))\,d\lambda_{\tilde{\tau}}(s)\right].
\end{aligned}$$

Observe that the total mass of $\mu^{\tilde{\tau}}$ equals $1/\alpha$ and the masses of each $\nu_k^{\tilde{\tau}}$ are bounded by 1. Also notice that the measure $\nu_k^{\tilde{\tau}}$ has no mass accrued for $\tau_k=\infty$ so it incorporates $I_{\{\tau_k<\infty\}}$ implicitly. In addition, for each k, the measure $\nu_k^{\tilde{\tau}}$ can be identified with a measure on $(x_l,x_r)\times\{0,1\}$ since $Z(s-)=k$ in the definition of $\nu_k^{\tilde{\tau}}$. We also observe that when k is even, $Y(s-)=0$ and $Y(s-)=1$ when k is odd so $\nu_k^{\tilde{\tau}}$ can be further identified with a measure solely on (x_l,x_r). In fact, the restriction on the finite admissible switching times that $X(\tau_{2k})>\overline{w}$ and $X(\tau_{2k-1})<\overline{w}$ implies that the support of $\nu_{2k}^{\tilde{\tau}}$ is in $[\overline{w},x_r)\times\{0\}$ while the support of $\nu_{2k-1}^{\tilde{\tau}}$ is in $(x_l,\overline{w}]\times\{1\}$; these restrictions imply that the measure $\nu_1^{\tilde{\tau}}\times\nu_2^{\tilde{\tau}}\times\cdots\times\nu_{2N}^{\tilde{\tau}}$ has its support in $\Xi_{\overline{w}}$. We initially take advantage of the reduction to a measure on $(x_l,x_r)\times\{0,1\}$ and later will consider each $\nu_k^{\tilde{\tau}}$ as a measure on (x_l,x_r).

The identity (2.2) is expressed in terms of these measures as

$$0 = f(x_0, y_0, z_0) + \int [\tilde{A}f(x,y,z) - \alpha f(x,y,z)]\, \mu^{\tilde{\tau}}(dx \times dy \times dz) \\ + \sum_{k=1}^{2N} \int [f(x, 1-y, k-1) - f(x,y,k)] \nu_k^{\tilde{\tau}}(dx \times dy). \tag{2.3}$$

We turn now to an examination of the objective function (1.6). First, we continue to utilize the single set of indices in order to simplify the presentation. For $k = 1, \ldots, N$, let $g_{2k} = g_{N-k+1}^{(in)}$ and $g_{2k-1} = g_{N-k+1}^{(out)}$; again, this choice of subscript indicates the number of remaining decisions so, for example, $g_{2N} = g_1^{(in)}$, $g_{2N-1} = g_1^{(out)}$, $g_2 = g_N^{(in)}$ and $g_1 = g_N^{(out)}$. Recall, the measure $\nu_k^{\tilde{\tau}}$ incorporates $I_{\{\tau_k < \infty\}}$ implicitly so as a result (1.6) can be restated as

$$\begin{aligned} J(\tilde{\tau}; x_0, y_0) &= \sum_{k=1}^{N} \mathbb{E}\left[\int_{(\tau_{2k+1}, \tau_{2k}]} e^{-\alpha s} g_{2k}(X(s), Y(s-))\, d\lambda_{\tilde{\tau}}(s) \right] \\ &\quad + \sum_{k=1}^{N} \mathbb{E}\left[\int_{(\tau_{2k}, \tau_{2k-1}]} e^{-\alpha s} g_{2k-1}(X(s), Y(s-))\, d\lambda_{\tilde{\tau}}(s) \right] \\ &= \sum_{k=1}^{2N} \int g_k(x,y)\, \nu_k^{\tilde{\tau}}(dx \times dy). \end{aligned} \tag{2.4}$$

Summarizing the analysis so far, we note that the original multiple-decision stochastic problem consists of determining a finite collection $\tilde{\tau} = \{\tau_k : k = 1, \ldots, 2N\} \in \mathcal{A}_0$ of stopping times so as to maximize (1.6) for a process X satisfying (1.2) and Y being the indicator process corresponding to $\tilde{\tau}$. Each choice $\tilde{\tau}$ of such stopping times defines the random process $\lambda_{\tilde{\tau}}$ which in turn is used to define the measures $\nu_k^{\tilde{\tau}}$, $k = 0, \ldots, 2N$, and the measure $\mu^{\tilde{\tau}}$. The choice of these measures are therefore at the discretion of the manager. The objective (1.6) to be maximized is expressed in terms of the measures by (2.4), with the collection of measures satisfying (2.3) for all $f \in C_c^2((x_l, x_r) \times \{0,1\} \times \{0, \ldots, 2N\})$, the bounds on their total masses and the restrictions on their support. The original problem is therefore imbedded in the linear program

$$\begin{cases} \text{Maximize} \quad \displaystyle\sum_{k=1}^{2N} \int g_k(x,y)\, \nu_k(dx \times dy) \\ \text{Subject to} \quad f(x_0, y_0, z_0) = -\displaystyle\int [\tilde{A}f(x,y,z) - \alpha f(x,y,z)]\, \mu(dx \times dy \times dz) \\ \qquad\qquad - \displaystyle\sum_{k=1}^{2N} \int [f(x, 1-y, k-1) - f(x,y,k)] \nu_k(dx \times dy), \\ \qquad\qquad\qquad \forall f \in C_c^2, \\ \qquad \displaystyle\int 1\, \nu_k(dx) \le 1, \qquad k = 0, \ldots, 2N, \\ \qquad \displaystyle\int 1\, \mu(dx) = 1/\alpha, \\ \qquad \mu, \nu_k \text{ measures with } \operatorname{supp}(\nu_1 \times \cdots \times \nu_{2N}) \subset \Xi_{\overline{w}}. \end{cases} \tag{2.5}$$

Let $V_{lp}(x_0, y_0, z_0)$ denote the optimal value of (2.5). We therefore have the following theorem comparing the two value functions.

Theorem 2.1 *For each x_0 with $x_l < x_0 \leq \overline{w}$, $V(x_0, y_0) \leq V_{lp}(x_0, y_0, z_0)$.*

Proof. The discussion of the section shows that the measures $\mu^{\tilde{\tau}}$ and $\nu_k^{\tilde{\tau}}$, $k = 0, \ldots, 2N$, induced by each $\tilde{\tau} \in \mathcal{A}$ are feasible for (2.5) and moreover that the value of $J(\tilde{\tau}; x_0, y_0)$ is given by the objective function of (2.5). □

3 Auxiliary Program and Nonlinear Optimization

We now simplify the linear programming problem by limiting the number of constraints and as a consequence eliminating one of the measures. The resulting auxiliary linear program is then shown to have an optimal solution that arises as the solution to a nonlinear optimization problem.

Recall, ϕ_0, ϕ_1, ψ_0 and ψ_1 are functions satisfying Condition 1.2. Only ψ_0 and ϕ_1 play a role in the sequel so to simplify notation, we set $\psi := \psi_0$ and $\phi := \phi_1$. The increasing function ψ will only be used to determine a time to restart production during "good times" when it is mothballed ($Y(t) = 0$) and the decreasing function ϕ is only involved in identifying times to mothball the process when times are sufficiently "bad" and production is running ($Y(t) = 1$). We remind the reader that when x_l is either a natural or an entry-not-exit boundary point $\phi(x_l+) = \infty$ and $\psi(x_l) \geq 0$ with a similar statement applying for the boundary point x_r in which the roles of ϕ and ψ reversed.

The simple idea underlying the derivation of the nonlinear function to be optimized is to use the functions $f_k(x, y, z) = \phi(x)I_{\{1\}}(y)I_{\{k\}}(z) + \psi(x)I_{\{0\}}(y)I_{\{k\}}(z)$ as the functions in the constraints of (2.5). The derivation of this linear program, however, requires the function f to be bounded, a condition that ϕ does not satisfy near x_l and which ψ fails at x_r. We therefore temporarily introduce a further restriction on the admissible stopping times.

Restriction 3.1 *Let $\mathcal{A}_1 \subset \mathcal{A}_0$ be the subcollection of stopping times $\tilde{\tau}$ for which there exists some $a > x_l$ and $b < x_r$ such that $X(\tau_{2k}) \in [\overline{w}, b]$and $X(\tau_{2k-1}) \in [a, \overline{w}]$, for $k = 1, 2, \ldots, N$.*

This condition is important in two ways. First, for any switching policy $\tilde{\tau} \in \mathcal{A}_1$, $Y(t) = 1$ whenever $X(t) < a$ and $Y(t) = 0$ when $X(t) > b$. As a result, the measure $\mu^{\tilde{\tau}}$ will have no mass on $(x_l, a) \times \{0\} \times \{1, \ldots, N\}$ and similarly $\mu^{\tilde{\tau}}((b, x_r) \times \{1\} \times \{1, \ldots, N\}) = 0$. The second advantage to switching rules $\tilde{\tau} \in \mathcal{A}_1$ is that for $k = 1, \ldots, N$, the measures ν_{2k} have their support in $[\overline{w}, b] \times \{0\}$ while the measures ν_{2k-1} have their support in $[a, \overline{w}] \times \{1\}$. Define the set $\Xi_R = \{(x_1, y_1, x_2, y_2, \ldots, x_{2n}, y_{2N}) : x_{2k-1} \in [a, \overline{w}], y_{2k-1} = 1, x_{2k} \in [\overline{w}, b], y_{2k} = 0, k = 1, 2, \ldots, N\}$. The second observation implies that $\nu_1^{\tilde{\tau}} \times \nu_2^{\tilde{\tau}} \times \cdots \times \nu_{2n}^{\tilde{\tau}}$ has its support in Ξ_R.

Let $\tilde{\phi} \in C^2(x_l, x_r)$ be a bounded function with bounded derivatives, such that $\tilde{\phi}(x) = \phi(x)$ for $x \in [a, x_r)$. Similarly let $\tilde{\psi} \in C^2(x_l, x_r)$ be a mollification of ψ which is bounded with bounded derivatives such that $\tilde{\psi}(x) = \psi(x)$ for $(x_l, b]$.

Define $f(x, y) = \tilde{\psi}(x)I_{\{0\}}(y) + \tilde{\phi}(x)I_{\{1\}}(y)$ and, for $k = 1, \ldots, 2N$, define $f_k(x, y, z) = f(x, y)I_{\{k\}}(z)$. Using a localization argument, if necessary, the identity (2.3) holds for this collection of functions. The significance of this choice of functions is twofold. First, for each k, $\tilde{A}f_k - \alpha f_k \equiv 0$ where $\mu^{\tilde{\tau}}$ has its support and hence the identity does not involve the measure $\mu^{\tilde{\tau}}$. Second for $k = 1, \ldots, N$, $f_{2k}(x, 0, z) = \psi(x) I_{\{2k\}}(z)$ and $f_{2k-1}(x, 1, z) = \phi(x) I_{\{2k-1\}}(z)$ on the supports of the measures $\nu_{2k}^{\tilde{\tau}}$ and $\nu_{2k-1}^{\tilde{\tau}}$, respectively. Define the auxiliary linear program to be

$$\begin{cases} \text{Maximize} & \displaystyle\sum_{j=1}^{2N} \int g_j(x,y)\, \nu_j(dx \times dy) \\ \text{Subject to} & -\displaystyle\sum_{j=1}^{2N} \int [f_k(x, 1-y, j-1) - f_k(x,y,j)] \nu_j(dx \times dy) \\ & \qquad\qquad = f_k(x_0, y_0, z_0), \quad k = 1, \ldots, 2N, \\ & \nu_j \text{ are finite measures with } \mathrm{supp}(\nu_1 \times \cdots \times \nu_{2N}) \subset \Xi_R. \end{cases} \tag{3.1}$$

Notice that the auxiliary linear program only requires the main constraint of (2.5) to be satisfied by the collection $\{f_k : k = 1, \ldots, 2N\}$, removes any dependence on the measure μ and relaxes the mass conditions. It immediately follows that each set of feasible measures for (2.5) is also feasible for (3.1).

It will be helpful to examine the auxiliary linear program (3.1) more carefully before proceeding to develop the nonlinear optimization problem whose solution will determine optimal times for the original stochastic problem. It is at this point that we seek to reduce the problem to one involving measures on (x_l, x_r). First observe that the summation in the constraint of (3.1) simplifies since $f_k(x, y, z) = 0$ when $z \neq k$ so the only indices of j for which the integral is non-zero are $j = k$ and $j = k+1$. Furthermore, $\tilde{\phi}(x) = \phi(x)$ on $[a, \overline{w}]$ and $\tilde{\psi}(x) = \psi(x)$ on $[\overline{w}, b]$ so the expressions can all be written in terms of ϕ or ψ. Also, the right-hand-side $f_k(x_0, y_0, z_0)$ is only non-zero for $k = 2N$ and since $y_0 = 0$ we have $f(x_0, y_0, z_0) = \psi(x_0)$. The set of constraints is therefore

$$\begin{array}{rcl} \int \phi(x_1)\, \nu_1(dx_1) - \int \phi(x_2)\, \nu_2(dx_2) & = & 0 \\ \int \psi(x_2)\, \nu_2(dx_2) - \int \psi(x_3)\, \nu_3(dx_3) & = & 0 \\ \int \phi(x_3)\, \nu_3(dx_3) - \int \phi(x_4)\, \nu_4(dx_4) & = & 0 \\ \vdots & & \\ \int \phi(x_{2N-1})\, \nu_{2N-1}(dx_{2N-1}) - \int \phi(x_{2N})\, \nu_{2N}(dx_{2N}) & = & 0 \\ \int \psi(x_{2N})\, \nu_{2N}(dx_{2N}) & = & \psi(x_0). \end{array} \tag{3.2}$$

Define the $2N \times 2N$-dimensional matrix of integrands $\hat{A} = ((a_{ij}))$ by

$$a_{ij}(x_j) = \begin{cases} \phi(x_j), & i = 2k-1, j = i, k = 1, \ldots, N, \\ -\phi(x_j), & i = 2k-1, j = i+1, k = 1, \ldots, N, \\ \psi(x_j), & i = 2k, j = i, k = 1, \ldots, N, \\ -\psi(x_j), & i = 2k, j = i+1, k = 1, \ldots, N, \\ 0, & \text{otherwise}, \end{cases}$$

and define the $2N \times 1$-dimensional column vector $b = (0, \ldots, 0, \psi(x_0))'$. Then this system of constraints takes the form

$$\sum_{j=1}^{2N} \int a_{ij}(x_j)\, \nu_j(dx_j) = b_j, \qquad i = 1, \ldots, 2N. \tag{3.3}$$

Turning to the objective function, we have observed that $Y(t) = 0$ when $Z(t)$ is even and $Y(t) = 1$ when $Z(t)$ is odd. Since ν_k has mass only when $Z(t-) = k$, it follows that the support of ν_k is in $[\overline{w}, b] \times \{0\}$ when k is even and in $[a, \overline{w}] \times \{1\}$ when k is odd. The even-indexed functions $g_{2k}(x, y) = f_r(x, y) - f_r(x, 1-y) - c^{(in)}_{N-k+1}$, $k = 1, \ldots, N$, are therefore only evaluated when $y = 0$; similarly for $k = 1, \ldots, N$, $g_{2k-1}(x, y) = f_r(x, y) - f_r(x, 1-y) - c^{(out)}_{N-k+1}$ are only evaluated when $y = 1$. Slightly abusing notation, we let $g_{2k}(x) := g_{2k}(x, 0)$ and $g_{2k-1}(x) := g_{2k-1}(x, 1)$. The objective function of (3.1) then becomes

$$\sum_{j=1}^{2N} \int g_j(x_k)\, \nu_j(dx_j). \tag{3.4}$$

The linear program (3.1) now takes the form of maximizing (3.4) over a collection of measures $\{\nu_j : j = 1, \ldots, 2N\}$ on (x_l, x_r) satisfying the constraints (3.3) with each measure having total mass bounded by 1 and the product measure having its support in Ξ_R. For $j = 1, \ldots, N$, letting $S_{2j-1} = [a, \overline{w}]$ and $S_{2j} = [\overline{w}, b]$ for $j = 1, \ldots, 2N$, the problem has the form of the linear program (A.1) of the Appendix.

Let $x = (x_1, \ldots, x_{2N})$ denote a generic point in $S = \prod_{j=1}^{2N} S_j$. Let $\bar{x}_j$ denote a generic point in $\bar{S}_j := \prod_{i \neq j} S_i$, the $2N-1$ product which omits the j^{th} component. Theorem A.1 implies the following result.

Theorem 3.2 *Under Conditions 1.1, 1.2 and 1.5, the stochastic multiple-intervention problem of maximizing (1.1) over admissible sets of stopping times $\tilde{\tau} \in \mathcal{A}_1$ is equivalent to the nonlinear optimization problem of maximizing*

$$J_{nl}(x) = (\det \hat{A}(x))^{-1} \sum_{j=1}^{2N} g_j(x_j) \det \hat{A}_j(\bar{x}_j) \tag{3.5}$$

over $x \in S$. Using an optimizer $x^* = (x_1^*, \ldots, x_{2N}^*)$ of J_{nl}, the intervention times

$$\tau_k^* = \inf\{t : (X(t-), Z(t-)) = (x_k^*, k)\}, \qquad k = 1, \ldots, 2N, \tag{3.6}$$

are optimal for the stochastic problem.

Proof. Let $V_{nl}(x_0, y_0) = \sup_{x \in S} J_{nl}(x)$. Theorem A.1 of the Appendix establishes the fact that $V_{nl}(x_0, y_0)$ is an upper bound on $V_{aux}(x_0, y_0, z_0)$. A maximizer x^* exists since the optimization is over a compact set. Now for $j = 1, \ldots, N$, define $\theta_{2j-1} = \phi$ and $\theta_{2j} = \psi$. Also define $x_{2N+1} = x_0$. Then for $j = 1, 2, \ldots, 2N$, for $x^* = (x_1^*, \ldots, x_{2N}^*)$, the ratio $\det \hat{A}_j(x_j^*)/\det \hat{A}(x^*) = \prod_{k=j}^{2N} \theta_k(x_{k+1}^*)/\theta_k(x_k^*)$. Recall from (1.3) that the individual ratios give the expected discount factor for hitting times of the X process. Using the strong Markov property, the value $J_{nl}(x^*)$ is therefore seen to be the value corresponding to the switching times $\{\tau_k^*\}$ in (3.6), and hence these switching times are optimal. $\square$

We observe that $P(\tau_k^* = \tau_{k+1}^*) = 0$ for $k = 1, \ldots, 2N-1$.

The next theorem removes the restriction that $\tilde{\tau} \in \mathcal{A}_1$; its proof involves analysis of the stochastic problem having objective function (2.4).

Theorem 3.3 *Under Conditions 1.1, 1.2 and 1.5, the stochastic multiple-intervention problem of maximizing (1.1) over admissible sets of stopping times* $\tilde{\tau} \in \mathcal{A}_0$ *is equivalent to the nonlinear optimization problem of maximizing (3.5), the intervention times (3.6) are optimal for the stochastic problem, and the optimal value for the stochastic problem of maximizing (1.6) is*

$$V(x_0, 0) = J_{nl}(x^*)$$

where $x^* \in S$ *is a maximizer of* J_{nl}. *The optimal value for the* original *stochastic problem is* $V(x_0, 0) - f_r(x_0, 0)$.

Proof. Select $\tilde{\tau} \in \mathcal{A}$ arbitrarily. Let $\{a_n\}$ be a sequence such that $a_n \searrow x_l$ as $n \to \infty$ and similarly let $\{b_n\}$ be a sequence converging upward to x_r; we assume $a_1 < \overline{w} < b_1$. Define the switching times $\tau_{2N,b_n} = \inf\{t > 0 : X(t) = b_n\}$, $\tau_{2N-1,a_n} = \inf\{t > \tau_{2N} \wedge \tau_{2N,b_n} : X(t) = a_n\}$ and for $j = 1, \ldots, N-1$, $\tau_{2j,b_n} = \inf\{t > \tau_{2j+1} \wedge \tau_{2j+1,a_n} : X(t) = b_n\}$ and finally $\tau_{2j-1,a_n} = \inf\{t > \tau_{2j} \wedge \tau_{2j,b_n} : X(t) = a_n\}$. Observe that the collection $\tilde{\tau}_n = \{\tau_{2N} \wedge \tau_{2N,b_n}, \tau_{2N-1} \wedge \tau_{2N-1,a_n}, \ldots, \tau_1 \wedge \tau_{1,a_n}\}$ is an element of $\mathcal{A}_1$. Also since x_l and x_r are either natural or entrance-not-exit boundary points, $\tau_{2j-1,a_n}, \tau_{2j,b_n} \to \infty$ a.s. as $n \to \infty$ and thus $\tilde{\tau}_n \to \tilde{\tau}$ almost surely.

Since $\tilde{\tau}_n \in \mathcal{A}_1$, Theorem 3.2 implies the first inequality in

$$\begin{aligned}
V_{nl}(x_0, 0) &\geq J(\tilde{\tau}_n; x_0, 0) \\
&= \sum_{j=1}^{N} \mathbb{E}[I_{\{\tau_{2j}\wedge\tau_{2j,b_n}<\infty\}} e^{-\alpha\tau_{2j}\wedge\tau_{2j,b_n}} g_{2j}(X(\tau_{2j} \wedge \tau_{2j,b_n}), 0) I_{\{\tau_{2j}<\infty\}}] \\
&\quad + \sum_{j=1}^{N} \mathbb{E}[I_{\{\tau_{2j}\wedge\tau_{2j,b_n}<\infty\}} e^{-\alpha\tau_{2j}\wedge\tau_{2j,b_n}} g_{2j}(X(\tau_{2j} \wedge \tau_{2j,b_n}, 0) I_{\{\tau_{2j}=\infty\}}] \\
&\quad + \sum_{j=1}^{N} \mathbb{E}[I_{\{\tau_{2j-1}\wedge\tau_{2j-1,a_n}<\infty\}} e^{-\alpha\tau_{2j-1}\wedge\tau_{2j-1,a_n}} g_{2j-1}(X(\tau_{2j-1} \wedge \tau_{2j-1,a_n}), 1) I_{\{\tau_{2j-1}<\infty\}}] \\
&\quad + \sum_{j=1}^{N} \mathbb{E}[I_{\{\tau_{2j-1}\wedge\tau_{2j-1,a_n}<\infty\}} e^{-\alpha\tau_{2j-1}\wedge\tau_{2j-1,a_n}} g_{2j-1}(X(\tau_{2j-1} \wedge \tau_{2j-1,a_n}, 1) I_{\{\tau_{2j-1}=\infty\}}] \\
&= \sum_{j=1}^{N} \mathbb{E}[I_{\{\tau_{2j}\wedge\tau_{2j,b_n}<\infty\}} e^{-\alpha\tau_{2j}\wedge\tau_{2j,b_n}} g_{2j}(X(\tau_{2j} \wedge \tau_{2j,b_n}), 0) I_{\{\tau_{2j}<\infty\}}] \\
&\quad + \sum_{j=1}^{N} \mathbb{E}[I_{\{\tau_{2j,b_n}<\infty\}} e^{-\alpha\tau_{2j,b_n}} g_{2j}(X(\tau_{2j,b_n}), 0) I_{\{\tau_{2j}=\infty\}}]
\end{aligned}$$

$$\begin{aligned}
&+ \sum_{j=1}^{N} \mathbb{E}[I_{\{\tau_{2j-1}\wedge\tau_{2j-1,a_n}<\infty\}} e^{-\alpha\tau_{2j-1}\wedge\tau_{2j-1,a_n}} g_{2j-1}(X(\tau_{2j-1} \wedge \tau_{2j-1,a_n}), 1) I_{\{\tau_{2j-1}<\infty\}}] \\
&+ \sum_{j=1}^{N} \mathbb{E}[I_{\{\tau_{2j-1,a_n}<\infty\}} e^{-\alpha\tau_{2j-1,a_n}} g_{2j-1}(X(\tau_{2j-1,a_n}), 1) I_{\{\tau_{2j-1}=\infty\}}].
\end{aligned}$$

Consider now the first summation. On the set $\{\tau_{2j} < \infty\}$, the stopping time $\tau_{2j} \wedge \tau_{2j,b_n} \nearrow \tau_{2j}$ so the integrand in the expectations converge pointwise to $I_{\{\tau_{2j}<\infty\}} e^{-\alpha\tau_{2j}} g_k(X(\tau_{2j}), 0)$. Similarly the integrands in the third summation converge pointwise to $I_{\{\tau_{2j-1}<\infty\}} e^{-\alpha\tau_{2j-1}} g_k(X(\tau_{2j-1}), 1)$. Looking at the terms in the second summation,

$$\begin{aligned}\mathbb{E}[I_{\{\tau_{2j,b_n}<\infty\}} e^{-\alpha\tau_{2j,b_n}} g_{2j}(X(\tau_{2j,b_n}), 0) I_{\{\tau_{2j}=\infty\}}] &\leq \mathbb{E}[I_{\{\tau_{2j,b_n}<\infty\}} e^{-\alpha\tau_{2j,b_n}} g_{2j}(X(\tau_{2j,b_n}), 0)] \\ &\leq \frac{g_{2j}(b_n, 0)\psi(x_0)}{\psi(b_n)},\end{aligned}$$

with a similar bound of $\frac{g_{2j-1}(a_n,1)\phi(x_0)}{\phi(a_n)}$ applying to the terms in the fourth summations. Using Conditions 1.5(b,c) along with Fatou's Lemma, we obtain

$$V_{nl}(x_0, 0) \geq \sum_{k=1}^{2N} \mathbb{E}[I_{\{\tau_k<\infty\}} e^{-\alpha\tau_k} g_k(X(\tau_k), Y(\tau_k-))]$$

and taking the supremum over all $\tilde{\tau} \in \mathcal{A}_0$ implies that $V_{nl}(x_0, 0) \geq V(x_0, 0)$. Since $\mathcal{A}_1 \subset \mathcal{A}_0$, the opposite inequality is immediate. □

This optimality result can be best understood by considering the simple examples of a stochastic problem in which one is limited to a single increase and decrease in production and two increases and two decreases of production levels; we consider both single-cycle and double-cycle problems. Notice that the single-cycle problem can be viewed as an entry-and-exit problem in which one wishes to invest in a market and then disinvest. The double-cycle problem allows one to reenter and exit for a second (and last) time.

Example: One Cycle
The problem of interest consists of a single decision to open up production and one to close out production. To be consistent with the indexing of the stopping times, let τ_2 denote the time and x_2 denote the level at which production is started (hence there are two decisions to be made) and similarly τ_1 is the time and x_1 the level for closing down production (the last decision). The auxiliary linear program (3.1) for this single cycle problem is therefore

$$\begin{cases} \text{Maximize} & \int g_1(x_1)\,\nu_1(dx_1) + \int g_2(x_2)\,\nu_2(dx_2) \\ \text{Subject to} & \int \phi(x_1)\,\nu_1(dx_1) - \int \phi(x_2)\,\nu_2(dx_2) \;=\; 0, \\ & \qquad\qquad\qquad\qquad \int \psi(x_2)\,\nu_2(dx_2) \;= \psi(x_0). \end{cases}$$

From this we see that $\det\hat{A}(x_1, x_2) = \phi(x_1)\psi(x_2)$, $\det\hat{A}_1(x_2) = \psi(x_0)\phi(x_2)$ and $\det\hat{A}_2(x_1) = \phi(x_1)\psi(x_0)$. The nonlinear function to be optimized is therefore

$$J_{nl}(x_1, x_2) := \frac{g_1(x_1)\phi(x_2) + \phi(x_1)g_2(x_2)}{\phi(x_1)\psi(x_2)} \cdot \psi(x_0). \tag{3.7}$$

For ease of understanding, we illustrate the problem using the following toy problem. Let X be a (non-standard) Brownian motion with initial position $x_0 = 0$ and having diffusion coefficient $\sigma > 0$ so $X(t) = \sigma W(t)$. In this case, the dynamics do not actually depend on the process Y. The generator of the pair (X, Y) is $Af(x,y) = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(x,y)$ and thus the solutions to $Af - \alpha f = 0$ are $\psi(x,y) = e^{\rho x} I_{\{0\}}(y)$ and $\phi(x) = e^{-\rho x} I_{\{1\}}(y)$, where $\rho = \sqrt{\frac{2\alpha}{\sigma^2}}$ and $\alpha > 0$. We take $c_1 = c_2 =: c$ so the cost to start production is the same as the cost to shut down production. The function r does depend on Y; let $r(x,y) = xI_{\{1\}}(y)$ so a reward is received only while production is present. A function f_r satisfying the differential equation in Remark 1.4 is $f_r(x,y) = -\frac{x}{\alpha} I_{\{1\}}(y)$ and hence $g_1(x_1) = f_r(x_1,1) - f_r(x_1,0) - c_1 = -(\frac{x_1}{\alpha} + c)$ and $g_2(x_2) = f_r(x_2,0) - f_r(x_2,1) - c_2 = \frac{x_2}{\alpha} - c$. The nonlinear function to be optimized is

$$J_{nl}(x_1,x_2) = \frac{-(\frac{x_1}{\alpha}+c)e^{-\rho x_2} + (\frac{x_2}{\alpha} - c)e^{-\rho x_1}}{e^{\rho(x_2 - x_1)}} \cdot e^{\rho x_0}$$

with $(x_1, x_2) \in (-\infty, 0] \times [0, \infty)$. Setting the partial derivatives equal to 0 results in the system

$$\begin{cases} 0 = -\frac{1}{\alpha} e^{\rho(x_1 - 2x_2)} - \rho(\frac{x_1}{\alpha} + c) e^{\rho(x_1 - 2x_2)} \\ 0 = 2\rho(\frac{x_1}{\alpha} + c) e^{\rho(x_1 - 2x_2)} + \frac{1}{\alpha} e^{-\rho x_2} - \rho(\frac{x_2}{\alpha} - c) e^{-\rho x_2} \end{cases}$$

from which we see that $x_1^* = -(\alpha c + \frac{1}{\rho})$ and x_2^* is the unique positive solution to the transcendental equation

$$e^{-(1+\alpha\rho c)} e^{-\rho x_2} = \frac{(1 + \alpha\rho c)}{2} - \frac{\rho}{2} x_2.$$

The optimal value for the original stochastic problem is

$$\begin{aligned} V(x_0, y_0) = J_{nl}(x_1^*, x_2^*) - f_r(x_0, y_0) &= \frac{-(\frac{x_1^*}{\alpha} + c)e^{-\rho x_2^*} + (\frac{x_2^*}{\alpha} - c)e^{-\rho x_1^*}}{e^{\rho(x_2^* - x_1^*)}} \cdot e^{\rho x_0} \\ &= \frac{(\rho x_2^* - \alpha\rho c) + 1}{2\alpha\rho} e^{-\rho x_2^*} \cdot \psi(x_0) \end{aligned}$$

and the optimal times to change production levels are

$$\tau_2^* = \inf\{t > 0 : (X(t-), Y(t-)) = (x_2^*, 0)\} \text{ and } \tau_1^* = \inf\{t > 0 : (X(t-), Y(t-)) = (x_1^*, 1)\}.$$

□

Example: Two Cycles
Now consider the case where production is started up, then mothballed, then restarted and finally closed down resulting in two cycles. Let τ_4 and x_4 denote the time and level of initial start-up, τ_3 and x_3 the time and level at which mothballing occurs and similarly for τ_2, x_2, τ_1 and x_1 for the second cycle. The auxiliary linear program for this two-cycle problem is

$$\begin{cases} \text{Max.} \int g_1(x_1)\,\nu_1(dx_1) + \int g_2(x_2)\,\nu_2(dx_2) + \int g_3(x_3)\,\nu_3(dx_3) + \int g_4(x_4)\,\nu_4(ds_4) \\ \text{S.t.} \quad \int \phi(x_1)\,\nu_1(dx_1) - \int \phi(x_2)\,\nu_2(dx_2) = 0, \\ \qquad \int \psi(x_2)\,\nu_2(dx_2) - \int \psi(x_3)\,\nu_3(dx_3) = 0, \\ \qquad \int \phi(x_3)\,\nu_3(dx_3) - \int \phi(x_4)\,\nu_4(dx_4) = 0, \\ \qquad \int \psi(x_4)\,\nu_4(dx_4) = \psi(x_0). \end{cases}$$

From this it is easily determined that det$\hat{A}(x_1, x_2, x_3, x_4) = \phi(x_1)\psi(x_2)\phi(x_3)\psi(x_4)$. Replacing the first column of $\hat{A}$ by the right-hand side yields det$\hat{A}_1(x_2, x_3, x_4) = \psi(x_0)\phi(x_2)\psi(x_3)\phi(x_4)$. The other determinants are obtained from det$\hat{A}$ in a similar fashion. Let $x = (x_1, x_2, x_3, x_4)$. The nonlinear function to be optimized is

$$J_{nl}(x) = [g_1(x_1)\phi(x_2)\psi(x_3)\phi(x_4) + \phi(x_1)g_2(x_2)\psi(x_3)\phi(x_4) + \phi(x_1)\psi(x_2)g_3(x_3)\phi(x_4) \\ +\phi(x_1)\psi(x_2)\phi(x_3)g_4(x_4)] \cdot \frac{\psi(x_0)}{\phi(x_1)\psi(x_2)\phi(x_3)\psi(x_4)} . \tag{3.8}$$

We revisit the toy problem from the single-cycle problem in which the process is a non-standard Brownian motion with initial value $x_0 = 0$ and diffusion coefficient $\sigma > 0$, $0 < \alpha < \sqrt{2}$, $r(x, y) = xI_{\{1\}}(y)$ and the costs to change production levels are all the same constant $c > 0$. For this problem, $g_1(x) = g_3(x) = -(\frac{x}{\alpha} + c)$, $g_2(x) = g_4(x) = \frac{x}{\alpha} - c$ and hence J_{nl} of (3.8) is

$$J_{nl}(x) = \left[-\left(\tfrac{x_1}{\alpha} + c\right) e^{-\rho(x_2-x_3+x_4)} + \left(\tfrac{x_2}{\alpha} - c\right) e^{-\rho(x_1-x_3+x_4)} - \left(\tfrac{x_3}{\alpha} + c\right) e^{-\rho(x_1-x_2+x_4)} \\ + \left(\tfrac{x_4}{\alpha} - c\right) e^{-\rho(x_1-x_2+x_3)}\right] \cdot \frac{e^{\rho x_0}}{e^{-\rho(x_1-x_2+x_3-x_4)}} .$$

As one can immediately see, determining the solution by optimizing the function J_{nl} of four variables is nontrivial even for the two cycle problem. Rather than proceed with a discussion of this optimization, we investigate the structure of the optimization problem more generally and see that an iterative scheme is possible that involves the same type of optimization for each iteration but relative to different payoff functions. □

4 Iterative Solution Approach

The structure of the nonlinear function enables an efficient iterative approach to be employed in computing an optimal solution $x^* = (x_1^*, \ldots, x_{2N}^*)$. To easily describe this method, we reconsider the single-cycle and double-cycle examples of the previous section.

Example: One Cycle Revisited
Observe the function J_{nl} of (3.7) can be rewritten as

$$J_{nl}(x_1, x_2) = \frac{\psi(x_0)}{\psi(x_2)} \cdot g_2(x_2) + \frac{\psi(x_0)}{\psi(x_2)} \cdot \frac{\phi(x_2)}{\phi(x_1)} \cdot g_1(x_1). \tag{4.1}$$

Recall from (1.3) that the ratio $\frac{\psi(x_0)}{\psi(x_2)}$ is the Laplace transform of the time τ_2 for the process X to reach level x_2 starting from x_0; this represents the expected discount factor until the time production is started. Similarly the factor $\frac{\phi(x_2)}{\phi(x_1)} = \mathbb{E}[e^{-\alpha(\tau_1-\tau_2)}]$ gives the expected discounting for the time $\tau_1 - \tau_2$ it takes for X to hit level x_1 starting at x_2. Thus the form (4.1) indicates that the problem involves optimizing the expected discounted reward over hitting levels x_1 and x_2.

Now consider the task of maximizing J_{nl} over $(x_1, x_2) \in (x_l, \overline{w}] \times [\overline{w}, x_r)$. The function J_{nl} depends on x_1, the level at which production is shut down, only in the last term and only through the ratio $\frac{g_1(x_1)}{\phi(x_1)}$. Since g_1 is differentiable, an optimizer x_1^* will satisfy the first-order condition

$$\phi(x_1)g_1'(x_1) - \phi'(x_1)g_1(x_1) = 0 \tag{4.2}$$

when $x_1^* \in (x_l, \overline{w})$. The possibility $x_1^* = \overline{w}$ exists but only means one needs to consider various cases so is omitted from this discussion. Similarly, we omit discussion below of the case where $x_2^* = \overline{w}$ and note that the it will never be the case that $x_1^* = \overline{w} = x_2^*$ due to the presence of the strictly positive cost for immediately switching twice.

The next task would be to optimize $J_{nl}(x_1^*, x_2)$ over values of $x_2 \in [\overline{w}, x_r)$. Since x_1^* is now fixed, the optimization problem requires maximizing

$$J_{nl}(x_1^*, x_2) = \frac{g_2(x_2)}{\psi(x_2)} + \frac{\phi(x_2)}{\psi(x_2)} \cdot \frac{g_1(x_1^*)}{\phi(x_1^*)} = \frac{g_2(x_2) + \frac{g_1(x_1^*)}{\phi(x_1^*)} \cdot \phi(x_2)}{\psi(x_2)}. \tag{4.3}$$

Again, the first-order conditions for optimality imply

$$\psi(x_2)\left[g_2'(x_2) + \tfrac{g_1(x_1^*)}{\phi(x_1^*)} \cdot \phi'(x_2)\right] - [g_2(x_2) + \tfrac{g_1(x_1^*)}{\phi(x_1^*)} \cdot \phi(x_2)]\psi'(x_2) = 0.$$

From these observations, we see that the two-dimensional optimization problem is actually two one-dimensional optimization problems that, in fact, have the same structure. Notice that the payoff function when optimizing over x_2 consists of the sum of the payoff $g_2(x_2)$ received at the time production is started and the discounted payoff $\frac{\phi(x_2)}{\phi(x_1^*)} \cdot g_1(x_1^*)$ received at the time production is stopped. The difference between optimizing over x_1 and x_2 lies in the additional payoff arising from having the extra decision opportunity. □

Remark 4.1 *The first-order condition (4.2) can be rewritten as*

$$\frac{x_1 g_1'(x_1)}{g_1(x_1)} = \frac{x_1\phi'(x_1)}{\phi(x_1)}$$

which indicates that an optimizer $(x_1^*, x_2^*) \in (x_l, \overline{w}) \times (\overline{w}, x_r)$ is at a location where the elasticity of g_1 equals the elasticity of ϕ. Due to the properties of elasticity, the partial elasticity of the discount factor $\mathbb{E}[e^{-\alpha(\tau_1-\tau_2)}] = \frac{\phi(x_2)}{\phi(x_1)}$ with respect to x_1 is the negative of the corresponding elasticity of ϕ. Hence an optimal level $x_1^* \in (x_l, x_r)$ at which to shut down, the elasticity of g_1 is the negative of the partial elasticity of the discount factor. A similar statement holds for the elasticity of the adjusted payoff function $g_2(x_2) + \frac{g_1(x_1^*)}{\phi(x_1^*)} \cdot \phi(x_2)$ and the discount factor corresponding to the level at which to start production.

Example: Two Cycles Revisited
Consider the function J_{nl} of (3.8). Rewriting the ratios as in (4.1) shows that

$$\begin{aligned} J_{nl}(x) = {} & \frac{\psi(x_0)}{\psi(x_4)} \cdot g_4(x_4) + \frac{\psi(x_0)}{\psi(x_4)} \cdot \frac{\phi(x_4)}{\phi(x_3)} \cdot g_3(x_3) + \frac{\psi(x_0)}{\psi(x_4)} \cdot \frac{\phi(x_4)}{\phi(x_3)} \cdot \frac{\psi(x_3)}{\psi(x_2)} \cdot g_2(x_2) \\ & + \frac{\psi(x_0)}{\psi(x_4)} \cdot \frac{\phi(x_4)}{\phi(x_3)} \cdot \frac{\psi(x_3)}{\psi(x_2)} \cdot \frac{\phi(x_2)}{\phi(x_1)} \cdot g_1(x_1). \end{aligned} \tag{4.4}$$

As in the previous example, the factors multiplying $g_1(x_1)$ give the Laplace transforms of the hitting times of the levels x_4 from x_0, x_3 from x_4, x_2 from x_3 and x_1 from x_2; these Laplace transforms represent the expected discount factor corresponding to the hitting times. Thus, the nonlinear problem involves optimizing over levels at which production is started up or mothballed.

The variable x_1 only appears in the last term of (4.4). Optimizing $J_{nl}(x)$ with respect to x_1 requires maximizing the ratio $\frac{g_1(x_1)}{\phi(x_1)}$ so is, in fact, the same optimization as for the single cycle problem, though the optimal values differ. Thus an optimal level x_1^* at which to close down production satisfies the first-order conditions (4.2). Let x_1^* denote an optimizer of $\frac{g_1(x_1)}{\phi(x_1)}$. Turning to the optimization with respect to x_2. The x_2-dependence of $J_{nl}(x_1^*, x_2, x_3, x_4)$ requires optimizing (4.3); that is, the same optimization problem must be solved as for the single cycle problem. We therefore observe that the optimization for the last cycle is the same regardless of the number of cycles allowed. The only difference is that having additional cycles means the value associated with the optimal decision levels are more discounted. Let x_2^* be an optimizer of J_{nl} with x_1^* fixed.

Now looking at the optimization of $J_{nl}(x_1^*, x_2^*, x_3, x_4)$ with respect to x_3, we see that the function to be maximized is

$$J_{nl}(x_1^*, x_2^*, x_3, x_4) = \frac{g_3(x_3) + \left(\frac{g_2(x_2^*)}{\psi(x_2^*)} + \frac{g_1(x_1^*)\phi(x_2^*)}{\phi(x_1^*)\psi(x_2^*)}\right)\psi(x_3)}{\phi(x_3)} \cdot \frac{\phi(x_4)}{\psi(x_4)} \cdot \psi(x_0). \tag{4.5}$$

Again by separation of variables, the optimization over x_3 is independent of x_4. Moreover, an inspection the numerator of the first factor indicates that the optimization problem involves solving for an optimal level x_3^* which maximizes the ratio of the sum of three payoff functions and the function ϕ. This sum of payoff functions gives the return $g_3(x_3)$ received arising from the third-from-the-end decision plus the payment $g_2(x_2^*)$ at the time of the second-from-the-end intervention discounted by $\frac{\psi(x_3)}{\psi(x_2^*)}$, the expected discounted factor corresponding to the time the process X takes to move from x_3 to x_2^*, and a similar payoff $g_1(x_1^*)$ from the last intervention that is discounted by the factors $\frac{\phi(x_2^*)}{\phi(x_1^*)}$ and $\frac{\psi(x_3)}{\psi(x_2^*)}$. Finally, after finding an optimizer x_3^*, the last maximization problem only depends on x_4; the optimization problem has the same form with the more complex numerator formed from the sum of four appropriately discounted payoffs.

For certain relations between the mothballing and start-up fixed costs, one additional observation can be made about the relation of x_1^* and x_3^*, and x_2^* and x_4^* for an optimal $x^* \in S$. Suppose $c_1 \geq c_3$, where $c_3 := c_1^{(out)}$ and $c_1 := c_2^{(out)}$. Recall $g_{2j-1} = f_r(x, 1) - f_r(x, 0) - c_{N-j+1}^{(out)}$ so $g_3(x) = g_1(x) + c_1 - c_3$. Consider the optimization of the first factor of (4.5) with respect to x_3. The derivative is

$$\frac{\phi(x_3)g_3'(x_3) - \phi'(x_3)g_3(x_3)}{\phi(x_3)^2}$$
$$+\frac{\phi(x_3)\left(\frac{g_2(x_2^*)}{\psi(x_2^*)} + \frac{g_1(x_1^*)\psi(x_2^*)}{\phi(x_1^*)\psi(x_2^*)}\right)\psi'(x_3) - \phi'(x_3)\left(\frac{g_2(x_2^*)}{\psi(x_2^*)} + \frac{g_1(x_1^*)\phi(x_2^*)}{\phi(x_1^*)\psi(x_2^*)}\right)\psi(x_3)}{\phi(x_3)^2}$$

$$= \frac{\phi(x_3)g_1'(x_3) - \phi'(x_3)g_1(x_3)}{\phi(x_3)^2} - \frac{\phi'(x_3)(c_1 - c_3)}{\phi(x_3)^2}$$
$$+ \frac{\phi(x_3)\left(\frac{g_2(x_2^*)}{\psi(x_2^*)} + \frac{g_1(x_1^*)\psi(x_2^*)}{\phi(x_1^*)\psi(x_2^*)}\right)\psi'(x_3) - \phi'(x_3)\left(\frac{g_2(x_2^*)}{\psi(x_2^*)} + \frac{g_1(x_1^*)\phi(x_2^*)}{\phi(x_1^*)\psi(x_2^*)}\right)\psi(x_3)}{\phi(x_3)^2}.$$

Notice when $x_3 = x_1^*$ the first term on the right-hand side is 0 due to the optimality of x_1^*, the second term is nonnegative since ϕ is strictly decreasing and $c_1 \geq c_3$ and the third term is strictly positive. Hence at $x_3 = x_1^*$, the function is increasing and therefore a maximizer x_3^* must be greater than x_1^*. A similar analysis for the optimization with respect to x_4 indicates $x_4^* < x_2^*$ provided $c_2 \geq c_4$. □

These examples illustrate the structure of the auxiliary linear program for the N-cycle production level problem. The nonlinear function to be optimized consists of a sum of $2N$ terms with the x_1 variable entering only one term, the x_2-variable being in only two terms with one of those being the one depending on x_1 and so forth. As a result, the $2N$-dimensional optimization problem can be solved by an interated sequence of $2N$ one-dimensional nonlinear problems. We state this observation in the following theorem.

Theorem 4.2 *A maximizer $x^* = (x_1^*, \ldots, x_{2N}^*)$ of*

$$J_{nl}(x) = (\det\hat{A}(x))^{-1} \sum_{j=1}^{2N} g_j(x_j)\det\hat{A}_j(\bar{x}_j)$$

is obtained by sequentially solving $2N$ one-dimensional nonlinear optimization problems of the form $\frac{h_j(x_1^*, \ldots, x_{j-1}^*, x_j)}{\theta_j(x_j)}$ for $j = 1, \ldots, 2N$, in which $\theta_{2k}(x_{2k}) = \psi(x_{2k})$ and $\theta_{2k-1}(x_{2k-1}) = \phi(x_{2k-1})$ for $j = 1, \ldots, N$ and for $j = 1, \ldots, 2N$,

$$h_j(x_1, \ldots, x_j) = g_j(x_j) + \frac{\theta_{j-1}(x_j)}{\theta_{j-1}(x_{j-1})} g_{j-1}(x_{j-1}) + \cdots + \prod_{i=1}^{j-1} \frac{\theta_i(x_{i+1})}{\theta_i(x_i)} \cdot g_1(x_1).$$

Proof. For $k = 1, \ldots, 2N$, define the submatrices B_j of $\hat{A}$ by

$$B_j(x_j, \ldots, x_{2N}) = \begin{pmatrix} a_{jj}(x_j) & a_{j(j+1)}(x_{j+1}) & \cdots & a_{j(2N)}(x_{2N}) \\ a_{(j+1)j}(x_j) & a_{(j+1)(j+1)}(x_{j+1}) & \cdots & a_{(j+1)(2N)}(x_{2N}) \\ \vdots & \vdots & \ddots & \vdots \\ a_{(2N)j}(x_j) & a_{(2N)(j+1)}(x_{j+1}) & \cdots & a_{(2N)(2N)}(x_{2N}) \end{pmatrix}$$

and observe that only the diagonal and one-above-the-diagonal entries are non-zero. For $j = 1, \ldots, 2N$, define $\overline{B}_j$ by

$$\overline{B}_j(x_j, \ldots, x_{2N}) = \begin{pmatrix} 0 & a_{j(j+1)}(x_{j+1}) & \cdots & a_{j(2N)}(x_{2N}) \\ 0 & a_{(j+1)(j+1)}(x_{j+1}) & \cdots & a_{(j+1)(2N)}(x_{2N}) \\ \vdots & \vdots & \ddots & \vdots \\ \psi(x_0) & a_{(2N)(j+1)}(x_{j+1}) & \cdots & a_{(2N)(2N)}(x_{2N}) \end{pmatrix}$$

which is obtained from B_j by replacing the first column by the $(2N-j+1)$-vector $(0,\dots,0,\psi(x_0))'$. Note that $\overline{B}_{j+1}$ is a submatrix of $\hat{A}_j$. Observe

$$\det\hat{A}(x) = \prod_{k=1}^{2N}\theta_k(x_k) = \prod_{k=1}^{j-1}\theta_k(x_k)\cdot\det B_j(x_j,\dots,x_{2n})$$

and similarly,

$$\det\hat{A}_j(\bar{x}) = \prod_{k=1}^{j-1}\theta_k(x_k)\cdot\det\overline{B}_{j+1}(x_{j+1},\dots,x_{2n}).$$

As a result the function $J_{nl}(x)$ can be rewritten as

$$J_{nl}(x) = \sum_{k=1}^{2N}\frac{g_k(x_k)\cdot\det\overline{B}_{k+1}(x_{k+1},\dots,x_{2N})}{\det B_k(x_k,\dots,x_{2N})} = \sum_{k=1}^{2N}\frac{g_k(x_k)}{\theta_k(x_k)}\cdot\frac{\det\overline{B}_{k+1}(x_{k+1},\dots,x_{2N})}{\det B_{k+1}(x_{k+1},\dots,x_{2N})}$$

from which the structure of the summands is evident and the optimality of the iteration procedure of one-dimensional nonlinear problems is established. □

The next example illustrates how well the solution method described in this paper works for non-trivial problems. The mean-reverting dynamics of the process X are such that the functions ϕ and ψ are expressed in terms of the Kummer functions [1] which can be expressed either as series or an integral for which there is no closed-form expression. It is therefore necessary to employ numerical techniques.

Example: N-cycle Problem for Mean-Reverting Processes
We now examine a different dynamics for the process X. For $y\in\{0,1\}$, let the constants $\gamma(y)$, $\mu(y)$ and $\sigma(y)$ be positive. X is a solution of the stochastic differential equation

$$dX(t) = \mu(Y(t))(1-\gamma(Y(t))X(t))\,dt + \sigma(Y(t))\sqrt{X(t)}\,dW(t), \qquad X(0)=x_0>0 \quad (4.6)$$

so has generator $Af(x,y) = \frac{\sigma^2(y)}{2}x\frac{\partial^2 f}{\partial x^2}(x,y) + \mu(y)(1-\gamma(y)x)\frac{\partial f}{\partial x}(x,y)$ for all $f\in C^2(\mathbb{R}^+\times\{0,1\})$. In particular, observe that whenever $X(t)<\frac{1}{\gamma(Y(t))}$, the drift rate is positive so the diffusion tends to rise toward $\frac{1}{\gamma(Y(t))}$ but when $X(t)>\frac{1}{\gamma(Y(t))}$, the drift is negative so the diffusion tends to fall toward $\frac{1}{\gamma(Y(t))}$. The diffusion coefficient $\sigma(y)\sqrt{x}$ implies that the process X will never reach 0 and the positive drift at 0 means that 0 is an entrance-not-exit boundary point.

The increasing solution of the eigenfunction equation $Af(x,0)=\alpha f(x,0)$ in Condition 1.2 is given by

$$\psi(x) = K_M\left(\frac{\alpha}{\gamma(0)\mu(0)}, \frac{2\mu(0)}{\sigma(0)^2}, \frac{2\gamma(0)\mu(0)}{\sigma(0)^2}x\right), \quad (4.7)$$

in which $K_M(a,b,z)$ denotes the Kummer M-function

$$K_M(a,b,z) = 1 + \frac{az}{b} + \frac{(a)_2}{(b)_2}\frac{z^2}{2!} + \cdots + \frac{(a)_n}{(b)_n}\frac{z^n}{n!} + \cdots,$$

where $(c)_n := c(c+1)(c+2)\cdot\ldots\cdot(c+n-1)$, $(c)_0 := 1$ denotes the rising factorial (see [1] for details). For instance, $K_M(a,b,z)$ is a solution of the ordinary differential equation

$$zf''(z) + (b-z)f'(z) - af(z) = 0;$$

an alternative notation for the Kummer M-function is $K_M(a,b,z) = {}_1F_1(a,b;z)$ in [1]. The decreasing solution of $Af(x,1) - \alpha f(x,1) = 0$ is

$$\phi(x) = K_U\left(\frac{\alpha}{\mu(1)\gamma(1)}, \frac{2\mu(1)}{\sigma(1)^2}, \frac{2\mu(1)}{\sigma(1)^2}\gamma(1)x\right),$$

where the Kummer U-function can be expressed in terms of the Kummer M-function as

$$K_U(a,b,z) = \frac{\pi}{\sin(\pi b)}\left(\frac{K_M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\cdot\frac{K_M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right).$$

Let $r_i(x) = r(x,i) = k_i x - c_{fixed_i}$, $i = 0,1$, k_i and c_{fixed_i} given constants. One can easily check that the solution f_{r_i} of $Af - \alpha f = r_i$ is $f_{r_i}(x) = -\frac{k_i}{\alpha+\gamma\mu}x - \frac{1}{\alpha}\left(\frac{k_i\mu}{\alpha+\gamma\mu} - c_{fixed_i}\right)$. Note, the parameters γ and μ may also depend on i.

To illustrate the iteration procedure we look at two examples which are adaptations of an application considered by Dixit and Pindyck (see [8, p. 223]), namely when to invest and disinvest in a copper production. The adaptations use one or two mean-reverting processes instead of a single geometric Brownian motion process, along with appropriate parameter choices for the different dynamics. For the first example, consider the case in which the dynamics of the price process X of copper is not affected by the processes Y. The process Y does, however, specify different reward functions and different switching costs. Table 1 indicates the convergence of the trigger prices $\{a_N^*\}$ and $\{b_N^*\}$ to limiting values. Recall, for

N	a_N^*	b_N^*	*value*
1	0.125109	1.58976	3.98052
2	0.321411	1.47036	4.49382
3	0.354076	1.44612	4.59896
⋮	⋮	⋮	⋮
8	0.363173	1.43929	4.62853
9	0.363177	1.43929	4.62854
10	0.363178	1.43929	4.62855

Table 1: Trigger prices for exit a_N^* and entry b_N^* and Values as functions of remaining cycles N for a mean reverting process ($\gamma_0 = \gamma_1 = 1$, $\mu_0 = \mu_1 = 0.1$, $\sigma_0 = \sigma_1 = 0.3$) when $\alpha = 0.04$, $c^{(in)} = 2$, $c^{(out)} = 0.2$, $k_0 = 0$, $k_1 = 1$, $c_{fixed_0} = 0$, $c_{fixed_1} = 0.8$, and $x_0 = 0.8$.

instance, if $N = 8$ so there are 8 cycles and the system is initially mothballed, the value b_8^* refers to the first hitting level that has to be reached by X before production is started; a_8^* then specifies the level that has to be reached afterwards to turn off production. Thus a_1^* determines the time when the production actually stops. Note, the numerical results confirm that $\{a_N^*\}$ is a monotone increasing sequence while $\{b_N^*\}$ should be a monotone decreasing sequence.

The numbers in the column "value" are the optimal expected profit values when X starts at x_0 and evolves according to the specified dynamics using the finite switching sequence $\tilde{\tau}^* = (\tau_1^*, \ldots, \tau_{2N}^*)$ given in (3.6). It is worthwhile to point out that $\lim_{N\to\infty} a_N^*$ and $\lim_{N\to\infty} b_N^*$ coincide with the optimal trigger values of the infinite cycle problem and that the limit of the monotone increasing values equals the optimal value of that problem; moreover, all limits agree (up to 5 decimal places) with the numbers in row 10 (see [14]).

Our second example illustrates the case when the dynamics of X change following entry and exit decisions. We consider mean-reverting processes with two different long term average values $1/\gamma_0 = 10/7$ versus $1/\gamma_1 = 1$. The higher value of $1/\gamma_0$ reflects the situation that the long term average price is higher if a major producer is out of the market.

Comparing Table 2 with Table 1 reveals an import result. A producer whose presence in the market depresses the overall price level can take advantage of his market power. For each N he is able to increase his expected profit by employing a proper entry-and-exit strategy.

N	a_N^*	b_N^*	*value*
1	0.125109	1.88987	4.77835
2	0.3775	1.72465	5.37991
3	0.421476	1.68965	5.50733
⋮	⋮	⋮	⋮
8	0.434551	1.6792	5.54526
9	0.434558	1.67919	5.54528
10	0.43456	1.67919	5.54528

Table 2: Trigger prices for exit a_N^* and entry b_N^* and Values for two mean-reverting processes ($\gamma_0 = 0.7$ and $\gamma_1 = 1$, $\mu_0 = \mu_1 = 0.1$, $\sigma_0 = \sigma_1 = 0.3$) as functions of the remaining cycles N, when $\alpha = 0.04$, $c^{(in)} = 2$, $c^{(out)} = 0.2$, $k_0 = 0$, $k_1 = 1$, $c_{fixed_0} = 0$, $c_{fixed_1} = 0.8$, and $x_0 = 0.8$.

□

A Appendix

We prove the general optimization result in this appendix. To begin, let $n \in \mathbb{N}$ and for $j = 1, \ldots, N$, let $(S_j, \mathcal{X}_j)$ be a measurable space. For $j = 1, \ldots, n$, let $g_j : S_j \to \mathbb{R}$ and for $i, j = 1, \ldots, n$, let $a_{ij} : S_j \to \mathbb{R}$. Define the general linear program to be

$$\begin{cases} \text{Maximize} \quad \sum_{j=1}^{n} \int g_j(s_j)\, \nu_j(ds_j) \\ \text{Subject to} \quad \sum_{j=1}^{n} \int a_{ij}(s_j)\, \nu_j(ds_j) = b_j, \quad i = 1, \ldots, n \\ \qquad\qquad\quad \nu_j \text{ a finite measure on } (S_j, \mathcal{X}_j), \quad j = 1, \ldots, n. \end{cases} \tag{A.1}$$

Define the matrix $\hat{A}(s_1, \ldots, s_n) = (a_{ij}(s_j))$ and for $j = 1, \ldots, N$, let $\hat{A}_j$ be the matrix $\hat{A}$ with its jth column replaced by $(b_1, \ldots, b_n)^t$. Note that $\hat{A}_j$ is a function of the $n-1$ variables $(s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)$; denote these variables by $\bar{s}_j$. Finally define the product measure $\nu = \nu_1 \times \cdots \times \nu_n$ on $S := S_1 \times \cdots \times S_n$. Let $\hat{a} = \int \det\hat{A}(s)\, d\nu(s)$.

The following result provides an upper bound on the value of (A.1) and derives a nonlinear optimization problem whose solution determines a collection of optimal measures for (A.1).

Theorem A.1 *Suppose* $\det\hat{A} > 0$ *and* $\hat{a} < \infty$. *Then*

$$\sum_{j=1}^{n} \int g_j(s_j)\, \nu_j(ds_j) \leq \sup_{s \in S} \left\{ (\det\hat{A}(s))^{-1} \sum_{j=1}^{n} g_j(s_j) \det\hat{A}_j(\bar{s}_j) \right\}. \tag{A.2}$$

If the supremum is achieved at a point $s^* = (s_1^*, \ldots, s_n^*)$, *then for* $j = 1, \ldots, n$, *the measures* ν_j^* *which are concentrated on* $\{s_j^*\}$ *are optimal for the linear program (A.1).*

Proof. Define $\bar{S}_j = S_1 \times \cdots \times S_{j-1} \times S_{j+1} \times \cdots \times S_n$ so that $\bar{S}_j$ omits the jth coordinate space from S. Also define $\bar{\nu}_j$ to be the product measure on $\bar{S}_j$ of the measures $\{\nu_l : l \neq j\}$ which excludes the jth measure ν_j from the product. Observe that using the constraints of (A.1) along with Cramer's rule yields

$$\hat{a} = \int_S \det\hat{A}(s)\, \nu(ds) = \det\left(\int a_{ij}(s_j)\, \nu_j(ds) \right) = \int_{\bar{S}_j} \det\hat{A}_j(\bar{s})\, \bar{\nu}_j(d\bar{s}).$$

Therefore,

$$\begin{aligned} \int_S \sum_{j=1}^{n} g_j(s_j)\ \det\hat{A}_j(\bar{s})\, \nu(ds) &= \sum_{j=1}^{n} \int_{S_j} g_j(s_j)\, \nu_j(ds_j) \cdot \int_{\bar{S}_j} \det\hat{A}_j(\bar{s})\, \bar{\nu}_j(d\bar{s}) \\ &= \hat{a} \sum_{j=1}^{n} \int_{S_j} g_j(s_j)\, \nu_j(ds_j). \end{aligned}$$

Define the probability measure μ on S having density $\hat{a}^{-1}\det\hat{A}$ with respect to the product measure ν on S. Then

$$\begin{aligned} \sum_{j=1}^{n} \int_{S_j} g_j(s_j)\, \nu_j(ds_j) &= \int_S \hat{a}^{-1} \sum_{j=1}^{n} g_j(s_j) \det\hat{A}_j(\bar{s})\, \nu(ds) \\ &= \int_S (\det\hat{A}(s))^{-1} \sum_{j=1}^{n} g_j(s_j) \det\hat{A}_j(\bar{s})\, \mu(ds). \end{aligned}$$

Since μ is a probability measure on S, both the upper bound (A.2) and the optimality of the measures ν_j^*, $j = 1, \ldots, n$, immediately follow. □

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