Sequences and Limits

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Definition

A (real-valued) sequence is a function $a : \mathbb{N} \to \mathbb{R}$
Notation: $(a_n)_{n\in\mathbb{N}}$, $(a_n, n \in \mathbb{N})$, $(a_n)_{n=0}^{\infty}$, …

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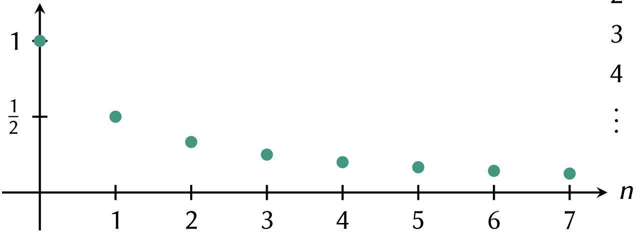
n	a_n
0	1
1	$\frac{1}{2}$
2	$\frac{1}{3}$
3	$\frac{1}{4}$
4	$\frac{1}{5}$
$\vdots$	$\vdots$

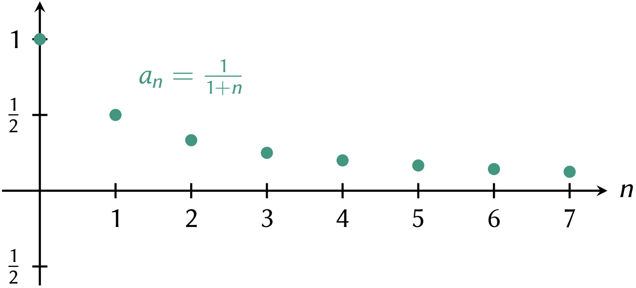


Definition

A number $a \in \mathbb{R}$ is called limit of the sequence $(a_n)_{n\in\mathbb{N}}$ if

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} : |a_n - a| < \epsilon \quad \forall n \geq N.$$

We write $\lim_{n\to\infty} a_n = a$.



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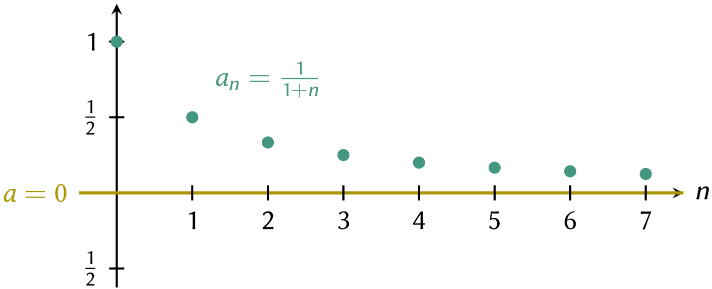

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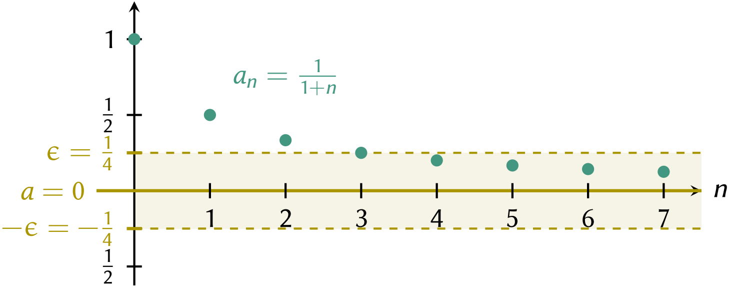

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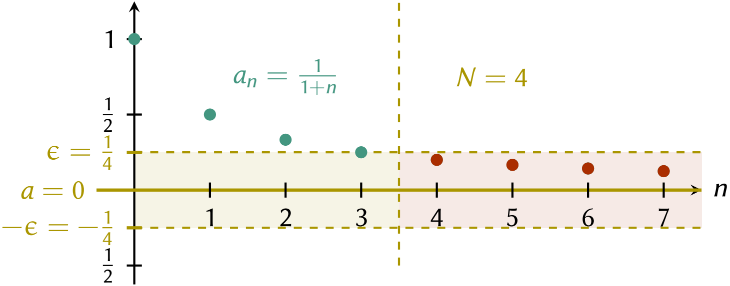

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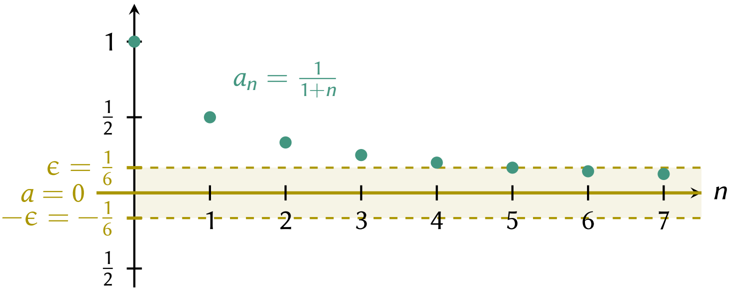

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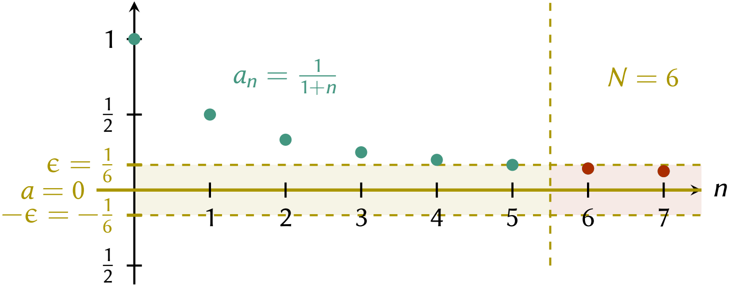

Theorem

Let $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, *and* $c \in \mathbb{R}$. *Then*

- $\lim_{n\to\infty} c \cdot a_n = c \cdot a$
- $\lim_{n\to\infty} (a_n + b_n) = a + b$
- $\lim_{n\to\infty} (a_n \cdot b_n) = a \cdot b$
- $\lim_{n\to\infty} \left(\frac{1}{a_n}\right) = \frac{1}{a}$ *if* $a_n \neq 0$ *and* $a \neq 0$
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$$\lim_{n\to\infty} \frac{1}{n^p} = 0 \qquad \text{for all } p > 0.$$

Divergent Sequences

- A sequence without a limit is called **divergent**.
- If for every $M \in R$ there is $N \in \mathbb{N}$ with

$$a_n \geq M \qquad \forall n \geq N$$

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- $\lim\limits_{n \to \infty} n^p = \infty$ for all $p > 0$
- If $\lim\limits_{n \to \infty} a_n = a \in \mathbb{R} \setminus \{0\}$, $\lim\limits_{n \to \infty} b_n = 0$, and $b_n > 0$ then

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \text{“}\frac{a}{0}\text{”} = \begin{cases} \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0. \end{cases}$$