Exercises

Sequences and Limits – Solutions

Exercise 1.

1.

$$a_n = \frac{4n^2+3n-27}{8n^2-24n+108} = \frac{n^2\left(4+3\frac{1}{n}+27\frac{1}{n^2}\right)}{n^2\left(8-24\frac{1}{n}+108\frac{1}{n^2}\right)}$$

$$= \frac{4+3\overbrace{\frac{1}{n}}^{\xrightarrow{n\to\infty}0}+27\overbrace{\frac{1}{n^2}}^{\xrightarrow{n\to\infty}0}}{8-24\underbrace{\frac{1}{n}}_{\xrightarrow{n\to\infty}0}+108\underbrace{\frac{1}{n^2}}_{\xrightarrow{n\to\infty}0}} \xrightarrow{n\to\infty} \frac{4}{8} = \frac{1}{2}$$

2.

$$b_n = \frac{5n^3-6n}{8n^4-3} = \frac{n^4(5\frac{1}{n}-6\frac{1}{n^3})}{n^4(8-3\frac{1}{n^4})}$$

$$= \frac{5\frac{1}{n}-6\frac{1}{n^3}}{8-3\frac{1}{n^4}} \xrightarrow{n\to\infty} \frac{0}{8} = 0$$

3.

$$c_n = \frac{n^2-n+5}{n+8} = \frac{n^2(1-\frac{1}{n}+\frac{5}{n})}{n^2(\frac{1}{n}+\frac{8}{n})}$$

$$= \frac{1-\frac{1}{n}+\frac{5}{n}}{\frac{1}{n}+\frac{8}{n}} \xrightarrow{n\to\infty} \infty$$

since the numerator converges to 1 and the denominator converges to 0.

4.

$$
\begin{aligned}
d_n &= \sqrt{n^2+n+1} - \sqrt{n^2+1} \\
&= \frac{(\sqrt{n^2+n+1} - \sqrt{n^2+1}) \cdot (\sqrt{n^2+n+1} + \sqrt{n^2+1})}{\sqrt{n^2+n+1} + \sqrt{n^2+1}} \\
&= \frac{(\sqrt{n^2+n+1})^2 - (\sqrt{n^2+1})^2}{\sqrt{n^2+n+1} + \sqrt{n^2+1}} \\
&= \frac{n}{\sqrt{n^2+n+1} + \sqrt{n^2+1}} \\
&= \frac{n}{\sqrt{n^2(1+\frac{1}{n}+\frac{1}{n^2})} + \sqrt{n^2(1+\frac{1}{n^2})}} \\
&= \frac{n}{\sqrt{n^2}\left(\sqrt{1+\frac{1}{n}+\frac{1}{n^2}} + \sqrt{1+\frac{1}{n^2}}\right)} \\
&= \frac{1}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}} + \sqrt{1+\frac{1}{n^2}}} \overset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{1+0+0} + \sqrt{1+0}} = \frac{1}{2}
\end{aligned}
$$

Exercise 2.

1. The sequences $a_n = n$ and $b_n = -n$ are divergent (to ∞ and $-\infty$) but $a_n + b_n = 0$ and thus converges to 0.

2. The sequences $a_n = 2n \overset{n\to\infty}{\longrightarrow} +\infty$ and $b_n = -n \overset{n\to\infty}{\longrightarrow} -\infty$ fulfil

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} n = \infty.$$

3. The sequence $a_n = -\frac{1}{n}$ is strictly increasing and converges to 0.

4. $a_n = \frac{1}{n} \overset{n\to\infty}{\longrightarrow} 0$ but with $b_n = n^2$ we have $\lim_{n\to\infty} a_n b_n = \infty$.

5. The sequence $a_n = (-1)^n$ is bounded (by -1 and 1) but a_n is not convergent.

Exercise 3.

1. Define the geometric sequence $a_n := C_0 \cdot (1+p)^n$.

2. We want to find C_0 such that $a_{10} = 10000$. In general we have

$$a_n = C_0 \cdot (1+p)^n \Leftrightarrow C_0 = \frac{a_n}{(1+p)^n}$$

so in this case we compute

$$C_0 = \frac{10000}{1.05^{10}} \approx 6139.13 \approx 6140.$$

3. The capital after 20 years is

$$a_{20} = 10000 \cdot (1.04)^{20} \approx 26532.97$$

4. (a) With the formula from (c) we get that it takes $n \approx 23.44$, i.e. about 24 years to double the capital.

   (b) With the formula from (c) we get that it takes $n \approx 11.89$, i.e. about 12 years to double the capital.

   (c) For a general $p \in [0, 1]$ we have

$$2 = (1+p)^n \Leftrightarrow \ln(2) = n \cdot \ln(1+p) \Leftrightarrow n = \frac{\ln 2}{\ln(1+p)}$$

Exercise 4.

1. The Fibonacci numbers are

n	0	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2	3	5	8	13	21	34	55	89

2. (a) For $n = 0$ we clearly have $a_0 = 0 \leq 1$.
   Suppose that $a_n \leq 1$ for some $n \in \mathbb{N}$. Then

$$a_{n+1} = \frac{1}{2}(\underbrace{a_n}_{\leq 1} + 1) \leq \frac{1}{2}(1+1) = 1.$$

   Thus $a_n \leq 1$ for all $n \in \mathbb{N}$ by induction.

   (b) Using (a) we have

$$a_{n+1} = \frac{1}{2}(\underbrace{a_n}_{\leq 1} + \underbrace{1}_{\geq 1}) \geq \frac{1}{2}(a_n + a_n) = a_n$$

   and thus a_n is increasing.

(c) With (a) and (b) we have that a_n is convergent (since a_n can not diverge to $\pm\infty$ since it is bounded nor alternate (as $(-1)^n$ for example) since it is increasing[1]). So we can use the rules for the computation of limits and obtain

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(\frac{1}{2}(a_n + 1)\right) = \frac{1}{2}\left(\lim_{n\to\infty} a_n + 1\right) = \frac{1}{2}(a+1)$$

Thus the limit a satisfies the equation $a = \frac{1}{2}(a+1)$ and we get $a = 1$.

[1]This can be proven formally, see e.g. https://en.wikipedia.org/wiki/Monotone_convergence_theorem.