Exercises

Determinants

Exercise 1. Compute the determinants of the following matrices.

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $\begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & -2 \\ 1 & 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 5 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$

Exercise 2. Compute the determinants of the following matrices.

(a) $A_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 1 & 4 & 2 \end{pmatrix}$ (b) $A_2 = \begin{pmatrix} 5 & 5 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Exercise 3. Find a counterexample to show that the statement

$$\det(A+B) = \det(A) + \det(B)$$

is *incorrect.*

Exercise 4. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & a & 3 \\ 1 & 2 & a \end{pmatrix}$ and $B = \begin{pmatrix} b & 1 & 2 \\ -1 & b & 0 \\ 2 & 0 & -1 \end{pmatrix}$.

(a) For which $a \in \mathbb{R}$ do we have $\det A = 0$, $\det A > 0$, $\det A < 0$?

(b) For which $b \in \mathbb{R}$ do we have $\det(B) = 0$?

(c) For which $b \in \mathbb{R}$ is $\det(B)$ maximal?

Exercise 5 (optional). Let $A \in \mathbb{R}^{n \times n}$ be a quadratic matrix and $b \in \mathbb{R}^n$ some vector. Then we define

$$A_i := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1i-1} & b_1 & a_{1i+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i-1} & b_2 & a_{2i+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni-1} & b_n & a_{ni+1} & \cdots & a_{nn} \end{pmatrix}$$

the matrix A where the i-th column is replaced by the vector b. Then **Cramer's rule** states that the vector x with

$$x_i := \frac{\det A_i}{\det A}$$

is the unique solution of $Ax = b$ (as long as $\det A \neq 0$).

1. Compute the solution of

   (a) $\begin{pmatrix} 1 & 4 \\ 2 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} x = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

   with Cramer's rule.

2. Compute the solution of

   $$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ -1 & 1 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$$

   with Cramer's rule.