How to... Compute a Taylor polynomial

Given: A degree $n \in \mathbb{N}$, a (at least) n-times ($n+1$ times if the error term is wanted) differentiable, real-valued function $f(x)$, and a point $x_0 \in \mathbb{R}$.

Wanted: The taylor polynomial T_{n,x_0} of degree n in the point x_0 of f.
Further, we are often interested in the *error term* and a bound for the error.

Example

We consider the function

$$f(x) = e^{x^2}$$

and want to compute the taylor polynomial of order $n = 2$ in $x_0 = 0$. Further, we want to compute the error term and find a bound for the error for $x \in [-1, 1]$.

1 Compute the derivatives

Compute the first n derivatives of the function f (including the zeroth derivatives, i.e. f itself) and compute the values of the derivatives at x_0. This can be done, e.g., in table form:

k	$f^{(k)}(x)$	$f^k(x_0)$
0	$f(x)$	$f(x_0)$
1	$f'(x)$	$f'(x_0)$
2	$f''(x)$	$f''(x_0)$
$\vdots$	$\vdots$	$\vdots$
n	$f^{(n)}(x)$	$f^{(n)}(x_0)$

The first n derivatives (and the $n+1$-th derivative for the error term) are

k	$f^{(k)}(x)$	$f^k(0)$
0	e^{x^2}	$e^0 = 1$
1	$2x\, e^{x^2}$	0
2	$2e^{x^2} + 4x^2\, e^{x^2}$	2
3	$12x\, e^{x^2} + 8x^2\, e^{x^2}$	–

The last row is needed for the error term only.

2 Set up the taylor polynomial

Compute the terms of the taylor polynomials as

$$\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

for $k = 0, 1, \ldots, n$. The values $f^{(k)}(x_0)$ were computed in the table in step 1 and $k! := 1 \cdot 2 \cdot 3 \cdots k$ is the factorial of k (with $0! = 1$).
Then the sum of all these terms is the taylor polynomial in the variable x

$$T_{n,x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

The taylor polynomial is

$$\begin{aligned} T_{2,0}(x) &= \frac{1}{0!}(x-0)^0 + \frac{0}{1!}(x-0)^1 + \frac{2}{2!}(x-0)^2 \\ &= 1 + 0x + 1x^2 = x^2 + 1. \end{aligned}$$

3 Compute the error term

Compute one additional (the $n+1$-th) derivative $f^{(n+1)}(x)$ of f. Then the error term is

$$R_{n,x_0}(x,\xi) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$

The error term depends on *two variables*! The x-value where the error is evaluated and some ξ-value. There is no information on ξ other than that ξ lies between the x-value and the x_0-value.
Note: The error term satisfies $f(x) = T_{n,x_0}(x) + R_{n,x_0}(x,\xi)$ for some (unknown) ξ between x and x_0 and for all x.

The $n+1$-th (in this case third) derivative in dependence of ξ is

$$f'''(\xi) = 12\xi\, e^{\xi^2} + 8\xi^2\, e^{\xi^2}.$$

Thus, the error term is

$$R_{3,0}(x,\xi) = \frac{12\xi\, e^{\xi^2} + 8\xi^2\, e^{\xi^2}}{3!}(x-0)^3 = \frac{12\xi\, e^{\xi^2} + 8\xi^2\, e^{\xi^2}}{6}x^3.$$

4 Find a bound for the error

Given some range for x, we can find a bound for the error term $R_{n,x_0}(x, \xi)$ by using the fact that ξ is from the same domain as x (as ξ is always in between x and x_0). By bounding the terms $f^{(n+1)}(\xi)$ and $(x - x_0)^{n+1}$, one may obtain a bound for the whole error term. This step highly depends on $f^{(n+1)}(\xi)$.

We want to find a bound for $R_{3,0}(x, \xi)$ and $x \in [-1, 1]$, i.e. we want to find a constant C such that

$$\left|R_{3,0}(x, \xi)\right| \leq C.$$

Using the rules for the absolute value (in particular the triangle inequality $|a+b| \geq |a| + |b|$) and the fact, that $\xi \in [-1, 1]$, we obtain

$$\begin{aligned} \left|R_{3,0}(x, \xi)\right| &= \left|\frac{12\xi\, e^{\xi^2} + 8\xi^2\, e^{\xi^2}}{6}\, x^3\right| \\ &\leq \frac{12|\xi|\left|e^{\xi^2}\right| + 8|\xi|^2\left|e^{\xi^2}\right|}{6}\left|x^3\right| \\ &\leq \frac{12 \cdot 1 \cdot e^1 + 8 \cdot 1 \cdot e^1}{6} \cdot 1 \\ &= \frac{10}{3} \underbrace{e^1}_{\leq 3} \leq 10. \end{aligned}$$

Hence, the error when using $T_{2,0}$ instead of f for $x \in [-1, 1]$ is at most 10.