Taylor Approximation

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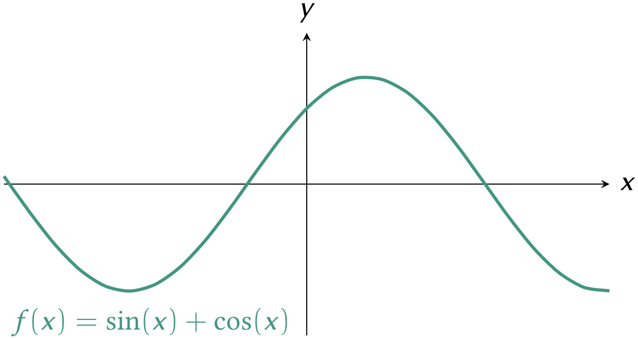
October 1, 2019

Motivation: Approximate some differentiable function f by simple functions (e.g. linear functions, polynomials)



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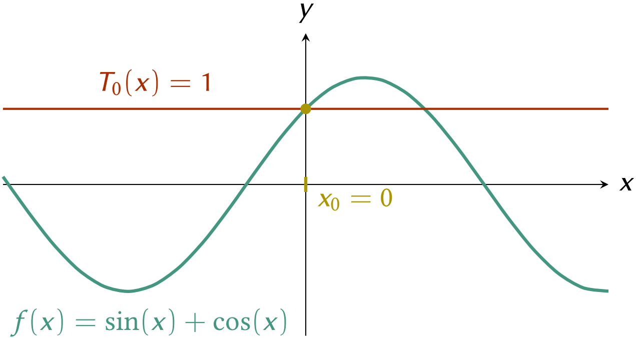

The function

$$T_0(x) := f(x_0)$$

satisfies $T_1(x_0) = f(x_0)$.

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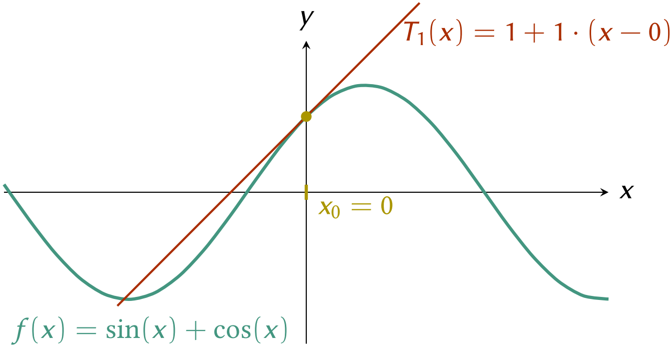

The function

$$T_1(x) := f(x_0) + f'(x_0) \cdot (x - x_0)$$

satisfies $T_1(x_0) = f(x_0)$ and $T_1'(x_0) = f'(x_0)$.

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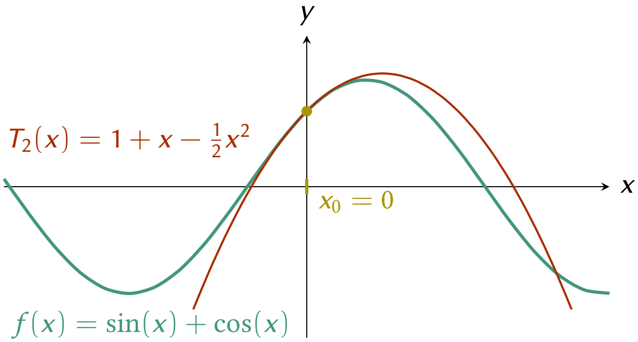

The function

$$T_2(x) := f(x_0) + f'(x_0) \cdot (x - x_0) + \frac{1}{2} f''(x_0) \cdot (x - x_0)^2$$

satisfies $T_2(x_0) = f(x_0)$, $T_2'(x_0) = f'(x_0)$, and $T_2''(x_0) = f''(x_0)$.

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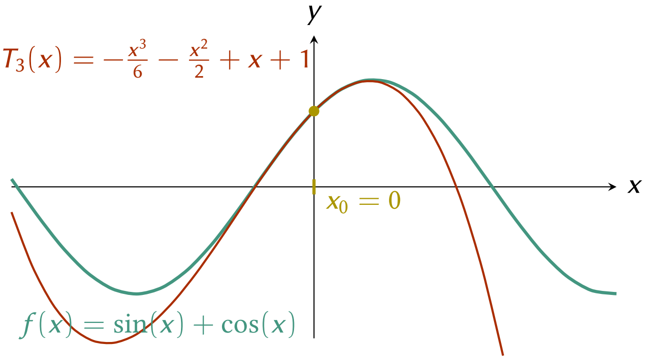

The function

$$T_3(x) := f(x_0) + f'(x_0)\cdot(x - x_0) + \frac{1}{2} f''(x_0)\cdot(x - x_0)^2 + \frac{1}{6} f''(x_0)\cdot(x - x_0)^3$$

satisfies $T_3^{(i)}(x) = f^{(i)}(x)$ for $i = 0, 1, 2, 3$.

Definition

The function

$$T_{n,x_0}(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

(where $f^{(k)}$ is the k-th derivative of f and $k! := 1 \cdot 2 \cdot \ldots \cdot (k-1) \cdot k$) is called the **$n$-th Taylor polynomial in x_0 of f**.

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Theorem

If f is $n+1$ times differentiable then there is a $\xi \in [x_0, x]$ such that

$$f(x) = T_{n,x_0}(x) + R_{n,x_0}(x, \xi)$$

where

$$R_{n,x_0}(x, \xi) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \text{ for some } \xi \text{ between } x \text{ and } x_0$$