Exercises

Optimization with side constraints – Solutions

Exercise 1.

1. The Lagrange function is

$$\mathcal{L}(x_1, x_2, \lambda) = \frac{1}{2}x_1^2 + x_2^2 + 2x_2 + 1000 + \lambda(x_1 + x_2 - 80)$$

and thus the partial derivatives are

$$\frac{\partial}{\partial x_1}\mathcal{L} = x_1 + \lambda \qquad \frac{\partial}{\partial x_2}\mathcal{L} = 2x_2 + 2 + \lambda \qquad \frac{\partial}{\partial \lambda}\mathcal{L} = x_1 + x_2 - 80$$

Solving $\nabla \mathcal{L} = 0$ yields the following equations

$$\begin{aligned} x_1 &= -\lambda \\ x_2 &= -1 - \frac{\lambda}{2} = 1 + \frac{x_1}{2} \\ x_1 + x_2 &= 80 \end{aligned}$$

Combining all equations gives

$$x_1 - 1 + \frac{x_1}{2} = 80 \Leftrightarrow \frac{3}{2}x_1 = 81 \Leftrightarrow x_1 = 54$$

and thus

$$x_2 = 26 \text{ and } \lambda = -54.$$

Thus $(54, 26)$ is a (possible) extreme point.

2. The Lagrange function is

$$\mathcal{L}(x, y, z, \lambda) = x + y + z + \lambda(xyz - 8)$$

and thus the partial derivatives are

$$\frac{\partial}{\partial x}\mathcal{L} = 1 + \lambda yz \qquad \frac{\partial}{\partial y}\mathcal{L} = 1 + \lambda xz \qquad \frac{\partial}{\partial z}\mathcal{L} = 1 + \lambda xy \qquad \frac{\partial}{\partial \lambda}\mathcal{L} = xyz - 8$$

Solving $\nabla \mathcal{L} = 0$ yields the following equations

$$\begin{aligned} \lambda yz &= 1 \\ \lambda xz &= 1 \\ \lambda xy &= 1 \\ xyz &= 8 \end{aligned}$$

Thus in particular $yz = xz = xy$ from which we can infer $x = y = z$ (since $x, y, z \neq 0$ by the last equation). With the last equality we thus get $x^3 = 8$ and thus $x = y = z = 2$.
The point $(2, 2, 2)$ is the only (possible) extreme point in this case.

3. The Lagrange function is

$$\mathcal{L}(x, y, \lambda) = 5x + 8y + \lambda(xy - 100)$$

and thus the partial derivatives are

$$\frac{\partial}{\partial x}\mathcal{L} = 5 + \lambda y$$
$$\frac{\partial}{\partial y}\mathcal{L} = 8 + \lambda x$$
$$\frac{\partial}{\partial \lambda}\mathcal{L} = xy - 1000$$

Solving $\nabla\mathcal{L} = 0$ yields the following equations

$$\lambda = -\frac{5}{y}$$
$$\lambda = -\frac{8}{x}$$
$$xy = 1000$$

The first two equations yields

$$\frac{5}{y} = \frac{8}{x} \Leftrightarrow \frac{y}{5} = \frac{x}{8} \Leftrightarrow y = \frac{5x}{8}$$

and this we get with the last equation

$$x \cdot \frac{5x}{8} = 1000 \Leftrightarrow x^2 = 1600 \Leftrightarrow x = \pm 40$$

and we get $y = \frac{5x}{8} = \pm 25$. So $(40, 25)$ and $(-40, -25)$ are the possible extreme points.

Exercise 2.

1. We want to solve the optimization problem

$$\max f(x_1, x_2) = 120\sqrt{x_1} + 160\sqrt{x_2}$$
$$\text{s.t. } x_1 + x_2 = 4 \cdot 10^6$$

The Lagrange function is

$$\mathcal{L}(x_1, x_2, \lambda) = 120\sqrt{x_1} + 160\sqrt{x_2} + \lambda(x_1 + x_2 - 4 \cdot 10^6)$$

and thus the partial derivatives are

$$\frac{\partial}{\partial x_1}\mathcal{L} = 60\frac{1}{\sqrt{x_1}} + \lambda \qquad \frac{\partial}{\partial x_2}\mathcal{L} = 80\frac{1}{\sqrt{x_2}} + \lambda \qquad \frac{\partial}{\partial \lambda}\mathcal{L} = x_1 + x_2 - 4 \cdot 10^6$$

Solving $\nabla\mathcal{L} = 0$ yields the following equations

$$\begin{aligned} \lambda &= -60\frac{1}{\sqrt{x_1}} \\ \lambda &= -80\frac{1}{\sqrt{x_2}} \\ x_1 + x_2 &= 4 \cdot 10^6 \end{aligned}$$

The first two equations yield

$$60\frac{1}{\sqrt{x_1}} = 80\frac{1}{\sqrt{x_2}} \Leftrightarrow \sqrt{x_1} = \frac{3}{4}\sqrt{x_2} \Leftrightarrow x_1 = \frac{9}{16}x_2$$

Thus we get

$$\frac{9}{16}x_2 + x_2 = 4 \cdot 10^6 \Rightarrow x_2 = 2.56 \cdot 10^6 \Rightarrow x_1 = 1.44 \cdot 10^6$$

and $\lambda = -60 \cdot \frac{1}{\sqrt{x_1}} = -0.05$.

2. If the capital changes by $\Delta x = -100000$. Thus the profit approximatively changed by

$$\Delta P \approx -\lambda\Delta x = -(-0.05)(-10^5) = -5000.$$