Integration

Philipp Warode

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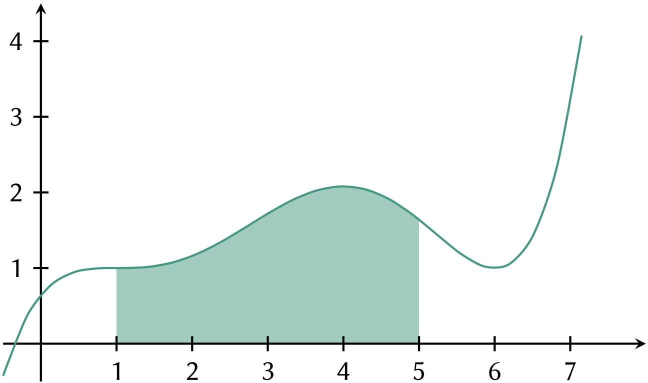
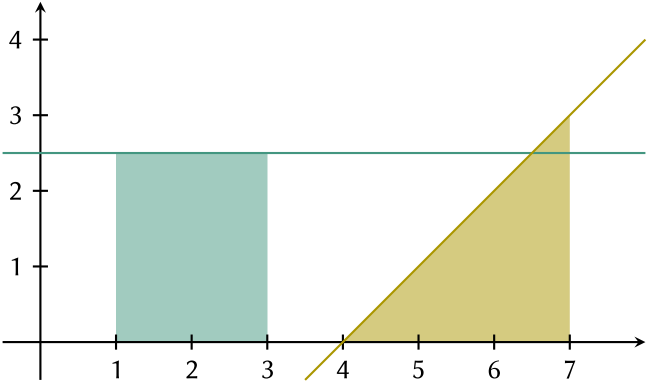
We want to find the area under some function.



Easy for simple geometric forms as squares or triangles, i.e. for constant or linear functions.



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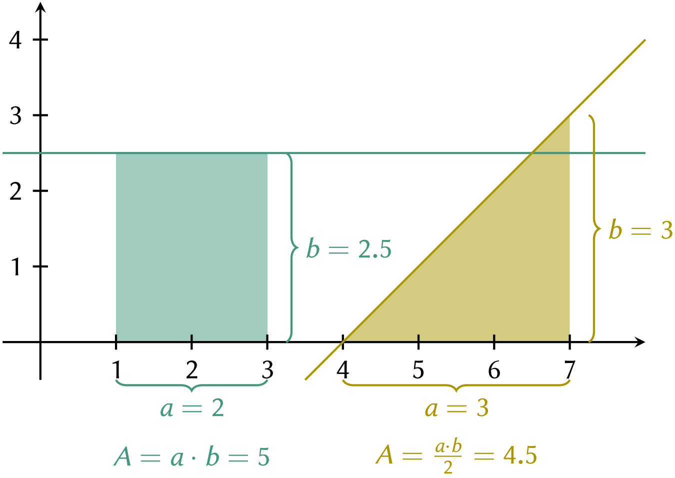
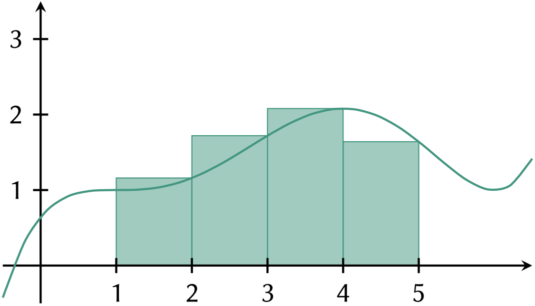


For general functions, we approximate the area with rectangles.



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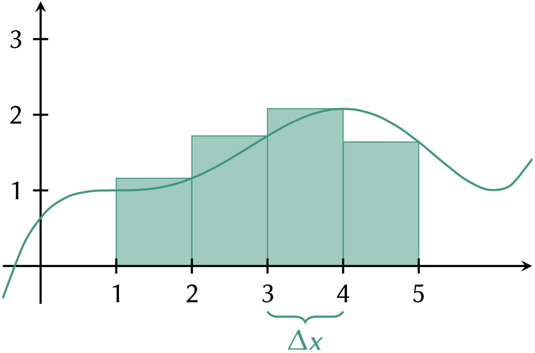

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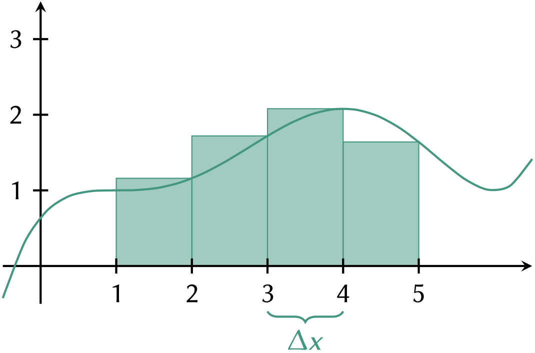

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$$A_n = \sum_{k=1}^{n} f(x_n) \cdot \Delta x \xrightarrow{\Delta x \to 0} \int_a^b f(x) dx$$

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- We often write

$$\int f(x)dx = F(x) + c$$

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Theorem (Fundamental theorem of calculus)

Let $f : [a, b] \to \mathbb{R}$ be continuous then

- *An antiderivative F of f exists*
- *For any antiderivative F of f it holds*

$$\int_a^b f(x)dx = F(b) - F(a)$$

The computation of an integral reduces to finding an antiderivative.

- Derivatives of elementary functions:

$f(x)$	0	1	x	x^p ($p \neq -1$)	$\frac{1}{x}$	e^x	$\sin x$	$\cos x$	$f(ax)$
$F(x)$	1	x	$\frac{1}{2}x^2$	$\frac{1}{p+1}x^{p+1}$	$\ln x$	e^x	$-\cos x$	$\sin x$	$\frac{1}{a}F(ax)$

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Theorem (Rules for integration)

$$\int_a^b c \cdot f(x) dx = c \cdot \int_a^b f(x) dx$$

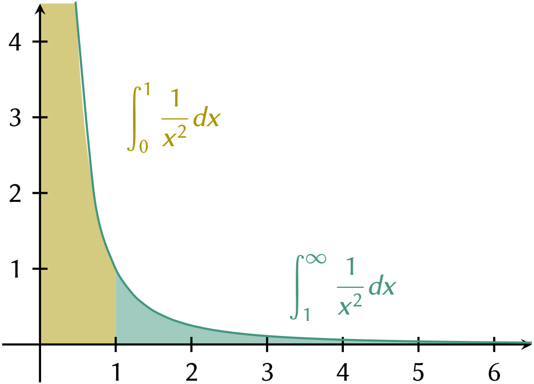
$$\int_a^b f(x) + g(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

Improper Integrals

- An integral with
  - at least one endpoint in $\{-\infty, \infty\}$
  - an endpoint a with $\lim_{x \to a} f(x) \in \{-\infty, \infty\}$

  is called **improper integral**.

Examples:



- To compute an improper integral
  - Replace the improper endpoint a by some variable α
  - Compute $A(\alpha) = \int_{\alpha}^{b} f(x)dx$
  - Compute $\lim\limits_{\alpha \to a} A(\alpha)$
- If the limit exits, we say the integral converges