Exercises

Vectors and Matrices – Solutions

Exercise 1.

a)

$$3u + 2v - w = 3\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ -11 \end{pmatrix}$$

b)

$$w - (\mathbf{e}_1 - \mathbf{e}_2) + \mathbf{e}_3 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} - \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

c)

$$\frac{1}{2}(u - \mathbf{1}) + 4(v - w) = \begin{pmatrix} -5 \\ -19.5 \\ -10 \end{pmatrix}$$

Exercise 2.

a) $A + B = \begin{pmatrix} 9 & -6 & 0 \\ -18 & 2 & 4 \end{pmatrix}$

b) AB is undefined.

$$AB^\top = \begin{pmatrix} 5 & -1 & 2 \\ -8 & 3 & 7 \end{pmatrix}\begin{pmatrix} 4 & -10 \\ -5 & -1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 21 & -55 \\ -61 & 56 \end{pmatrix}$$

$$BA^\top = (AB^\top)^\top = \begin{pmatrix} 21 & -61 \\ -55 & 56 \end{pmatrix}$$

c) $A\mathbf{1} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

$$\mathbf{e}_2^\top A = (-8, 3, 7)$$

d) $\mathbf{g}^\top A^\top = (1, 3, -2) \begin{pmatrix} 5 & -8 \\ -1 & 3 \\ 2 & 7 \end{pmatrix} = (-2, -13)$

$$\mathbf{g}^\top \mathbf{h} = (1, 3, -2) \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = -4$$

$$\mathbf{g}\,\mathbf{h}^\top = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} (-2, 0, 1) = \begin{pmatrix} -2 & 0 & 1 \\ -6 & 0 & 3 \\ 4 & 0 & -2 \end{pmatrix}$$

Exercise 3.

a) We want to find $a \in \mathbb{R}$ such that $\left\| \begin{bmatrix} a \\ -3a \end{bmatrix} \right\| = 1$. So we have to solve

$$\left\| \begin{pmatrix} a \\ -3a \end{pmatrix} \right\| = \left\| a \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right\| = |a| \left\| \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right\| = |a|\sqrt{1+9} = |a|\sqrt{10} \overset{!}{=} 1$$

and thus obtain the two solutions $a = \pm \frac{1}{\sqrt{10}}$.

b) We want to find vectors $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ that are orthogonal to u, i.e. vectors that satisfy $\langle u, x \rangle = 0$. We have to solve

$$0 \overset{!}{=} \left\langle \begin{bmatrix} 5 \\ -1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = 5x_1 - x_2$$

which yields $x_2 = 5x_1$. Thus all vectors $x = \begin{bmatrix} x_1 \\ 5x_1 \end{bmatrix}$ with $x_1 \in \mathbb{R}$ are orthogonal to u.

c) We compute the norms of the vectors as

$$\|v\| = \sqrt{4 + 16 + 25 + 4} = \sqrt{49} = 7$$
$$\text{and} \quad \|w\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$$

and thus obtain the normalized vectors

$$v_{\text{norm}} = \frac{v}{\|v\|} = \frac{1}{7} \begin{pmatrix} -2 \\ 4 \\ -5 \\ 2 \end{pmatrix}$$

$$w_{\text{norm}} = \frac{w}{\|w\|} = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

d) The vectors *orthogonal* to $(2,-3)^\top$ satisfy

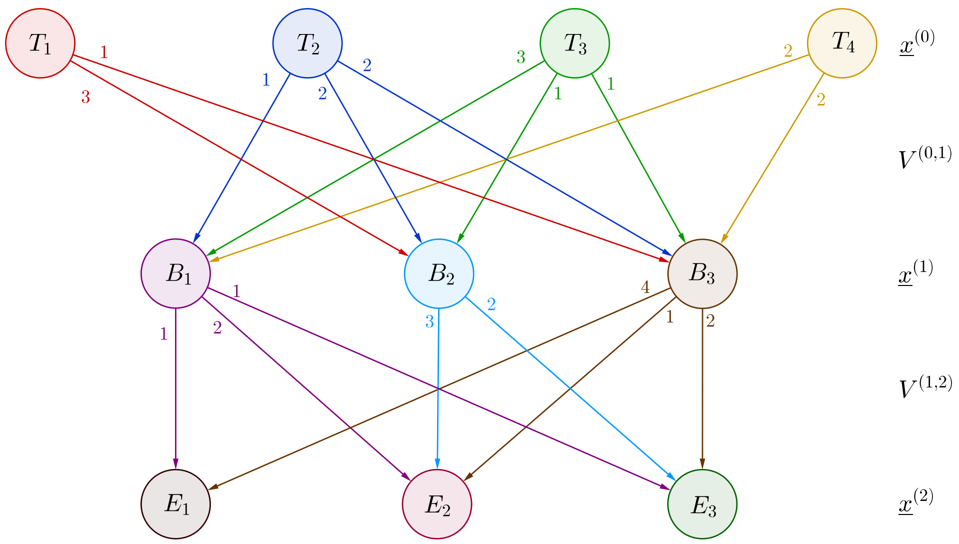
$$0 = (2,-3)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2x_1 - 3x_2 \quad \Longrightarrow \quad x_2 = \frac{2}{3}x_1$$

and are thus the vectors $x = \begin{bmatrix} x_1 \\ \frac{2}{3}x_1 \end{bmatrix}$ with $x_1 \in \mathbb{R}$. To be orthonormal, the vectors x must satisfy $1 = \|x\| = \sqrt{(1x_1)^2 + \left(\frac{2}{3}x_1\right)^2} = \sqrt{\frac{13}{9}x_1^2} = \frac{\sqrt{13}}{3}|x_1|$. Hence, $|x_1| = \frac{3}{\sqrt{13}}$ which implies that $x_1 = \frac{3}{\sqrt{13}}$ or $x_1 = -\frac{3}{\sqrt{13}}$. Thus,

$$x = \frac{1}{\sqrt{13}}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{or} \quad x = -\frac{1}{\sqrt{13}}\begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Exercise 4.

1. Scheme of the process



2. Productionmatrices

$$V^{(0,1)} = \begin{pmatrix} 0 & 3 & 1 \\ 1 & 2 & 2 \\ 3 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix} \qquad V^{(1,2)} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 4 & 1 & 2 \end{pmatrix}$$

Quantity vectors:

$$x^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \\ x_4^{(0)} \end{pmatrix} \triangleq \text{ vector of resources}$$

$$x^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} \triangleq \text{ vector of intermediate products}$$

$$x^{(2)} = \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{pmatrix} \triangleq \text{ vector of end products}$$

We have

$$x^{(0)} = V^{(0,1)} x^{(1)} = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{array}{c} \begin{array}{ccc} I_1 & I_2 & I_3 \end{array} \\ \begin{pmatrix} 0 & 3 & 1 \\ 1 & 2 & 2 \\ 3 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix} \end{array} \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix}$$

$$x^{(1)} = V^{(1,2)}\, x^{(2)} = \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \end{array} \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 4 & 1 & 2 \end{pmatrix} \end{array} \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{pmatrix}$$

This implies

$$x^{(0)} = V^{(0,1)}\, x^{(1)} = V^{(0,1)}\, V^{(1,2)}\, x^{(2)}$$

For $x^{(2)} = \begin{pmatrix} 50 \\ 100 \\ 200 \end{pmatrix}$ we get

$$x^{(1)} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 50 \\ 100 \\ 200 \end{pmatrix} = \begin{pmatrix} 450 \\ 700 \\ 700 \end{pmatrix}$$

Quantity of required intermediate products

$$x^{(0)} = \begin{pmatrix} 0 & 3 & 1 \\ 1 & 2 & 2 \\ 3 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 450 \\ 700 \\ 700 \end{pmatrix} = \begin{pmatrix} 2800 \\ 3250 \\ 2750 \\ 2300 \end{pmatrix} \quad \text{Quantity of required resources}$$