How to... Diagonalize matrices

Given: A quadratic matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$.

Wanted: A matrix $\mathbf{V} \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\mathbf{D} \in \mathbb{R}^{n\times n}$ such that

$$\mathbf{A} = \mathbf{V}\,\mathbf{D}\,\mathbf{V}^{-1}.$$

Example

We consider the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 3 & 3 \\ 3 & 5 & -3 \\ 6 & -6 & 2 \end{pmatrix}.$$

1 Computation of eigenvalues and -vectors

Compute all eigenvalues $\lambda_1, \dots, \lambda_n$ and the eigenvectors of $\mathbf{A}$.

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 8$, $\lambda_2 = 8$, $\lambda_3 = -4$ and the eigenspaces are

$$E_{\lambda=8} = \left\{ \alpha \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}, E_{\lambda=-4} = \left\{ \alpha \cdot \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}.$$

2 Check the algebraic and geometric multiplicities

Compute the algebraic and geometric multiplicities $\mu_{\mathbf{A}}(\lambda_i)$ and $\gamma_{\mathbf{A}}(\lambda_i)$ for all eigenvalues λ_i.

a $\mu_{\mathbf{A}}(\lambda_i) \neq \gamma_{\mathbf{A}}(\lambda_i)$ for some eigenvalue λ_i

If there is at least one eigenvalue λ_i with geometric multiplicity less than algebraic multiplicity ($\mu_{\mathbf{A}}(\lambda_i) < \gamma_{\mathbf{A}}(\lambda_i)$), then no diagonalization exists.

b $\mu_{\mathbf{A}}(\lambda_i) = \gamma_{\mathbf{A}}(\lambda_i)$ for all eigenvalues λ_i

If, for all eigenvalues λ_i, the geometric multiplicity is equal to the algebraic multiplicity ($\mu_{\mathbf{A}}(\lambda_i) = \gamma_{\mathbf{A}}(\lambda_i)$), then proceed with step 3.

The algebraic and geometric multiplicities are

$$\begin{aligned} \mu_{\mathbf{A}}(8) &= 2 \qquad & \mu_{\mathbf{A}}(-4) &= 1 \\ \gamma_{\mathbf{A}}(8) &= 2 \qquad & \gamma_{\mathbf{A}}(-4) &= 1. \end{aligned}$$

Thus the multiplicities match for every eigenvalue and a diagolaization exists.

3 Create the matricies $\mathbf{D}$ and $\mathbf{V}$

Set

$$\mathbf{D} := \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Take n linear independent eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ (Note: For every eigenvalue λ_i you can find $\gamma_{\mathbf{A}}(\lambda_i)$ linear independent eigenvectors, and since $\mu_{\mathbf{A}}(\lambda_i) = \gamma_{\mathbf{A}}(\lambda_i)$ for every eigenvalue, you can find n linear independent eigenvectors in total). Then set

$$\mathbf{V} := \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{pmatrix},$$

i.e. create a matrix with columns that are the linear independent eigenvectors. *Important:* The eigenvector in the first column must belong to the first eigenvalue in the diagonal matrix $\mathbf{D}$, the eigenvector in the second column must belong to the second eigenvalue in $\mathbf{D}$, and so on.

We create the diagonal matrix of eigenvalues

$$\mathbf{D} := \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

and the matrix

$$\mathbf{V} := \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 2 & 2 \end{pmatrix}.$$

Then we compute the inverse

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

and obtain the representation

$$\mathbf{A} = \begin{pmatrix} 5 & 3 & 3 \\ 3 & 5 & -3 \\ 6 & -6 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$