How to... Compute extrema of real-valued functions

Given: A differentiable, real-valued function $f : \mathbb{R} \to \mathbb{R}$.

Wanted: All local and global extrema.

Example

We consider the function

$$f(x) = x^3 - 6x^2 + 9x - 1.$$

1 Find possible extreme points

Compute the first derivative f' of f, then find the roots x_i of f', i.e. solve

$$f'(x) = 0.$$

The first derivative of f is

$$f'(x) = 3x^2 - 12x + 9.$$

We compute the roots (in this case, one may use the "completing the square"-method (*quadratische Ergänzung*) or the p-q-formula) as follows:

$$\begin{aligned} f'(x) &= 0 \\ \Leftrightarrow 3x^2 - 12x + 9 &= 0 \\ \Leftrightarrow x^2 - 4x + 3 &= 0 \\ \Leftrightarrow x_1 = 1 \text{ or } x_2 &= 3 \end{aligned}$$

The points $x_1 = 1$ and $x_2 = 3$ are *possible* extreme points.

2 Determine the type of extreme points

There multiple ways of determining if a possible extreme point is a maximum or a minimum (or neither). Sometimes there might be other, more easy arguments to proof that a point is a maximum/minimum than the two presented in the following.

a Option A: Compute higher derivatives

Compute the second derivative and check the following criterion at the possible extreme points:
If $f''(x_i) > 0$ (while $f'(x_i) = 0$) then x_i is a minimum.
If $f''(x_i) < 0$ (while $f'(x_i) = 0$) then x_i is a minimum.

Note: If $f''(x_i) = 0$ then nothing is known about x_i.

b Option B: Study the monotonicity of the function

We can use the following facts to find maxima and minima of a function.

f is increasing in x if $f'(x) \geq 0$.
f is decreasing in x if $f'(x) \leq 0$.
f has a maximum at x_i if f is increasing before and decreasing after x_i.
f has a minimum at x_i if f is decreasing before and increasing after x_i.

By determining the intervals where $f'(x) > 0$ and $f'(x) < 0$ we can easily infer where the maxima and minima of f lie.

a Option A:

We compute the second derivative as

$$f''(x) = 6x - 12.$$

Thus, $f''(x_1) = f''(1) = 6 - 12 = -6 < 0$ and $f''(x_2) = f''(3) = 16 - 12 = 6 > 0$ and, hence, there is a local maximum at $x_1 = 1$ and a local minimum at $x_2 = 3$.

b Option B:

The first derivative can be written as

$$f'(x) = (x-1)(x-3).$$

(Note that it is easy to obtain this form in the most cases, as you already know the roots of your function.) For $x < 1$ both terms are negative ($(x-1) < 0$ and $(x-3) < 0$), for $1 < x < 3$ only the second term is negative ($(x-1) > 0$ and $(x-3) < 0$, and for $x > 3$ both terms are positive ($(x-1) > 0$ and $(x-3) > 0$).
Thus, we have
f is increasing ($f'(x) > 0$) for $x < 1$
f is decreasing ($f'(x) < 0$) for $1 < x < 3$
f is increasing ($f'(x) > 0$) for $x > 3$
So, there is a local maximum at $x = 1$ and a local minimum at $x = 3$.

3 Find the global maxima/minima

To find the global extrema, compare the function values of all maxima and the limits for $x \to \pm\infty$ to find the global maximum (or to find that none exists) and compare the function values of all minima and the limits for $x \to \pm\infty$ to find the global minimum (or to find that none exists).

There is only one maximum at $x_1 = 1$ with value $f(x_1) = 3$. But this maximum is not global as $\lim_{x\to\infty} f(x) = +\infty$, hence $f(x_1) = 3$ is not the largest possible value of x.
There is only one minimum at $x_2 = 3$ with value $f(x_2) = -1$. But this maximum is not global as $\lim_{x\to-\infty} f(x) = -\infty$, hence $f(x_2) = -1$ is not the largest possible value of x.
Thus f has neither a global maximum nor a global minimum.