Exercises

Derivatives

Exercise 1.

(a) Use the chain-rule and the product-rule to prove for two differentiable functions f, g the quotient-rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

(b) Use the properties of the exponentiation (in particular that $a = e^{\ln(a)}$ and $(e^a)^b = e^{a \cdot b}$) to compute the derivatives of the following functions:

(i) $f(x) = a^x$ for $a \in \mathbb{R} \setminus \{0\}$ (ii) $f(x) = x^x$

Exercise 2. Compute the derivatives of the following functions

(a) $f(x) = e^{\sqrt{x}}$

(b) $f(x) = \frac{1}{x} - x^3 + 2\ln x + e$

(c) $f(x) = \sin^2 x \cdot \cos^2 x$

(d) $f(x) = 2^x + \frac{\ln x}{2} - \frac{1}{x}$

(e) $f(x) = \frac{x^2}{\sin x + x}$

(f) $f(x) = e^{(x+2)^2 - x}$

Exercise 3. Compute all local and global extrema of the following functions.

(a) $f(x) = 2x^3 + 3x^2 - 36x + 42$

(b) $g(x) = e^{(x+2)^2 - x}$

Exercise 4. Compute the following limits with L'Hôspital's rule.

(a) $\lim_{x \to 0} \frac{\sin x}{x}$

(b) $\lim_{x \to 0} \ln(x) \cdot x$

(c) $\lim_{x \to 0} \frac{e^x - x - 1 - \frac{1}{2}x^2}{\sin x - x}$

Exercise 5. Let f be a differentiable and invertible function. Then the following rule for the derivative of the inverse holds

$$\left(f^{-1}(x)\right)' = \frac{1}{f'(f^{-1}(x))}$$

for all x with $f'(f^{-1}(x)) \neq 0$.

(a) Prove this rule.
Hint: Use that $f(f^{-1}(x)) = x$ and compute the derivative of both sides of this equation (use the chain rule).

(b) Use this rule to prove that

$$(\ln(x))' = \frac{1}{x}$$

where $\ln(x)$ is the natural logarithm, i.e. the inverse of e^x.

(c) Consider the function $\tan(x) : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$ defined by $\tan(x) := \frac{\sin(x)}{\cos(x)}$.

(i) Use the quotient rule to obtain

$$(\tan(x))' = 1 + \tan^2(x)$$

(ii) Let $\arctan : \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ be the inverse function of tan (i.e. $\tan(\arctan(x)) = x$). Use the rule for the derivative of the inverse above to prove

$$(\arctan(x))' = \frac{1}{1 + x^2}.$$