Exercises

Systems of Linear Equations – Solutions

Exercise 1.

(a)

x_1	x_2	rhs
1	2	8
4	3	17
1	2	8
0	−5	−15
1	0	2
0	1	3

$$\Longrightarrow \quad x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad \mathcal{L} = \{x\}$$

Unique Solution.

(b)

x_1	x_2	rhs
2	4	17
1	2	8
0	0	1
1	2	8

$$\Longrightarrow \quad 0 \neq 1 \quad \text{No solution, i.e. } \mathcal{L} = \emptyset$$

(c)

x_1	x_2	x_3	rhs
−3	1	−2	4
−6	2	−4	8
−15	5	−10	20
−3	1	−2	4
0	0	0	0
0	0	0	0

$$\Longrightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad x_1, x_3 \in \mathbb{R}$$

Infinitely many solutions:

$$\mathcal{L} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad t_1,\, t_2 \,\in\, \mathbb{R} \right\}$$

(d)

x_1	x_2	x_3	rhs
1	2	3	0
4	5	6	3
7	8	9	6
1	2	3	0
0	−3	−6	3
0	−6	−12	6
1	0	−1	2
0	1	2	−1
0	0	0	0

$$\Longrightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad x_3 \in \mathbb{R}$$

Infinitely many solutions:

$$\mathcal{L} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad t_1, \,\in\, \mathbb{R} \right\}$$

Exercise 2.

u_1	u_2	u_3	e_1	e_2	e_3
2	−2	1	1	0	0
−1	**1**	0	0	1	0
2	1	−2	0	0	1
0	0	**1**	1	2	0
−1	1	0	0	1	0
3	0	−2	0	−1	1
0	0	1	1	2	0
−1	1	0	0	1	0
3	0	0	2	3	1
0	0	1	1	2	0
0	1	0	$\frac{2}{3}$	2	$\frac{1}{3}$
1	0	0	$\frac{2}{3}$	1	$\frac{1}{3}$
1	0	0	$\frac{2}{3}$	1	$\frac{1}{3}$
0	1	0	$\frac{2}{3}$	2	$\frac{1}{3}$
0	0	1	1	2	0

$$A^{-1} = \begin{pmatrix} \frac{2}{3} & 1 & \frac{1}{3} \\ \frac{2}{3} & 2 & \frac{1}{3} \\ 1 & 2 & 0 \end{pmatrix}$$

Exercise 3.

(a) Since $v_1 = -1 \cdot v_2$ the vectors are clearly linear dependent.

(b) The matrix

$$\begin{pmatrix} 1 & 3 & 5 \\ 3 & 9 & 10 \\ 0 & 2 & 7 \\ 2 & 8 & 12 \end{pmatrix}$$

has rank 3 and thus the vectors are linear independent.

Exercise 4.

x	y	z	rhs
2	$-a$	1	2
1	2	-1	b
0	$-a-4$	3	$2-2b$
1	2	-1	b
0	$-\frac{a+4}{3}$	1	$\frac{2-2b}{3}$
1	$2-\frac{a+4}{3}$	0	$b+\frac{2-2b}{3}$

With $2 - \frac{a+4}{3} = \frac{2-a}{3}$ and $b + \frac{2-2b}{3} = \frac{b+2}{3}$ we get

$$\begin{aligned} x &= \frac{b+2}{3} - \frac{2-a}{3} y \\ z &= \frac{2-2b}{3} + \frac{a+4}{3} y \\ y &\in \mathbb{R} \quad \text{free} \rightarrow (y = t,\ t \in \mathbb{R}), \quad \text{i. e.} \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{b+2}{3} \\ 0 \\ \frac{2-2b}{3} \end{pmatrix} + t \begin{pmatrix} -\frac{2-a}{3} \\ 1 \\ \frac{a+4}{3} \end{pmatrix}, \quad t \in \mathbb{R}$$

Exercise 5. Let $a \neq 0$. Then:

u_1	u_2	e_1	e_2
a	b	1	0
c	d	0	1
1	$\frac{b}{a}$	$\frac{1}{a}$	0
0	$d - \frac{bc}{a}$	$-\frac{c}{a}$	1
1	0	$\frac{d}{ad-bc}$	$-\frac{b}{ad-bc}$
0	1	$-\frac{c}{ad-bc}$	$\frac{a}{ad-bc}$

where we used

$$\frac{1}{a} + \frac{b}{a} \cdot \frac{c}{a\left(d - c\,\frac{b}{a}\right)} = \frac{d}{ad-bc}.$$

For the last step we need $ad - bc \neq 0$.

If $a = 0$ but $c \neq 0$ we get the same inverse as long as $ad - bc \neq 0$.
If $ad - ac$ (which in particular is the case if $a = c = 0$) then there is no inverse.
Overall, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \text{for } ad - bc \neq 0$$

Exercise 6. Solving the linear equations yields:

1	1	1	$2q$
2	-3	2	$4q$
3	-2	p	q
1	1	1	$2q$
0	-5	0	0
0	-5	$p-3$	$-5q$
1	0	1	$2q$
0	1	0	0
0	0	$p-3$	$-5q$

$$\iff \begin{aligned} x_2 &= 0 \\ x_1 &= 2q - x_3 \\ (p-3)x_3 &= -5q \end{aligned}$$

(i) The system has a unique solution $\iff p \neq 3$

(ii) The system has no solution $\iff p = 3\,,\ q \neq 0$

(iii) The system has infinitely many solutions $\iff p = 3\,,\ q = 0$