How to... Find possible extreme points with Lagrange Multipliers

Given: A real-valued objective function $f(x_1, \dots, x_n)$ and m equality constraints of the form $g_i(x_1, \dots, x_n) = 0$ for $i = 1, \dots, m$.

Wanted: Points $\mathbf{x} \in \mathbb{R}^n$ that satisfy the necessary conditions of an extreme point.

Example

We want to find possible for the following optimization problem.

$$\begin{aligned} \min \quad & f(x, y) && = x^3 - 9xy \\ \text{s.t.} \quad & x^2 + y^2 && = 1 \end{aligned}$$

1 Setup the Lagrange function

Introduce a Lagrangian multiplier variable λ_i for all constraints. Then, setup the Lagrange function

$$\mathcal{L}(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) := f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}),$$

i.e., add to the objective function f all constraint functions multiplied with the associated multiplier λ_i.
Note: It may be necessary, to first transform the constraints such that they are of the form $g(\mathbf{x}) = 0$!

First, we rephrase the constraint to $x^2 + y^2 - 1 = 0$ such that we can identify the constraint function $g(x, y) = x^2 + y^2 - 1$. Since we only have one constraint, there will be only one multiplier that we denote by λ. We obtain the Langrange function

$$\mathcal{L}(x, y, \lambda) = x^3 - 9xy + \lambda(x^2 + y^2 - 1).$$

2 Compute the partial derivatives

Compute all partial derivatives of the Lagrange function (with respect to all $x_1, \dots, x_n$ as well as all Langrange multipliers $\lambda_1, \dots, \lambda_m$,i.e, compute

$$\frac{\partial}{\partial x_i}\mathcal{L}(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) \quad \text{and} \quad \frac{\partial}{\partial \lambda_i}\mathcal{L}(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m).$$

Note: Computing the partial derivatives wrt. λ_i will yield the constraint $g_i(x)$, hence, these derivatives require no computation.

Since we have two variables x and y and one multiplier λ, we compute the following derivatives.

$$\begin{aligned}
\frac{\partial}{\partial x}\mathcal{L}(x, y, \lambda) &= 3x - 9y + 2\lambda \\
\frac{\partial}{\partial y}\mathcal{L}(x, y, \lambda) &= -9x + 2\lambda \\
\frac{\partial}{\partial \lambda}\mathcal{L}(x, y, \lambda) &= x^2 + y^2 - 1
\end{aligned}$$

3 Solve $\nabla\mathcal{L} = 0$

Finally solve the system $\nabla\mathcal{L}(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = 0$, i.e., set all partial derivatives from the previous step equal to zero and solve the system of equations. The resulting point(s) are possible extreme points with their multipliers λ.

Setting all partial derivatives to 0 yields the following system of equations.

$$\begin{aligned} 3x - 9y + 2\lambda &= 0 \\ -9x + 2\lambda &= 0 \\ x^2 + y^2 - 1 &= 0 \end{aligned}$$

By subtracting the first two equations we obtain $12x - 9y = 0$ which is equivalent to

$$y = \frac{4}{3}x.$$

Inserting this in the third equation gives

$$x^2 + \frac{16}{9}x^2 - 1 = 0 \Leftrightarrow \frac{25}{9}x^2 = 1 \Leftrightarrow x^2 = \frac{9}{25} \Leftrightarrow x = \pm\frac{3}{5}$$

and thus $y = \pm\frac{4}{5}$. Hence, the possible extreme points are

$$(x_1, y_1) = \left(\frac{3}{5}, \ \frac{4}{5}\right) \quad \text{and} \quad (x_2, y_2) = \left(-\frac{3}{5}, \ -\frac{4}{5}\right)$$

with Lagrange multipliers $\lambda_1 = \frac{9}{5}$ and $\lambda_2 = -\frac{9}{5}$ (obtained by using $x = \pm\frac{3}{5}$ and the second equation).